Quantized Feedback Stabilization of Linear Systems

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***Abstract*—This paper addresses feedback stabilization problems for linear time-invariant control systems with saturating quantized measurements. We propose a new control design methodology, which relies on the possibility of changing the sensitivity of the quantizer while the system evolves. The equation that describes the evolution of the sensitivity with time (discrete rather than continuous in most cases) is interconnected with the given system (either continuous or discrete), resulting in a hybrid system. When applied to systems that are stabilizable by linear time-invariant feedback, this approach yields global asymptotic stability.**

***Index Terms*—Feedback stabilization, hybrid system, linear control system, quantized measurement.**

I. INTRODUCTION

THIS PAPER deals with quantized feedback stabilization problems for linear time-invariant control systems. A quantizer, as defined here, acts as a functional that maps a real-valued function into a piecewise constant function taking on a finite set of values. Given a system that is stabilizable by linear time-invariant feedback, the problem under consideration is to find a quantized feedback control law that stabilizes the system. Problems of this kind arise, for example, when the output measurements to be used for feedback are transmitted via a digital communication channel.

A standard assumption in the literature on quantized control is that one is given a *fixed* quantizer representing some finite precision effects in the system to be controlled (see, among many sources, [5]–[7] and [17]). In this paper we adopt a different point of view. Namely, we treat the number of values of the quantizer as being fixed *a priori*, but we allow ourselves to alter other quantization parameters while the system evolves. This approach enables us to achieve asymptotic stability, a property that cannot be obtained with the schemes previously investigated. Some examples of situations where the present assumptions are meaningful will be discussed below.

We now introduce some notation and give a definition that makes the above concepts precise. We will denote by $\|x\|$ the standard Euclidean norm of a vector $x \in \mathbb{R}^n$ and by $\|A\|$ the induced norm of a matrix $A \in \mathbb{R}^{n\times n}$. We will also use the *maximum norm* on $\mathbb{R}^n$ defined by $\|x\|_\infty := \max\{|x_i|: 1 \le i \le n\}$, as well as the induced *infinity norm* on $\mathbb{R}^{n\times n}$ defined by $\|A\|_\infty := \max\{\sum_{j=1}^n |A_{ij}|: 1 \le i \le n\}$. The set of nonnegative integers will be denoted by $\mathbb{Z}_{\geq 0}$. We will let I_A denote the *indicator* of a set $A \in \mathbb{R}$, i.e.,

$$I_A(t) := \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{if } t \notin A \end{cases}$$

We find it convenient to use the following *floor* function: $\lfloor x \rfloor := \max\{k \in \mathbb{Z}: k < x\}$. Functions denoted by the capital letters F, G and H are assumed to be piecewise continuous in all their arguments.

Given a positive integer M and a nonnegative real number Δ, we define the *quantizer* q: $\mathbb{R} \to \mathbb{Z}$ with *sensitivity* Δ and *saturation value* M by the formula

$$q(x) = \begin{cases} M, & \text{if } x > (M+1/2)\Delta \\ -M, & \text{if } x \le -(M+1/2)\Delta \\ \left\lfloor \dfrac{x}{\Delta} + \dfrac{1}{2} \right\rfloor, & \text{if } -(M+1/2)\Delta < x \\ & \quad \le (M+1/2)\Delta \end{cases}$$

Thus on the interval $((k-1/2)\Delta, (k+1/2)\Delta]$ of length Δ, where $k \in \mathbb{Z}$ and $-M \le k \le M$, the function q takes on the value k. Suppose that we have n quantizers q_i: $\mathbb{R} \to \mathbb{Z}$ with sensitivities Δ_i and the same saturation value M ($i = 1, \cdots, n$). We define the quantizer q: $\mathbb{R}^n \to \mathbb{Z}^n$ with sensitivity $(\Delta_1, \cdots, \Delta_n)$ and saturation value M as follows: $q(x) := (q_1(x_1), \cdots, q_n(x_n))$, where $(x_1, \cdots, x_n)$ are the coordinates of x relative to a fixed orthonormal basis in $\mathbb{R}^n$. Geometrically, $\mathbb{R}^n$ is thereby divided into a finite number of rectilinear *quantization blocks*, each corresponding to a fixed value of q. We will sometimes refer to the boundaries between these quantization blocks as *switching hyperplanes*. If all q_i's have the same sensitivity Δ, we will call q a *uniform* quantizer with sensitivity Δ.

The above notation is similar to the one used by Delchamps in [6], but an essential feature that makes our definition different is that the set of values taken on by the quantizer here is finite rather than countable. In fact, we are especially interested in situations where the saturation value M is small. For example, we will consider the case when $M = 1$. The corresponding quantizer can be thought of as describing a sensor which determines whether the temperature of a certain object is "normal," "too high," or "too low."

The approach to be used here is based on the hypothesis that it is possible to change the sensitivity (but not the saturation value) of the quantizer on the basis of available quantized measurements. Such a quantizer can be viewed as a device consisting of a multiplier by an adjustable factor followed by an analog-to-digital converter. But this is not the only situation that can be alluded to as a motivation for the present work. For example, given

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a temperature sensor with limited capability of the kind mentioned above, it is reasonable to assume that one is allowed to adjust the threshold settings. As another example, a camera with zooming capability and a finite number of pixels can be modeled as a quantizer with varying sensitivity and a fixed saturation value. More generally, our approach fits into the framework of *control with limited information* ([12], [24]) in the sense that the state of the system is not completely known, but it is only known which one of a fixed number of quantization blocks contains the current state at each instant of time. Changing the size of these quantization blocks, one can extract more information about the behavior of the system, which appears to be a very natural thing to do when such manipulations are permitted.

The control policy will usually be divided into two stages. First, since the initial state is unknown, we will have to "zoom out," i.e., increase Δ until the state of the system can be adequately measured. Second, we will "zoom in," i.e., decrease Δ in such a way as to drive the state to 0. This can be formalized by introducing a discrete "zoom" variable z taking on the values 1 and -1. In essence, our goal is to demonstrate that if a linear system can be stabilized by linear time-invariant feedback, then it can also be stabilized by quantized feedback with the help of the approach described here.

For continuous-time systems, we will describe the evolution of Δ with time by an equation that might take the form

$$\Delta(t) = G(z, \lfloor t/\tau \rfloor, q(x(\lfloor t/\tau \rfloor \tau)), \Delta(\lfloor t/\tau \rfloor \tau))$$

where τ is a fixed positive number. The above equation defines a "strictly causal" function Δ that is continuous from the left everywhere and maintains a constant value on each interval $(k\tau, k\tau + \tau]$, $k \in \mathbb{Z}_{\geq 0}$. In the control policies considered below, such an equation for Δ will be coupled with the given linear system. This results in a "hybrid system" of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t, q(x(t)), \Delta(t)), \\ \Delta(t) &= G(z, \lfloor t/\tau \rfloor, q(x(\lfloor t/\tau \rfloor \tau)), \Delta(\lfloor t/\tau \rfloor \tau)). \end{aligned} \tag{1}$$

This system falls into the general framework for hybrid systems presented in [2]. Clearly, for every initial condition $(x(0), \Delta(0))$ there exists a unique solution trajectory. The system (1), as well as all other systems of differential-difference equations considered in this paper, is of "hereditary type," and as such is covered by the theory of hereditary systems developed in [10]. The logic governing the construction of closed-loop systems such as (1) will become clear later.

Two technical comments are in order. The first one concerns our usage of the term "asymptotic stability." The desired properties of the control policies to be considered below, which we will refer to loosely as "asymptotic stability," are that i) $x = 0$ is an equilibrium state of the first equation in (1), that ii) it is stable in the sense of Lyapunov, and that iii) we have $x(t), \Delta(t) \to 0$ as $t \to \infty$. However, this does not really mean that the system (1) is asymptotically stable because, as we will see, the state $x = \Delta = 0$ will typically not be an equilibrium state of the overall system (1). Since the validity of i) and ii) will usually be obvious, in the proofs to follow we will concentrate on verifying the property iii).

