CYCLIC PURSUIT WITHOUT COORDINATES: CONVERGENCE to REGULAR POLYGON FORMATIONS

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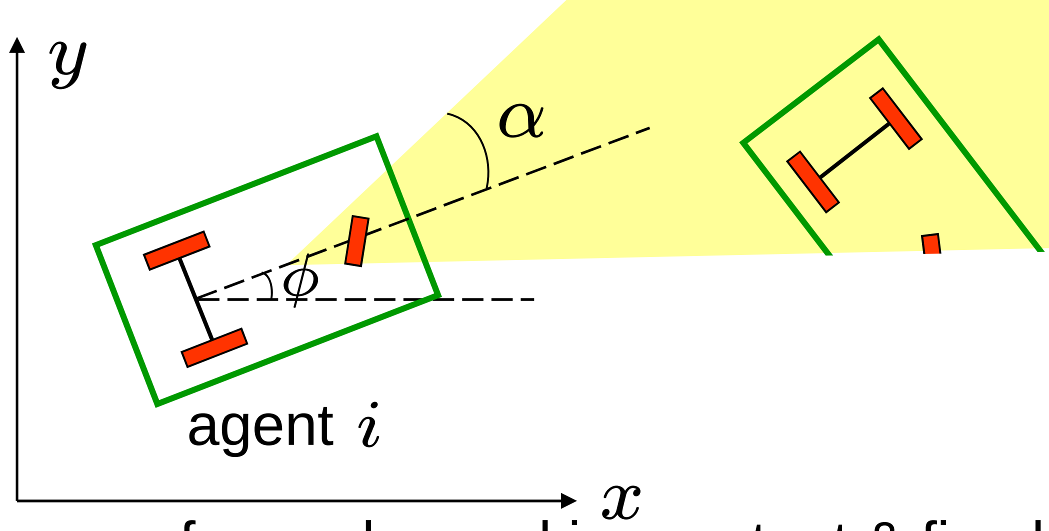
CDC, LA, Dec 2014

MINIMALISTIC CONTROL & SENSING MODEL

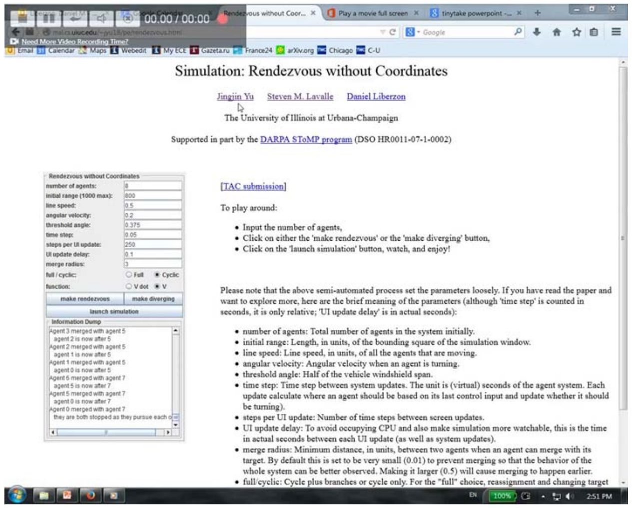


$$\dot{x}_i = u_i \cos \phi_i$$
$$\dot{y}_i = u_i \sin \phi_i$$
$$\dot{\phi}_i = w_i$$

- Assume forward speed is constant & fixed (say $u_i \equiv 1$)
- Angular speed $w_i \in \{-\bar{w}, 0, \bar{w}\}$
- Can detect presence of another agent in windshield sector $(-\alpha, \alpha)$ (unlimited sensing range)
- Agents cyclically arranged, with each agent i seeing its target agent $i+1$ in its windshield (connectivity) – can be maintained if $\bar{w}$ is large enough
- System behavior depends critically on windshield angle α

SMALL WINDSHIELD ANGLE: RENDEZVOUS

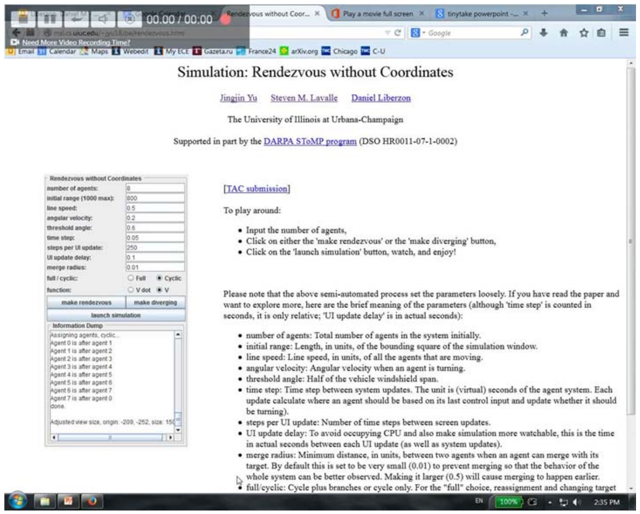
$\alpha < \pi/n$ where $n =$ # of agents