Secondly, in the continuous-time case quantized feedback control laws lead to differential equations with discontinuous right-hand sides. When the existence and uniqueness of solutions in the classical sense cannot be guaranteed, they are to be interpreted in the sense of Filippov [8]. This issue will arise in Section IV where we will use a sliding mode control law based on quantized output measurements for the case when the saturation value is small. Other control strategies described in this paper do not rely on chattering, and the analysis of the resulting closed-loop systems does not explicitly require a concept of generalized solution.

The outline of the paper is as follows. In Section II we develop techniques for stabilizing continuous-time linear systems with quantized state feedback. In Section III we present analogous results for discrete-time systems. Section IV deals with quantized output feedback stabilization. In Section V we describe control strategies that involve state observation. In Section VI we briefly discuss quantized feedback stabilization of nonlinear systems. We make some concluding remarks and sketch directions for future research in Section VII.

II. Quantized State Feedback Stabilization: Continuous Time

This section deals with state feedback stabilization problems for the continuous-time linear system

$$\dot{x} = Ax + Bu \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and A and B are matrices of suitable dimensions. If (2) is controllable in the unstable modes, then there exists a matrix K such that all eigenvalues of $A - BK$ have negative real parts (see [22, Sec. 6.3]). In this case it seems logical to try to implement a quantized state feedback control law of the form $u = -K\Delta q(x)$, where q is a uniform quantizer with sensitivity Δ. Our first result shows that this control law yields global asymptotic stability when combined with a suitable adjustment policy for Δ.

Theorem 1: Suppose that all eigenvalues of $A - BK$ have negative real parts. Then there exists a control policy of the form

$$\begin{aligned} \Delta(t) &= G(z, \lfloor t/\tau \rfloor, q(x(\lfloor t/\tau \rfloor \tau)), \Delta(\lfloor t/\tau \rfloor \tau)) \\ u(t) &= -KI_{[k_0\tau, \infty)}(t)\Delta(t)q(x(t)) \end{aligned}$$

where q is a uniform quantizer with sensitivity $\Delta(t)$ and k_0 is a positive integer, such that the solutions of the closed-loop system

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BKI_{[k_0\tau, \infty)}(t)\Delta(t)q(x(t)), \quad x(0) \textit{ arbitrary} \\ \Delta(t) &= G(z, \lfloor t/\tau \rfloor, q(x(\lfloor t/\tau \rfloor \tau)), \Delta(\lfloor t/\tau \rfloor \tau)), \quad \Delta(0) = 0 \end{aligned}$$

approachd 0 as $t \to \infty$.

Proof: Consider the system

$$\dot{x} = Ax - BK\Delta q(x)$$

which we can also write as

$$\dot{x} = (A - BK)x + BKs(x) \tag{3}$$

thus displaying the "error" vector $s(x) := x - \Delta q(x)$. When

$$\|x\|_\infty \le \left(M - \tfrac{1}{2}\right)\Delta \tag{4}$$

the quantizer does not saturate (i.e., x belongs to the union of the quantization blocks of finite size), so that we have

$$\|s(x)\| \le \Delta\sqrt{n}/2. \tag{5}$$

We will let $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the smallest and the largest eigenvalue of a symmetric matrix P, respectively. Recall that by the standard Lyapunov stability theory there exist positive definite symmetric matrices Q and D such that $(A - BK)^T Q + Q(A - BK) = -D$. Whenever (4) holds, the derivative of $x^T Qx$ along the solutions of (3) is given by

$$\begin{aligned}\frac{d}{dt}\, x^T Qx &= -x^T Dx + 2x^T QBKs(x)\\ &\le -\lambda_{\min}(D)\|x\|^2 + 2\|x\|\,\|QBK\|\Delta\sqrt{n}/2\\ &= -\|x\|(\lambda_{\min}(D)\|x\| - \|QBK\|\Delta\sqrt{n}).\end{aligned} \tag{6}$$

The last expression is negative outside the ball $\{x\colon\ \|x\| \le \Theta\Delta\sqrt{n}\}$, where

$$\Theta := \|QBK\|/\lambda_{\min}(D).$$

In what follows we will use the simple facts that the radius of the ball inscribed in an ellipsoid of the form $\{x\colon x^T Qx \le \gamma^2\}$ equals $\gamma/\sqrt{\lambda_{\max}(Q)}$ and the radius of the ball circumscribed about the same ellipsoid equals $\gamma/\sqrt{\lambda_{\min}(Q)}$. Fix an arbitrary $\epsilon > 0$. Define the *scaling factor* Ω by the formula

$$\Omega := (\Theta\sqrt{n} + \epsilon)\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}}\left(M - \frac{1}{2}\right)^{-1}$$

and take the saturation value M of q to be large enough so that $\Omega < 1$. Define

$$\tau := \frac{(M - 1/2)^2\lambda_{\min}(Q) - (\Theta\sqrt{n} + \epsilon)^2\lambda_{\max}(Q)}{(\Theta\sqrt{n} + \epsilon)\lambda_{\min}(D)\epsilon}. \tag{7}$$

Since $\Omega < 1$, it is easy to see that $\tau > 0$.

We now describe the "zooming-out" stage of the control strategy ($z = 1$). Set the control to 0 and let $\Delta(0) = 0$. Increase Δ fast enough to dominate the rate of growth of $\|e^{At}\|$, e.g., let $\Delta(t) = e^{2\|A\|\lfloor t/\tau\rfloor\tau}$. Then there exists a positive integer k such that

$$\|x(k\tau)\| \le \Delta(k\tau)\left(\left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}} - \sqrt{n}\right)$$

hence

$$\|q(x(k\tau))\| \le \left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}} - \frac{\sqrt{n}}{2}$$

by virtue of (5). We can thus define

$$k_0 := \min\left\{k \ge 1\colon \|q(x(k\tau))\| \le \left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}} - \frac{\sqrt{n}}{2}\right\}$$

which implies

$$\|x(k_0\tau)\| \le \Delta(k_0\tau)\left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}}.$$

Therefore,

$$x^T(k_0\tau)Qx(k_0\tau) \le (\Delta(k_0\tau))^2\left(M - \tfrac{1}{2}\right)^2\lambda_{\min}(Q). \tag{8}$$

Next, we come to the "zooming-in" stage ($z = -1$). Observe that (4) holds with $\Delta = \Delta(k_0\tau)$ for any x that belongs to the ellipsoid

$$R_1 := \left\{x\colon x^T Qx \le (\Delta(k_0\tau))^2\left(M - \tfrac{1}{2}\right)^2\lambda_{\min}(Q)\right\}.$$

Since $\Omega < 1$, we have in particular $(M - 1/2)\sqrt{\lambda_{\min}(Q)} > \Theta\sqrt{n}\sqrt{\lambda_{\max}(Q)}$. From this and (6) it follows that if we let $u(t) = -K\Delta(t)q(x(t))$ with $\Delta(t) = \Delta(k_0\tau)$ for $k_0\tau < t \le k_0\tau + \tau$, then x will not leave R_1, hence the quantizer will not saturate. We claim that

$$\begin{aligned}&x^T(k_0\tau + \tau)Qx(k_0\tau + \tau)\\ &\quad\le (\Delta(k_0\tau))^2\left(\Theta\sqrt{n} + \epsilon\right)^2\lambda_{\max}(Q)\\ &\quad= (\Omega\Delta(k_0\tau))^2\left(M - \tfrac{1}{2}\right)^2\lambda_{\min}(Q).\end{aligned} \tag{9}$$

Suppose that (9) is not true. Then we have

$$x^T(k_0\tau + \tau)Qx(k_0\tau + \tau) > (\Delta(k_0\tau))^2\left(\Theta\sqrt{n} + \epsilon\right)^2\lambda_{\max}(Q) \tag{10}$$

and therefore

$$\|x(t)\| > \Delta(k_0\tau)\left(\Theta\sqrt{n} + \epsilon\right) \quad \text{for all } t \in [k_0\tau,\ k_0\tau + \tau]. \tag{11}$$

But (6) and (11) imply that for $k_0\tau \le t \le k_0\tau + \tau$ we have

$$\frac{d}{dt}\, x^T Qx \le -(\Delta(k_0\tau))^2\left(\Theta\sqrt{n} + \epsilon\right)\lambda_{\min}(D)\epsilon.$$

Comparing the last inequality with (7), (8) and (10), we arrive at a contradiction, which establishes the validity of (9).

The basic idea that allows us to achieve asymptotic stability is to decrease Δ by means of multiplying it by the scaling factor Ω. After we do that, by virtue of (9) the state of the system will still belong to the union of the quantization blocks of finite size, and so we can continue the analysis as before. Thus we let

$u(t) = -K\Delta(t)q(x(t))$ with $\Delta(t) = \Omega\Delta(k_0\tau)$ for $k_0\tau + \tau < t \leq k_0\tau + 2\tau$, which yields

$$\begin{aligned} &x^T(k_0\tau + 2\tau)Qx(k_0\tau + 2\tau) \\ &\quad \leq (\Omega^2\Delta(k_0\tau))^2 \left(M - \tfrac{1}{2}\right)^2 \lambda_{\min}(Q). \end{aligned}$$

Similarly, we let $\Delta(t) = \Omega^2\Delta(k_0\tau)$ for $k_0\tau + 2\tau < t \leq k_0\tau + 3\tau$. Repeating this procedure, we obtain the desired control policy. Indeed, Lyapunov stability follows directly from the adjustment policy for Δ (note that the amount by which Δ needs to be increased initially is proportional to $\|x(0)\|$). Moreover, we have $\Delta(t) \to 0$ as $t \to \infty$, and by the above analysis the same is true for $x(t)$. □

The above quantized feedback control strategy calls for Δ taking on a countable set of values rather than a continuum of values. In fact, it is not hard to see from the proof that the proposed approach, suitably modified, will still work if Δ is restricted to take values in some given set S, provided that:

1) S contains a sequence $\Delta_{11}, \Delta_{21}, \cdots$ that increases to ∞.
2) Each Δ_{i1} from this sequence belongs to a sequence $\Delta_{i1}, \Delta_{i2}, \cdots$ in S that decreases to 0 and is such that we have $\Omega \leq \Delta_{i,j+1}/\Delta_{ij}$ for each $j \in \mathbb{Z}_{\geq 0}$.

In some applications there may only be a finite set of possible values for Δ (for example, if the values of Δ have to be passed through a quantizer with fixed sensitivity). Adjusting our control policy to this case, we would only obtain practical stability and not global asymptotic stability claimed in Theorem 1 (cf. Section IV below).

The control policy described above uses a variant of the so-called *dwell-time switching logic* [16] in the sense that the value of Δ is held constant on time intervals of fixed length τ. Another possibility is to change Δ every time $\|q(x(t))\|$ becomes smaller than or equal to a certain prescribed value. To demonstrate how this alternative method works, we will use it in proving the discrete-time counterpart of Theorem 1 (Theorem 3 in the next section). The main advantage of the dwell-time switching approach is that it can also be applied to quantized output feedback stabilization problems (cf. Sections IV and V below). In specific applications, one might want to compare the effectiveness of these two methods with respect to various performance characteristics, such as the speed of convergence of solution trajectories to zero (time-optimality) or the frequency of switching hyperplane crossings which cause the control function to change its value ("minimum attention control"—cf. [3], [4]).

We see from the proof of Theorem 1 that the state of the closed-loop system belongs, at equally spaced instants of time, to ellipsoids whose sizes decrease according to consecutive integer powers of Ω (where $0 < \Omega < 1$). Therefore, $x(t)$ converges to zero exponentially as $t \to \infty$. To make this argument precise, note that for $t > k_0\tau$ we have

$$\begin{aligned} \|x(t)\| &\leq \Omega^{\lfloor t/\tau \rfloor - k_0 - 1}\Delta(k_0\tau)\left(\Theta\sqrt{n} + \epsilon\right) \\ &\quad \cdot \sqrt{\lambda_{\max}(Q)/\lambda_{\min}(Q)} \\ &\leq e^{(\tau^{-1}\log\Omega)t}\Omega^{-k_0-1}\Delta(k_0\tau)\left(\Theta\sqrt{n} + \epsilon\right) \\ &\quad \cdot \sqrt{\lambda_{\max}(Q)/\lambda_{\min}(Q)}. \end{aligned}$$

This observation suggests that, at least qualitatively, there is no degradation in performance of the quantized feedback system compared with that of the linear time-invariant system. As can be seen from the simple example

$$\dot{x} = ax - k\Delta q(x), \qquad x \in \mathbb{R}, \quad k > a > 0 \tag{12}$$

the lower bound on the rate of convergence is smaller than in the absence of quantization, although for some values of x the convergence in the quantized system is actually faster.

We will now address in passing the issue of time sampling in the context of equation (12). Suppose that the values of $q(x(t))$ are not measured continuously, but instead they are sampled at times $0, \delta, 2\delta, \cdots$, where $\delta > 0$ is the *sampling period*. This results in the equation $\dot{x}(t) = ax(t) - k\Delta q(\lfloor x/\delta \rfloor \delta)$. Do we still have asymptotic stability? The answer is yes, provided that no "overshooting" occurs. Namely, we have to make sure that if, say, $x(0) < 0$, then $x(t)$ remains negative for all future times. This can be done by means of a simple calculation. Suppose that the sampling is performed at $t = 0$ and $x(0) = -\Delta/2$ (the most "dangerous" case). Then we will have $x(t) < 0$ for all $t > 0$ if $\delta < (1/a)\ln(2k/(2k - a))$, i.e., if the sampling is performed frequently enough (see [14, p. 23] for details). It is important to notice that this upper bound for δ does not depend on Δ, so we can still change the sensitivity in the way described above. In other words, we see that the sampling considerations are decoupled from the issues regarding the implementation of the quantized feedback stabilizing control policy. This basic idea was independently explored in [11] in the general context of the system (2). That paper also contains a detailed discussion of performance and robustness characteristics of the resulting quantized feedback control system.

The stabilization strategy of Theorem 1 employs a quantizer whose (fixed) saturation value is assumed to be sufficiently large. As we are about to see, it is possible to stabilize the system (2) with quantized state feedback even if the saturation value M of the quantizer is substantially smaller than that required in the above proof. In fact, let us show that we can achieve global asymptotic stability using a (nonuniform) quantizer q with $M = 1$. What we will do is basically design a sampled-data feedback control law using generalized hold functions. The procedure will be based on the following idea: if the state of the system at a given instant of time is known to belong to a certain rectilinear box, and if we pick the sensitivities so that the switching hyperplanes divide this box into smaller boxes, then on the basis of the corresponding quantized measurement we can immediately determine which one of these smaller boxes contains the state of the system, thereby improving our state estimate.

Theorem 2: Suppose that all eigenvalues of $A - BK$ have negative real parts. Then there exists a control policy of the form

$$\begin{aligned} \Delta_i(t) &= G_i(z, \lfloor t/\tau \rfloor, q(x(\lfloor t/\tau \rfloor \tau))), \quad i = 1, \cdots, n, \\ u(t) &= H(t - \lfloor t/\tau \rfloor \tau, q(x(\lfloor t/\tau \rfloor \tau))) \end{aligned}$$

where q is a quantizer with sensitivity $(\Delta_1(t), \cdots, \Delta_n(t))$ and saturation value 1, such that the solutions of the closed-loop system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BH(t - \lfloor t/\tau \rfloor \tau,\, q(x(\lfloor t/\tau \rfloor \tau))),\\ &\quad x(0)\ arbitrary\\ \Delta_i(t) &= G_i(z,\, \lfloor t/\tau \rfloor,\, q(x(\lfloor t/\tau \rfloor \tau))), \quad \Delta(0) = 0\end{aligned}$$

approach 0 *as* $t \to \infty$.

Proof: Fix a number $\epsilon \in (0, 1)$. Since $\|I\|_\infty = 1$, we can find a number $\tau > 0$ such that $\|e^{At}\|_\infty < 1+\epsilon$ for all $t \in [0, \tau]$. If we let $u(t) = 0$ and $\Delta(t) = e^{2\|A\| \lfloor t/\tau \rfloor \tau}$, then there exists a well-defined integer $k_0 := \min\{k \geq 1\colon q(x(k\tau)) = 0\}$. We have $\|x(k_0\tau)\|_\infty \leq E_0$, where $E_0 := e^{2\|A\| k_0 \tau}/2$. Thus 0 can be viewed as an estimate of $x(k_0\tau)$ with the estimation error whose maximum norm is at most E_0. Our goal is to construct a sequence of state estimates with estimation errors approaching 0 as $t \to \infty$.