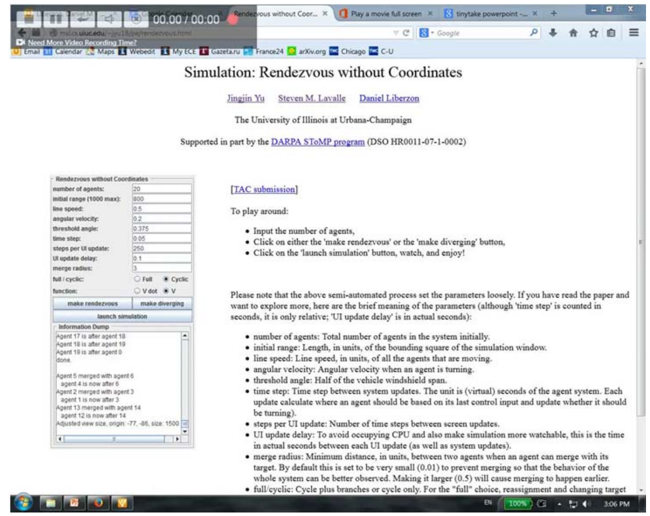


In this case all agents converge to a single point, with perimeter $P(t)$ of the formation polygon serving as a Lyapunov function [Yu–LaValle–L, IEEE TAC, 2012]

LARGE WINDSHIELD ANGLE: REGULAR POLYGONS?

$\alpha > \pi/n$ where $n =$ # of agents





Agents diverge but tend to form regular polygons

Our goal: theoretically justify this empirically observed phenomenon

RELATION to PRIOR WORK

- Many works on convergence of multi-agent formations to regular shapes, using different tools and different modeling assumptions:

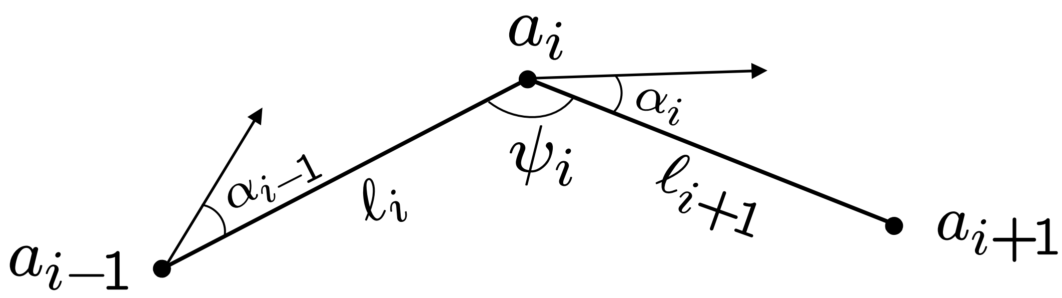
  Behroozi–Gagnon (1980s), Richardson (2001),
  Marshall–Broucke–Francis (2004, 2006), Pavone–Frazzoli (2007),
  Sinha–Ghose (2007), Arnold–Baryshnikov–LaValle (2012),
  Galloway–Justh–Krishnaprasad (2013), Sharma–Ramakrishnan–Kumar (2013)

- Unique feature of our model (same as in Yu–LaValle–L, 2012): neither inter-agent distance nor heading error available for feedback

- As in Marshall et al., we will use eigenvalue properties of block-circulant matrices to show convergence

- But unlike in Marshall et al., here regular shape is attractive while formation size grows, which makes analysis different

SYSTEM EQUATIONS



Angles are subject to constraint $\sum_{i=1}^{n} \psi_i = \pi(n-2k)$

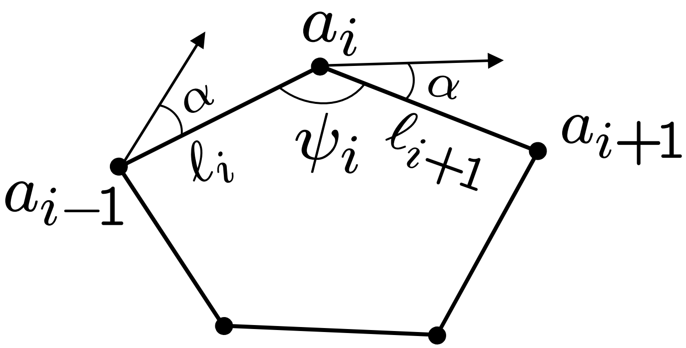
$$\dot{\psi}_i = (\sin \alpha_{i-1} + \sin(\psi_i + \alpha_i))/\ell_i \\ \quad - (\sin \alpha_i + \sin(\psi_{i+1} + \alpha_{i+1}))/\ell_{i+1}$$

$$\dot{\ell}_i = -\cos \alpha_{i-1} - \cos(\psi_i + \alpha_i)$$

"Constant-bearing" case: assume $\alpha_i \equiv \alpha \ \forall i$, i.e., each a_i maintains its target a_{i+1} exactly on its windshield boundary

This is sliding regime for angular velocity $w_i \in \{-\bar{w}, 0, \bar{w}\}$

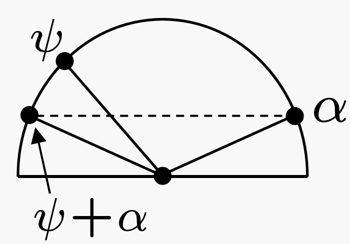
STATIONARY SHAPES ARE REGULAR POLYGONS



$$\dot{\psi}_i = (\sin\alpha + \sin(\psi_i + \alpha))/\ell_i$$
$$\quad - (\sin\alpha + \sin(\psi_{i+1} + \alpha))/\ell_{i+1}$$
$$\dot{\ell}_i = -\cos\alpha - \cos(\psi_i + \alpha)$$

- A shape is an equivalence class of polygons w.r.t. scaling & rigid motions
- A shape is stationary if it is invariant under system dynamics
- For a shape to be stationary we must have $\dot{\psi}_i \equiv 0 \ \forall i$
- Can show that then all angles must be equal: $\boxed{\psi_i \equiv \psi \ \forall i}$
  so all edges have the same derivative:
  $\ell_i(t) = \ell_i(0) - t\,(\cos\alpha + \cos(\psi + \alpha))$
- Case of interest here is when α is large enough s.t. $\cos\alpha + \cos(\psi + \alpha) < 0$
- Since $\ell_i(t) \to \infty$ and $\ell_i(t)/\ell_j(t) \to 1$,
  stationary shape must have equal edges: $\boxed{\ell_i(t) = \ell_j(t) \ \forall i, j}$

Example: convex regular n-gon



$$\psi = \pi - 2\pi/n$$

Need $\alpha > \pi/n$

SPACE–TIME COORDINATE CHANGE

- Similarly to [Galloway et al., 2013], instead of edge lengths $\ell_i(t)$ consider $\rho_i(t) := \frac{\ell_i(t)}{P(t)}$ where $P(t) := \sum_{i=1}^{n} \ell_i(t)$ is perimeter

- In rescaled coords, stationary shapes (regular polygons) $\longrightarrow$ stationary points, i.e., equilibria: $\psi_i \equiv \psi, \ \rho_i \equiv 1/n \quad \forall i$

- Also introduce time rescaling $\tau(t) := t/P(t)$ which allows us to write the system in (ψ_i, ρ_i) coords

- If equilibria are locally exponentially stable for this new system, then regular polygons are locally attractive for original system

EIGENVALUE ANALYSIS of LINEARIZED SYSTEM

- Rescaled coords: $(\psi_1, \rho_1, \ldots, \psi_n, \rho_n)$ where $\rho_i(t) := \ell_i(t)/P(t)$
- $2n \times 2n$ Jacobian matrix at equilibria $\psi_i = \psi$, $\rho_i = 1/n$ takes block-circulant form $J = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & A_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_{n-1} & A_0 \end{pmatrix}$
- It always has one 0 eigenvalue with eigenvector $(d, 0, \ldots, d, 0)$
  – not an admissible direction as all angles ψ_i cannot increase
- k-th pair of eigenvalues of J is eigenvalues of 2×2 matrix $A_0 + A_1\chi_k + A_2\chi_k^2 + \ldots + A_{n-1}\chi_k^{n-1}$ where $\chi_k := e^{2\pi ik/n}$
- By Routh-Hurwitz criterion for polynomials with complex coefficients can show that these eigenvalues have $\mathrm{Re}(\lambda) < 0$ if and only if

$$(3C^2 + AC + 1 + B)\cos(2\pi/n) < (6C^2 + A^2 + 5AC - 1 - B)$$

where $A := \cos\alpha$, $B := \cos\psi$, $C := \cos(\psi + \alpha)$

Sufficient condition for local attractivity of regular polygons

DISCUSSION

We have convergence to regular polygons if $A+C<0$ and

$$(3C^2+AC+1+B)\cos(2\pi/n) < (6C^2+A^2+5AC-1-B) \quad (\ast)$$

where $A := \cos\alpha$, $B := \cos\psi$, $C := \cos(\psi+\alpha)$

For convex regular n-gon we have $\psi = \pi(1-2/n)$ and can show that $(\ast)$ holds for $\alpha > \pi/n$ as long as α is not too large (recall that $\alpha < \pi/n$ gives rendezvous)