For $k_0\tau < t \leq k_0\tau + \tau$, let $u(t) = 0$. This gives $x(t) = e^{A(t-k_0\tau)}x(k_0\tau)$, hence $\|x(t)\|_\infty \leq (1+\epsilon)E_0$. The quantized measurement $q(x(k_0\tau + \tau))$ with $\Delta_i(k_0\tau+\tau) = (1+\epsilon)E_0$, $i = 1, \cdots, n$ singles out a rectilinear box with edges at most $(1+\epsilon)E_0$ which contains $x(k_0\tau+\tau)$. Denoting the center of this box by $\overline{x}(k_0\tau+\tau)$, we see that

$$\|x(k_0\tau+\tau) - \overline{x}(k_0\tau+\tau)\|_\infty \leq (1+\epsilon)E_0/2.$$

For $k_0\tau + \tau < t \leq k_0\tau + 2\tau$, let $u(t) = -Ke^{(A-BK)(t-k_0\tau-\tau)}\overline{x}(k_0\tau+\tau)$. This gives

$$\begin{aligned}&\frac{d}{dt}\left(x(t) - e^{(A-BK)(t-k_0\tau-\tau)}\overline{x}(k_0\tau+\tau)\right)\\ &\quad = A\left(x(t) - e^{(A-BK)(t-k_0\tau-\tau)}\overline{x}(k_0\tau+\tau)\right)\end{aligned}$$

hence

$$\begin{aligned}&\left\|x(t) - e^{(A-BK)(t-k_0\tau-\tau)}\overline{x}(k_0\tau+\tau)\right\|_\infty\\ &\quad = \left\|e^{A(t-k_0\tau-\tau)}(x(k_0\tau+\tau) - \overline{x}(k_0\tau+\tau))\right\|_\infty\\ &\quad \leq (1+\epsilon)^2 E_0/2.\end{aligned}$$

The quantized measurement $q(x(k_0\tau + 2\tau))$ with

$$\Delta_i(k_0\tau+2\tau) = \begin{cases} 2\left|\left(e^{(A-BK)\tau}\overline{x}(k_0\tau+\tau)\right)_i\right| & \text{if } \left(e^{(A-BK)\tau}\overline{x}(k_0\tau+\tau)\right)_i \neq 0\\ (1+\epsilon)^2E_0/2 & \text{if } \left(e^{(A-BK)\tau}\overline{x}(k_0\tau+\tau)\right)_i = 0\end{cases}$$

singles out a rectilinear box with edges at most $(1+\epsilon)^2E_0/2$ which contains $x(k_0\tau+2\tau)$. Denoting the center of this box by $\overline{x}(k_0\tau+2\tau)$, we see that

$$\|x(k_0\tau+2\tau) - \overline{x}(k_0\tau+2\tau)\|_\infty \leq (1+\epsilon)^2E_0/4.$$

For $k_0\tau + 2\tau < t \leq k_0\tau + 3\tau$, let $u(t) = -Ke^{(A-BK)(t-k_0\tau-2\tau)}\overline{x}(k_0\tau + 2\tau)$. Proceeding in this fashion, we obtain a piecewise continuous control function u such that $\|u(t) + Kx(t)\|_\infty \to 0$ as $t \to \infty$. The same statement is therefore true for the Euclidean norm $\|u(t) + Kx(t)\|$. This, combined with an argument of the type used in the proof of Theorem 1, implies that $x(t) \to 0$ as $t \to \infty$. □

Remark: Again, if the set of possible values for Δ_i is finite, global asymptotic stability is replaced by practical stability (see also Section IV below).

III. Quantized State Feedback Stabilization: Discrete Time

In this section we will establish counterparts of Theorems 1 and 2 for the discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \tag{13}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. For illustrative purposes, to prove the next theorem we use a different approach than that described in the proof of Theorem 1.

Theorem 3: Suppose that all eigenvalues of $A - BK$ lie inside the unit circle. Then there exists a control policy of the form

$$\begin{aligned}\Delta(k+1) &= G(z,\, k,\, q(x(k)),\, \Delta(k)),\\ u(k) &= -KI_{[k_0,\, \infty)}(k)\Delta(k)q(x(k))\end{aligned}$$

where q is a uniform quantizer with sensitivity $\Delta(k)$ and k_0 is a positive integer, such that the solutions of the closed-loop system

$$\begin{aligned}x(k+1) &= Ax(k) - BKI_{[k_0,\, \infty)}(k)\Delta(k)q(x(k)),\\ &\quad x(0)\ arbitrary\\ \Delta(k+1) &= G(z,\, k,\, q(x(k)),\, \Delta(k)), \quad \Delta(0) = 0\end{aligned}$$

approach 0 *as* $k \to \infty$.

Proof: Consider the system

$$x(k+1) = Ax(k) - BK\Delta q(x(k))$$

which we can also write as

$$x(k+1) = (A - BK)x(k) + BKs(x(k)) \tag{14}$$

with $s(x) = x - \Delta q(x)$ as before. By the standard Lyapunov stability theory for discrete-time linear systems, there exist positive definite symmetric matrices Q and D such that $(A-BK)^TQ(A-BK) - Q = -D$. If the inequality (4) holds, the bound (5) is valid. For the solutions of (14) this implies

$$\begin{aligned}&x^T(k+1)Qx(k+1) - x^T(k)Qx(k)\\ &\quad = -x^T(k)Dx(k) + 2x^T(k)(A-BK)^TQBK's(x(k))\\ &\qquad + s^T(x(k))K^TB^TQBKs(x(k))\\ &\quad \leq -\lambda_{\min}(D)\|x(k)\|^2 + 2\|x(k)\|\\ &\qquad \cdot \|(A-BK)^TQBK\|\Delta\sqrt{n}/2\\ &\qquad + \|K^TB^TQBK\|\Delta^2 n/4.\end{aligned}$$

The last expression is negative outside the ball $\{x\colon \|x\| \le \Theta\Delta\sqrt{n}\}$, where

$$\Theta := \frac{1}{2\lambda_{\min}(D)}\Bigg(\|(A-BK)^T QBK\| + \sqrt{\|(A-BK)^T QBK\|^2 + \lambda_{\min}(D)\|K^T B^T QBK\|}\Bigg)$$

(cf. [6, Proposition 2.3]). Define the scaling factor Ω by the formula

$$\Omega := \left((\Theta\sqrt{n}+\epsilon)\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} + \sqrt{n}\right) \cdot \sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}}\left(M - \frac{1}{2}\right)^{-1}$$

for some fixed $\epsilon > 0$, and take the saturation value M of q to be large enough so that $\Omega < 1$. If we let $u(k) = 0$ and $\Delta(k) = \|A\|^{2k}$ for $k \in \mathbb{Z}_{\ge 0}$, then there exists a well-defined number

$$k_0 := \min\left\{k \ge 1\colon \|q(x(k))\| \le \left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}} - \frac{\sqrt{n}}{2}\right\}.$$

We have

$$\|x(k_0)\| \le \Delta(k_0)\left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}}.$$

Therefore, $x(k_0)$ belongs to the ellipsoid

$$R_1 := \left\{x\colon x^T Qx \le (\Delta(k_0))^2\left(M - \tfrac{1}{2}\right)^2 \lambda_{\min}(Q)\right\}.$$

Observe that (4) holds with $\Delta = \Delta(k_0)$ for all $x \in R_1$. Since $\Omega < 1$, it follows that if we let $u(k) = -K\Delta(k)q(x(k))$ with $\Delta(k) = \Delta(k_0)$ for $k \ge k_0$, then x will never leave R_1. Moreover, x will approach the ellipsoid

$$R_0 := \{x\colon x^T Qx \le (\Delta(k_0))^2\Theta^2 n\lambda_{\max}(Q)\}.$$

Thus we can define

$$k_1 := \min\left\{k \ge k_0 + 1\colon \|q(x(k))\| \le (\Theta\sqrt{n}+\epsilon)\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} + \frac{\sqrt{n}}{2}\right\}$$

which implies

$$\|x(k_1)\| \le \Delta(k_0)\left((\Theta\sqrt{n}+\epsilon)\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} + \sqrt{n}\right).$$

When $k = k_1$, change the sensitivity to $\Delta(k_1) = \Omega\Delta(k_0)$. Arguing as before, we can show that if we let $u(k) = -K\Delta(k)q(x(k))$ with $\Delta(k) = \Delta(k_1)$ for $k \ge k_1$, then (4) will still hold, and there exists a well-defined number

$$k_2 := \min\left\{k \ge k_1 + 1\colon \|q(x(k))\| \le (\Theta\sqrt{n}+\epsilon)\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} + \frac{\sqrt{n}}{2}\right\}.$$

When $k = k_2$, change the sensitivity to $\Delta(k_2) = \Omega\Delta(k_1)$. Repeating this procedure, we obtain a sequence $\Delta(k_0), \Delta(k_1), \Delta(k_2), \Delta(k_3), \cdots \to 0$. We conclude that $x(k) \to 0$ as $k \to \infty$. □

Our analog of Theorem 2 for the discrete-time case contains one additional hypothesis which means, loosely speaking, that the state of the uncontrolled system $x(k+1) = Ax(k)$ is "not excessively unstable."