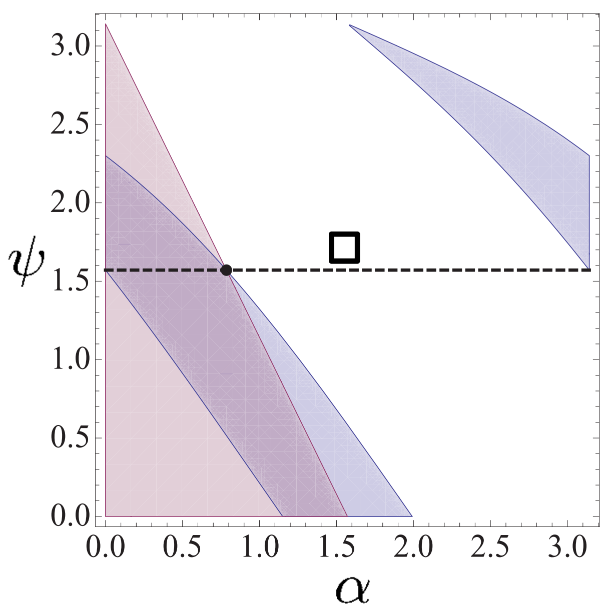
Example: $n = 4$

Pink region is where $A+C>0$

Blue region is where $(\ast)$ fails

White region is the 'good' region

Square ($\psi = \pi/2$) is locally attractive for $\alpha \in (\pi/4, \pi)$



DISCUSSION

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For convex regular n-gon we have $\psi=\pi(1-2/n)$ and can show that ($\ast$) holds for $\alpha>\pi/n$ as long as α is not too large (recall that $\alpha<\pi/n$ gives rendezvous)

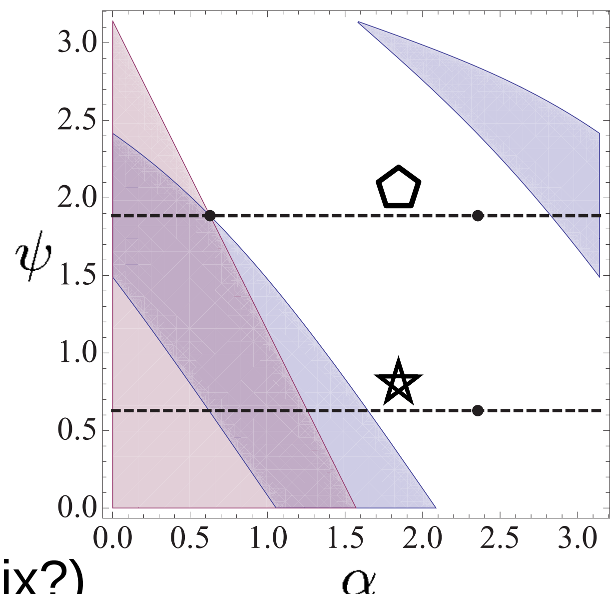
Example: $n=5$

Pink region is where $A+C>0$

Blue region is where ($\ast$) fails

White region is the 'good' region

Both pentagon ($\psi=3\pi/5$) and pentagram ($\psi=\pi/5$) are locally attractive for some range of α (separatrix?)



DISCUSSION

We have convergence to regular polygons if $A + C < 0$ and

$(3C^2 + AC + 1 + B)\cos(2\pi/n) < (6C^2 + A^2 + 5AC - 1 - B)$ **(*)**

where $A := \cos\alpha$, $B := \cos\psi$, $C := \cos(\psi + \alpha)$

Using 1 instead of $\cos(2\pi/n)$ in **(*)**, we obtain

$$3C^2 + A^2 + 4AC - 2 - 2B > 0$$

which can be simplified to

$$\cos\left(2\alpha + \frac{\psi}{2}\right) + 3\cos\left(2\alpha + \frac{3\psi}{2}\right) > 0$$

This sufficient condition is more conservative but works for any n

CONCLUSIONS

- Agents in cyclic pursuit with large fixed relative heading angle diverge but tend to form regular patterns

- Stationary shapes are regular polygons

- Block-circulant structure of linearization matrix gives a sufficient condition for convergence

- Need to better understand system behavior, especially far away from stationary shapes

- Extensions to non-constant heading angles and non-cyclic formations are also of interest