Theorem 4: Suppose that all eigenvalues of $A - BK$ lie inside the unit circle. Suppose also that $\|A\|_\infty \le 2$. Then there exists a control policy of the form

$$\begin{aligned} \Delta_i(k+1) &= G_i(z, k, q(x(k))), \qquad i = 1, \cdots, n, \\ u(k+1) &= H(q(x(k))) \end{aligned}$$

where q is a quantizer with sensitivity $(\Delta_1(k), \cdots, \Delta_n(k))$ and saturation value 1, such that the solutions of the closed-loop system

$$\begin{aligned} x(k+1) &= Ax(k) + BH(q(x(k))), \qquad x(0) \ \textit{arbitrary} \\ \Delta_i(k+1) &= G_i(z, k, q(x(k))), \qquad \Delta(0) = 0 \end{aligned}$$

approach 0 as $k \to \infty$.

Proof: If we let $u(k) = 0$ and $\Delta_i(k) = \|A\|^{2k}$ for $k \in \mathbb{Z}_{\ge 0}$, then there exists $k_0 := \min\{k \ge 1\colon q(x(k)) = 0\}$. We have $\|x(k_0)\|_\infty \le E_0$, where $E_0 := \|A\|^{2k_0}/2$. We will construct a sequence of state estimates $\overline{x}(k)$ such that $\|x(k) - \overline{x}(k)\|_\infty \to 0$ as $k \to \infty$.

Let $u(k_0) = 0$. Then $\|x(k_0+1)\|_\infty = \|Ax(k_0)\|_\infty \le \|A\|_\infty E_0$. The quantized measurement $q(x(k_0+1))$ with $\Delta_i(k_0+1) = \|A\|_\infty E_0$ singles out a rectilinear box with edges at most $\|A\|_\infty E_0$ which contains $x(k_0+1)$. Denoting the center of this box by $\overline{x}(k_0+1)$, we obtain

$$\|x(k_0+1) - \overline{x}(k_0+1)\|_\infty \le \|A\|_\infty E_0/2.$$

Next, let $u(k_0+1) = -K\overline{x}(k_0+1)$. We have $x(k_0+2) = Ax(k_0+1) - BK\overline{x}(k_0+1)$ hence $\|x(k_0+2) - (A-BK)\overline{x}(k_0+1)\|_\infty \le \|A\|_\infty^2 E_0/2$. The quantized measurement $q(x(k_0+2))$ with

$$\Delta_i(k_0+2) = \begin{cases} 2|((A-BK)\overline{x}(k_0+1))_i| & \text{if } ((A-BK)\overline{x}(k_0+1))_i \ne 0 \\ \|A\|_\infty^2 E_0/2 & \text{if } ((A-BK)\overline{x}(k_0+1))_i = 0 \end{cases}$$

singles out a rectilinear box with edges at most $\|A\|_\infty^2 E_0/2$ which contains $x(k_0+2)$. Denoting the center of this box by $\overline{x}(k_0+2)$, we obtain

$$\|x(k_0+2)-\overline{x}(k_0+2)\|_\infty \le \|A\|_\infty^2 E_0/4.$$

Next, let $u(k_0+2) = -K\overline{x}(k_0+2)$. Repeating the above procedure for each k, we obtain a control function u such that $\|u(k)+Kx(k)\|_\infty \to 0$ as $k \to \infty$. The statement of the theorem follows. □

IV. Quantized Output Feedback Stabilization

We now turn to the problem of stabilizing the system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned} \tag{15}$$

with quantized measurements of the output. Assume (without loss of generality) that $\|C\|_\infty \le 1$. Suppose that there exists a matrix K such that all eigenvalues of $A-BKC$ have negative real parts. If $y \in \mathbb{R}^p$ with $p<n$, then the initial "zooming-out" stage of the stabilizing control policy of Theorem 1 cannot be implemented. For this reason, in the next theorem the asymptotic stability is local.

Theorem 5: Suppose that all eigenvalues of $A-BKC$ have negative real parts. For any number $E_0>0$, there exists a control policy of the form

$$\begin{aligned}\Delta(t) &= G(\Delta(\lfloor t/\tau \rfloor \tau)),\\ u(t) &= -K\Delta(t)q(y(t))\end{aligned}$$

where q is a uniform quantizer with sensitivity $\Delta(t)$, such that the solutions of the closed-loop system

$$\begin{aligned}\dot{x}(t) &= Ax(t) - BK\Delta(t)q(Cx(t)), \qquad \|x(0)\| \le E_0\\ \Delta(t) &= G(\Delta(\lfloor t/\tau \rfloor \tau)), \qquad \Delta(0) = F(E_0)\end{aligned}$$

approach 0 as $t \to \infty$.

Proof: As before, there exist positive definite symmetric matrices Q and D such that $(A-BKC)^TQ+Q(A-BKC) = -D$. Fix an arbitrary $\epsilon>0$. Define Θ, Ω and τ as in the proof of Theorem 1, and take the saturation value M of q to be large enough so that $\Omega<1$. Let $u=-K\Delta q(Cx)$, where q is a uniform quantizer with sensitivity Δ. The closed-loop system can be written as

$$\dot{x} = Ax - BK\Delta q(Cx) = (A-BKC)x + BKs(Cx) \tag{16}$$

where $s(Cx) = Cx - \Delta q(Cx)$. Whenever $\|y\|_\infty \le (M-1/2)\Delta$, the upper bound given by (6) is valid for the derivative of x^TQx along the solutions of (16).

Suppose that $\|x(0)\| \le E_0$. Let us choose the initial sensitivity $\Delta(0)$ large enough to have

$$E_0 \le \Delta(0)\left(M-\frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}}$$

hence

$$x^T(0)Qx(0) \le (\Delta(0))^2\left(M-\tfrac{1}{2}\right)^2\lambda_{\min}(Q).$$

This is completely analogous to (8) (with $k_0=0$), and the rest of the proof carries over from Theorem 1 with obvious minor modifications. □

The discrete-time case can be treated similarly. We now turn our attention to quantizers with saturation value 1. For simplicity, let us consider the single-input, single-output system

$$\begin{aligned}\dot{x} &= Ax + bu,\\ y &= c^Tx\end{aligned} \tag{17}$$

[generalization to the case of (15) is straightforward]. Suppose that there exists a feedback gain k such that all eigenvalues of $A-bkc^T$ have negative real parts. All such gains k can be found by using the well-known Nyquist criterion (see, e.g., [18]). Without loss of generality we may assume that $\|c\| \le 1$. We will develop a sliding mode control policy that yields asymptotic stability. It will be described by a differential equation which makes the sensitivity change fast enough so as to dominate the dynamics of the underlying linear system.

Theorem 6: Suppose that all eigenvalues of $A-bkc^T$ have negative real parts. For any number $E_0>0$, there exists a control policy of the form

$$\begin{aligned}\dot{\Delta}(t) &= G(q(y(t)))\\ u(t) &= -k\Delta(t)q(y(t))/2\end{aligned}$$

where q is a quantizer with sensitivity $\Delta(t)$ and saturation value 1, such that the solutions of the closed-loop system

$$\begin{aligned}\dot{x}(t) &= Ax(t) - bk\Delta(t)q(c^Tx(t))/2, \qquad \|x(0)\| \le E_0\\ \dot{\Delta}(t) &= G(q(y(t))), \qquad \Delta(0) = 0\end{aligned}$$

(interpreted in the Filippov's sense) approach 0 as $t \to \infty$.

Proof: We know that there exist positive definite symmetric matrices Q and D such that $(A-bkc^T)^TQ+Q(A-bkc^T) = -D$. Consider the system

$$\begin{aligned}\dot{x} &= Ax - bk\Delta q(c^Tx)/2, \qquad \|x(0)\| \le E_0\\ \dot{\Delta} &= (2|q(c^Tx)|-1)\beta, \qquad \Delta(0) = 0\end{aligned} \tag{18}$$

where

$$\beta := 2(\|A\| + \|b\|k)E + 2$$

and

$$E := \max\left(\frac{\|Qbk\|2E_0}{\lambda_{\min}(D)}, E_0\right)\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}}.$$

The right-hand side of (18) is discontinuous when $|c^Tx| = \Delta/2$, and we will interpret solutions of (18) in the Filippov's sense [8].

We have $\big||c^Tx(0)| - \Delta(0)/2\big| \le E_0$, and it is not hard to check that $(d/dt)(\big||c^Tx(t)| - \Delta(t)/2\big|) \le -1$ for $t \ge 0$ as long as $|c^Tx(t)| \ne \Delta(t)/2$ and $\|x(t)\| \le E$. But from the analysis of Section II it follows that the solution trajectory will never leave the region $\{x: \|x\| \le E\}$. This means that there is a time $t_0 > 0$ such that $|c^T x(t_0)| = \Delta(t_0)/2$. It is not difficult to show that the solution trajectory stays on the discontinuity locus $\{(x, \Delta): |c^T x| = \Delta/2\}$ for all $t \ge t_0$. To complete the proof, it remains to use the fact that the system $\dot{x} = (A - bkc^T)x$ is asymptotically stable. □

The results of Theorems 5 and 6, although local, are of "semiglobal" nature: given an *a priori* upper bound E_0 on the norm of $x(0)$, we can find a control law that drives the state to 0. A drawback of the above solution is that Δ is changing continuously, which might be undesirable in some applications. As the most restrictive case, consider a situation where Δ is only allowed to take on values in the set $S := \{-N\sigma, (-N+1)\sigma, \cdots, -\sigma, 0, \sigma, \cdots, (N-1)\sigma, N\sigma\}$, where σ is a fixed positive real number and N is a fixed positive integer. Then we can replace (18) by a hybrid system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) - bk\Delta(t)q(c^T x(t))/2, \qquad \|x(0)\| \le E_0 \\ \Delta(t) &= \Delta(\lfloor t/\tau \rfloor \tau) + (2|q(c^T x(\lfloor t/\tau \rfloor \tau))| - 1)\sigma, \\ &\quad \Delta(0) = 0 \end{aligned} \tag{19}$$

where $\tau := \sigma((\|A\| + \|b\|k)N\sigma + 2)^{-1}$. Note that Δ changes its value by σ every τ units of time. Using our earlier developments, we can easily establish the following stability property of (19).

Proposition 7: Let Q, D, σ and N be as in the foregoing. Suppose that N is large enough and $E_0 \in \mathbb{R}$ is small enough so that

$$\max\left(\frac{\|Qbk\|(2E_0 + \sigma)}{\lambda_{\min}(D)}, E_0\right)\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} \le N\sigma/2.$$

Then the solutions of the system (19) eventually enter the region

$$R_0 := \left\{x: x^T Q x \le \left(\frac{\|Qbk\|2\sigma}{\lambda_{\min}(D)}\right)^2 \lambda_{\max}(Q)\right\}$$

and stay there for all future time. Moreover, we have $\Delta(t) \in S$ for all $t \ge 0$.

One could also consider a situation where σ can take on different values from a certain finite set. This would increase the domain of stability for (19) and make the attracting invariant region R_0 smaller.

V. Observability and Quantized Feedback Stabilization

We will now show that, by employing somewhat more sophisticated techniques than those presented in Section IV, it is possible to design a quantized output feedback control policy that makes *all* solutions of the system (15) approach 0. At the beginning of Section IV we made the assumption that (15) is stabilizable by linear static output feedback. This is well known to imply that (15) is *detectable*, i.e., observable in the unstable modes (see [22, Sect. 6.4]). Since we are concerned with feedback stabilization, it makes sense to assume that (A, C) is actually an observable pair. As it turns out, the hypothesis that (15) is stabilizable by linear static output feedback can then be relaxed considerably. Namely, we will only require that (15) be stabilizable by linear static *state* feedback. This means that there exists a matrix K such that all eigenvalues of $A - BK$ have negative real parts, but there might not exist any K such that all eigenvalues of $A - BKC$ have negative real parts.

Since only quantized measurements of the output, and not of the state, are available, we have to develop a method for constructing state estimates which we will denote by $\overline{x}(t)$. We will be using the history of quantized output measurements over a time interval, in contrast with the simpler techniques of the previous sections. The evolution of $\overline{x}(t)$ can thus be described by a Volterra integral equation (this construction is based on a standard technique and will become more transparent in the course of the proof).

Theorem 8: Suppose that (A, C) is an observable pair, and that all eigenvalues of $A - BK$ have negative real parts. Then there exists a control policy of the form

$$\begin{aligned} \overline{x}(t) &= \iiint_{t-\tau \le s_i \le t,\ i=1,2,3} \cdot F(t, q(y(s_1)), \\ &\qquad \Delta(s_2), u(s_3))\, ds_1\, ds_2\, ds_3, \\ \Delta(t) &= G(z, \lfloor t/\tau \rfloor, \overline{x}(\lfloor t/\tau \rfloor \tau), q(x(\lfloor t/\tau \rfloor \tau)), \Delta(\lfloor t/\tau \rfloor \tau)), \\ u(t) &= H(t, \overline{x}(t)) \end{aligned}$$

where q is a uniform quantizer with sensitivity $\Delta(t)$, such that the solutions of the closed-loop system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BH(t, \overline{x}(t)), \qquad x(0) \ \textit{arbitrary} \\ \Delta(t) &= G(z, \lfloor t/\tau \rfloor, \overline{x}(\lfloor t/\tau \rfloor \tau), q(x(\lfloor t/\tau \rfloor \tau)), \\ &\qquad \Delta(\lfloor t/\tau \rfloor \tau)), \qquad \Delta(0) = 0 \end{aligned}$$

approach 0 as $t \to \infty$.

Proof: Let Q, D and Θ be as in the proof of Theorem 1. Fix an arbitrary $\epsilon > 0$. Let $\tau_u > 0$ be such that $\|e^{At}\| < 1 + \epsilon$ and $\|e^{A^T t}\| < 1 + \epsilon$ for all $t \in [0, \tau_u]$. Denote by W the *observability Gramian*, i.e., the full-rank matrix $\int_0^{\tau_u} e^{A^T t} C^T C e^{At}\, dt$ (see, e.g., [1]). Define

$$\Omega := \Psi\left(\Theta\sqrt{n} + \epsilon\right)\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}}\left(M - \frac{1}{2}\right)^{-1}$$

where

$$\Psi := \|W^{-1}\|\tau_u(1 + \epsilon)^2\|C^T\|.$$

Take the saturation value M of q to be large enough so that $\Omega < 1$. Define

$$\tau_\Delta = \frac{(M - 1/2)^2 \lambda_{\min}(Q) - \Psi^2\left(\Theta\sqrt{n} + \epsilon\right)^2 \lambda_{\max}(Q)}{\Psi^2\left(\Theta\sqrt{n} + \epsilon\right)\lambda_{\min}(D)\epsilon}.$$

We will be updating the value of Δ every τ_Δ units of time while updating the value of u every τ_u units of time. If we let $u(t) = 0$ and $\Delta(t) = e^{2\|A\|\lfloor t/\tau_\Delta \rfloor \tau_\Delta}$, then $\exists\, k_0 > 0$ such that

$$\|q(y(t))\| \le \left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}} - \frac{\sqrt{n}}{2} \quad \text{for all } t \in [k_0\tau_\Delta,\, k_0\tau_\Delta + \tau_u].$$

We have

$$\int_{k_0\tau_\Delta}^{k_0\tau_\Delta + \tau_u} e^{A^T(t - k_0\tau_\Delta)} C^T y(t)\, dt = W x(k_0\tau_\Delta).$$

Define

$$\overline{x}(k_0\tau_\Delta) = W^{-1} \int_{k_0\tau_\Delta}^{k_0\tau_\Delta + \tau_u} e^{A^T(t - k_0\tau_\Delta)} C^T \Delta(t) q(y(t))\, dt.$$

We have

$$\begin{aligned}&\|x(k_0\tau_\Delta) - \overline{x}(k_0\tau_\Delta)\| \\ &\quad \le \|W^{-1}\| \tau_u (1+\epsilon) \|C^T\| \Delta(k_0\tau_\Delta + \tau_u)\sqrt{n}/2.\end{aligned}$$

Denoting $e^{A\tau_u}\overline{x}(k_0\tau_\Delta)$ by $\overline{x}(k_0\tau_\Delta + \tau_u)$, we obtain

$$\|x(k_0\tau_\Delta + \tau_u) - \overline{x}(k_0\tau_\Delta + \tau_u)\| \le \Psi \Delta(k_0\tau_\Delta + \tau_u)\sqrt{n}/2.$$

Let $u(k_0\tau_\Delta + \tau_u) = -K\overline{x}(k_0\tau_\Delta + \tau_u)$. Pick a number $\overline{\Delta} \ge \Delta(k_0\tau_\Delta + \tau_u)$ such that

$$\begin{aligned}&\|\overline{x}(k_0\tau_\Delta + \tau_u)\| \\ &\quad \le \overline{\Delta}\left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}} - \Psi\Delta(k_0\tau_\Delta + \tau_u)\frac{\sqrt{n}}{2}.\end{aligned}$$

This implies

$$\|x(k_0\tau_\Delta + \tau_u)\| \le \overline{\Delta}\left(M - \frac{1}{2}\right)\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}}.$$

For $k_0\tau_\Delta + \tau_u < t \le k_0\tau_\Delta + \tau_u + \tau_\Delta$, let $\Delta(t) = \overline{\Delta}$.

Now, take any $t_1 \in (k_0\tau_\Delta, k_0\tau_\Delta + \min\{\tau_u,\, \tau_\Delta\}]$, and assume that $u(t)$ has already been defined for $t < t_1 + \tau_u$. We have

$$\int_{t_1}^{t_1+\tau_u} e^{A^T(t - t_1)} C^T y(t)\, dt = W x(t_1) + v_1$$

where v_1 is a known vector that depends on $u(t)$ for $t_1 \le t < t_1 + \tau_u$. Letting

$$\overline{x}(t_1) = W^{-1}\left(\int_{t_1}^{t_1+\tau_u} e^{A^T(t-t_1)} C^T \Delta(t) q(y(t))\, dt - v_1\right)$$

we obtain

$$\|x(t_1) - \overline{x}(t_1)\| \le \|W^{-1}\| \tau_u(1+\epsilon)\|C^T\|\overline{\Delta}\sqrt{n}/2.$$

Denoting by $\overline{x}(t_1 + \tau_u)$ the solution at time $t_1 + \tau_u$ of the equation $\dot{x} = Ax + Bu$ with $x(t_1) = \overline{x}(t_1)$, we have

$$\|x(t_1 + \tau_u) - \overline{x}(t_1 + \tau_u)\| \le \Psi\overline{\Delta}\sqrt{n}/2.$$

We then let $u(t_1 + \tau_u) = -K\overline{x}(t_1 + \tau_u)$. Proceeding in this way, we obtain a control function u such that for all $t \in [k_0\tau_\Delta + \tau_u,\, k_0\tau_\Delta + \tau_u + \tau_\Delta]$ we have

$$\|u(t) + Kx(t)\| \le \|K\|\Psi\overline{\Delta}\sqrt{n}/2.$$

Using the same techniques as in the proof of Theorem 1, we can show that

$$\begin{aligned}&x^T(k_0\tau_\Delta + \tau_u + \tau_\Delta) Q x(k_0\tau_\Delta + \tau_u + \tau_\Delta) \\ &\quad \le (\Psi\overline{\Delta})^2 \left(\Theta\sqrt{n} + \epsilon\right)^2 \lambda_{\max}(Q).\end{aligned}$$

Thus we let $\Delta(t) = \Omega\overline{\Delta}$ for $k_0\tau_\Delta + \tau_u + \tau_\Delta < t \le k_0\tau_\Delta + \tau_u + 2\tau_\Delta$, $\Delta(t) = \Omega^2\overline{\Delta}$ for $k_0\tau_\Delta + \tau_u + 2\tau_\Delta < t \le k_0\tau_\Delta + \tau_u + 3\tau_\Delta$, etc., while updating $u(t)$ in the same way as before. This gives $x(t) \to 0$ as needed. □

Remark: Another possibility, suggested to us by Steve Morse, is to implement a dynamic observer for $t \ge k_0\tau_\Delta + \tau_u$.

The next theorem is a counterpart of Theorem 8 for the discrete-time linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k). \end{aligned} \tag{20}$$

Its proof proceeds along the same lines and will not be given.

Theorem 9: Suppose that $(A,\, C)$ is an observable pair, and that all eigenvalues of $A - BK$ lie inside the unit circle. Then there exists a control policy of the form

$$\begin{aligned} \overline{x}(k) &= \sum_{i,\,j,\,l=0}^{n-1} F_{ijl}(q(y(k-i)),\, \Delta(k-j),\, u(k-l)), \\ \Delta(k+1) &= G(z,\, k,\, \overline{x}(k),\, q(x(k)),\, \Delta(k)), \\ u(k) &= H(k,\, \overline{x}(k)) \end{aligned}$$

where q is a uniform quantizer with sensitivity $\Delta(k)$, such that the solutions of the closed-loop system

$$\begin{aligned} x(k+1) &= Ax(k) + BH(k,\, \overline{x}(k)), \quad x(0)\ \textit{arbitrary} \\ \Delta(k+1) &= G(z,\, k,\, \overline{x}(k),\, q(x(k)),\, \Delta(k)), \quad \Delta(0) = 0 \end{aligned}$$

approach 0 as $t \to \infty$.

VI. A Remark on Nonlinear Systems

It can be shown via a linearization argument that by using our approach one can obtain local asymptotic stability for a nonlinear system, provided that the corresponding linearized system is stabilizable (see [11]). Here we briefly discuss the problem of achieving global or semiglobal asymptotic stability for nonlinear systems with quantized measurements. Working with a given nonlinear system directly, one gains an advantage even if only local asymptotic stability is sought, because the linearization of a stabilizable nonlinear system may fail to be stabilizable. As we will see, the intrinsic difficulty that lies in the way of extending the ideas presented above to the nonlinear case is the need to find a control law that is input-to-state stabilizing with respect to measurement errors.

Consider the system

$$\dot{x} = f(x, u). \tag{21}$$

Suppose that there exists a feedback control law $u = k(x)$ that makes the system

$$\dot{x} = f(x, k(x+e))$$

input-to-state stable (ISS) with respect to a measurement disturbance e, in the sense of Sontag [19]. According to [21], a necessary and sufficient condition for ISS in this case is the existence of a positive definite, radially unbounded, smooth function $V\colon \mathbb{R}^n \to \mathbb{R}$ such that for some continuous, positive definite, strictly increasing functions $\alpha_1, \alpha_2, \rho\colon [0, \infty) \to [0, \infty)$, for all $x \neq 0$, and for all e we have

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

and

$$\|x\| \ge \rho(\|e\|) \Rightarrow \nabla V(x) f(x, k(x+e)) < 0. \tag{22}$$

The problem of finding feedback control laws that achieve ISS with respect to measurement errors has received considerable attention in the literature. In particular, it was shown in [9] that the class of systems that admit such control laws includes single-input plants in strict feedback form. It also includes systems that admit *globally Lipschitz* control laws achieving ISS with respect to *actuator* errors, although this condition is quite restrictive.

Let q be a quantizer with sensitivity Δ and saturation value M. The problem under consideration is to find a quantized state feedback law that makes the system (21) asymptotically stable. Assume for the moment that a bound on the initial state is known: $\|x(0)\| \le E_0$. The idea that we propose is to use the above control law k, which results in the closed-loop system

$$\dot{x} = f(x, k(\Delta q(x))).$$

We can rewrite this as

$$\dot{x} = f(x, k(x - s(x)))$$

thus displaying the "error" vector $s(x) := x - \Delta q(x)$. When the inequality (4) holds, the quantizer does not saturate, and the bound (5) is valid. Fix a positive number ϵ, and define the functions

$$\gamma_1(\Delta) := \alpha_1^{-1} \circ \alpha_2 \circ \rho\left(\Delta\left(\sqrt{n}/2 + \epsilon\right)\right) + \Delta\sqrt{n}$$

and

$$\gamma_2(\Delta) := \alpha_2^{-1} \circ \alpha_1((M - 1/2)\Delta).$$

Suppose first that the following condition is satisfied.

Condition 1: $(\alpha_2^{-1} \circ \alpha_1)'(0) > 0$ and $\rho'(0) < \infty$.

In this case, for any given number $\Delta_0 > 0$ there exists a positive integer M such that we have

$$\gamma_2(\Delta) > \gamma_1(\Delta) \qquad \forall\, \Delta \in (0, \Delta_0]. \tag{23}$$

Furthermore, we can take M large enough to have

$$\gamma_2(\Delta_0) \ge E_0.$$

The quantized feedback control strategy can then be described as follows. Set Δ equal to Δ_0. Using (22) in much the same way as the inequality (6) has been used in the previous sections, we can show that there exists a time t_1 with the property that

$$\|q(x(t_1))\| \le \alpha_1^{-1} \circ \alpha_2 \circ \rho\left(\Delta\left(\sqrt{n}/2 + \epsilon\right)\right)/\Delta + \sqrt{n}/2$$

hence

$$\|x(t_1)\| \le \alpha_1^{-1} \circ \alpha_2 \circ \rho\left(\Delta\left(\sqrt{n}/2 + \epsilon\right)\right) + \Delta\sqrt{n} = \gamma_1(\Delta).$$

When $t = t_1$, set Δ equal to $\gamma_2^{-1} \circ \gamma_1(\Delta_0)$, and repeat the procedure. This gives asymptotic stability.

Now suppose that Condition 1 is not satisfied. In this case for any given numbers $\Delta_0 > \delta > 0$ there exists a positive integer M such that we have

$$\gamma_2(\Delta) > \gamma_1(\Delta) \qquad \forall\, \Delta \in (\delta, \Delta_0].$$

It is not hard to see that using the same procedure we only obtain practical stability and not asymptotic stability.

One reason why the above is not satisfactory is the presence of the technical Condition 1. Even when this condition holds, it is not clear whether we can in general achieve *global* (as opposed to just semiglobal) asymptotic stability, because the saturation value M of the quantizer must be chosen a priori and cannot be changed. The "zooming-out" technique does allow us to obtain a global result if for some M the inequality (23) holds with Δ_0 replaced by ∞. The paper [13] contains an example of a system for which this can be shown to be the case.

The class of systems for which control laws achieving ISS with respect to measurement disturbances are known to exist is relatively small. Thus the problem considered here to a large extent reduces to the problem of finding such control laws, which is interesting and important in its own right and is a subject of ongoing research. An alternative approach to semiglobal stabilization can be based on using stabilizing control laws that are robust with respect to *small* measurement errors [20]. These issues will be treated in greater detail elsewhere.

VII. Conclusions

This paper addressed quantized feedback stabilization problems for linear time-invariant control systems. The approach taken here was based on the hypothesis that it is possible to change the sensitivity (but not the saturation value) of the quantizer on the basis of available quantized measurements. We developed a number of techniques, for both continuous- and discrete-time systems, which enable one to achieve global asymptotic stability.

Many other quantized feedback control strategies, in particular those related to the material of Section V, can be found in the literature (see, e.g., [17]). One could try to improve them using our approach. It would also be interesting to extend the ideas presented here to situations where the quantization regions need not be rectilinear but instead can have arbitrary shapes (as in [15]).

In the particular case when $y \in \mathbb{R}^2$, the problem considered in this paper can be interpreted as the problem of finding a stabilizing feedback based on the output measurements obtained using an orthographic projection camera with zooming capability. This observation suggests that the above results may have applications to certain problems arising in vision-based control.

Several topics can be viewed as naturally extending the material presented in this paper. They include control and estimation in the presence of noise, finite communication rate constraints, and/or time delays. Some recent developments in these areas are reported in [14], [23] and [24].

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REFERENCES

[1] R. W. Brockett, *Finite Dimensional Linear Systems*. New York, NY: Wiley, 1970.
[2] ——, "Hybrid models for motion control systems," in *Essays on Control: Perspectives in the Theory and Its Applications*, H. Trentelman and J. C. Willems, Eds. Boston, MA: Birkhäuser , 1993, vol. 14, pp. 29–53.
[3] ——, "Minimum attention control," in *Proc. 36th Conf. Decision and Control*, San Diego, CA, 1997, pp. 2628–2632.
[4] R. W. Brockett and D. Liberzon, "Quantized feedback systems perturbed by white noise," in *Proc. 37th Conf. Decision and Control*, Tampa, FL, 1998, pp. 1327–1328.
[5] J.-H. Chou, S.-H. Chen, and I.-R. Horng, "Robust stability bound on linear time-varying uncertainties for linear digital control systems under finite wordlength effects," *JSME Int. J. Series C*, vol. 39, pp. 767–771, 1996.
[6] D. F. Delchamps, "Stabilizing a linear system with quantized state feedback," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 916–924, 1990.
[7] X. Feng and K. A. Loparo, "Active probing for information in control systems with quantized state measurements: A minimum entropy approach," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 216–238, 1997.
[8] A. F. Filippov, "Differential equations with discontinuous righthand sides," in *Mathematics and Its Applications*. Dordrecht: Kluwer, 1988, p. 18.
[9] R. Freeman and P. V. Kokotovic, "Global robustness of nonlinear systems to state measurement disturbances," in *Proc. 32nd Conf. Decision and Control*, 1993, pp. 1327–1328.
[10] J. K. Hale and M. A. Cruz, "Existence, uniqueness and continuous dependence for hereditary systems," *Ann. Mat. Pura Appl*, vol. 85, no. 4, pp. 63–81, 1970.
[11] B. Hu, Z. Feng, and A. N. Michel, "Quantized sampled-data feedback stabilization for linear and nonlinear control systems," in *Proc. 38th Conf. Decision and Control*, 1999, pp. 4392–4397.
[12] D. Hristu and K. Morgansen, "Limited communication control," *Syst. Contr. Lett.*, vol. 37, pp. 193–205, 1999.
[13] Z.-P. Jiang, I. Mareels, and D. Hill, "Robust control of uncertain nonlinear systems via measurement feedback," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 807–812, 1999.
[14] D. Liberzon, "Asymptotic properties of nonlinear feedback control systems," Ph.D. dissertation, Brandeis Univ., Waltham, MA, Feb 1998.
[15] J. Lunze, B. Nixdorf, and J. Schröder, "Deterministic discrete-event representations of linear continuous-variable systems," *Automatica*, vol. 35, pp. 395–406, 1999.
[16] A. S. Morse, "Supervisory control of families of linear set-point controllers—Part 1: Exact matching," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1413–1431, 1996.
[17] J. Raisch, "Control of continuous plants by symbolic output feedback," in *Hybrid Systems II*, P. Antsaklis *et al.*, Eds. Berlin: Springer-Verlag, 1995, pp. 370–390.
[18] C. E. Rohrs, J. L. Melsa, and D. G. Schultz, *Linear Contr. Syst.*. New York, NY: McGraw-Hill, 1993.
[19] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 435–443, 1989.
[20] ——, "Clocks and insensitivity to small measurement errors," in *Proc. 38th Conf. Decision Control*, 1999, pp. 2661–2666.
[21] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Syst. Contr. Lett.*, vol. 24, pp. 351–359, 1995.
[22] W. A. Wolovich, *Linear Multivariable Systems*. New York, NY: Springer-Verlag, 1974.
[23] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints I: State estimation problems," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1294–1299, 1997.
[24] ——, "Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 1049–1053, 1999.

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