Stabilizing Uncertain Systems with Quantization

Linh Vu Daniel Liberzon

***Abstract*—We consider the problem of stabilizing uncertain systems with quantization. The plant uncertainty is dealt with by the supervisory adaptive control framework, which employs switching among a finite family of candidate controllers. For a static quantizer, we quantify a relationship between the quantization range and the quantization error bound that guarantees closed loop stability. For a dynamic quantizer which can vary the quantization parameters in real time, we show that the closed loop is asymptotically stabilized provided that additional conditions on the quantization range and the quantization error bound is satisfied. This work extends previous results on stabilization of known systems with quantization to the case of uncertain systems.**

I. Introduction

Control with limited information has attracted growing interest in the control research community recently, largely motivated by the *control over network* paradigm. Unlike the classical control setting in which signals take values in a continuum and are available at every time, in networked control systems, information is limited in the sense that control and sensor signals are quantized/digitized before being sent over a communication channel, the information is only available at a certain rate and with delay, and there is a possibility of information loss during data transmission; see, for example, [6] for a recent survey on networked control systems.

Most of the work in control with limited information deals with known plants (see the references in [6], [18]), and only recently, attempts have been made to study control of *uncertain systems* with limited information. While there are several aspects in control with limited information as outlined in the previous paragraph, dealing with both plant uncertainty and limited information at the same time is rather challenging. As a first step, we treat limited information as quantization only. Quantized control systems with known plants have been considered, for example, in [3], [4], [15], [19], [24]. In this paper, we consider the problem of *stabilizing uncertain systems* with *quantization*. This problem has been studied by Hayakawa et al in [5], where the authors provided a solution using a (static) logarithmic quantizer and a Lyapunov-based adaptive algorithm.

Contacts: linhvu@u.washington.edu, liberzon@uiuc.edu.

Work of Vu is supported by AFOSR grant FA9550-07-10528. Work of Liberzon is supported by grant NSF ECCS-0701676.

We are interested in the case of uncertain systems with large uncertainty so that robust control is not sufficient, and adaptive control is required. Adaptive control is a classic control topic where various tools are available (see, e.g., [17, Section I] for a literature review on adaptive control). One of the recently developed tools in adaptive control is the *supervisory control* framework [8], which employs switching among a finite family of candidate controllers. The controllers are designed using (a finite number of) nominal parameters in the uncertainty set, and the switching is orchestrated by a switching logic based on comparison of the estimation errors coming out of a multi-estimator. Benefits of this adaptive control scheme include modularity in controller design and the ability to handle large uncertainty sets; see [8] for further discussions on advantages and applications of supervisory control. In this paper, we employ the supervisory control framework to deal with plant uncertainty.

For a static quantizer, we want to find a relationship between the quantization range and the quantization error bound to guarantee closed loop stability. While it has been shown [7, Proposition 6] that supervisory control is robust to measurement noise, extending this result to quantization is not trivial because one needs to ensure that the information to be quantized does not exceed the quantization range. In this work, we give a condition on the quantizer parameters to guarantee closed loop stability.

To achieve asymptotic stability, we utilize the *dynamic quantizers* in [2], [12], which have the capability of varying the quantization parameters in real time (in particular, the quantizer can zoom in and zoom out while keeping the number of alphabets fixed). In the works [2], [12], the authors have applied dynamic quantization to asymptotically stabilize known linear plants (see also [11] for performance analysis of dynamic quantization). For known linear plants, asymptotic stability can also be achieved with a logarithmic quantizer [3] (which indeed was the quantizer employed in [5] for the case of uncertain systems). Compared to logarithmic quantizers, which have infinite alphabets, a dynamic quantizer has a finite alphabet. We show that for uncertain systems with quantization, asymptotic stability is achievable with supervisory control and dynamic quantization, provided that the quantizer satisfies a certain condition. While the tools for analyzing supervisory control and dynamic quantization have been reported separately [7], [12], the analysis of the combination of both is far from a trivial extension of [7] and [12].

A. Notations

The notations in this paper are fairly standard: $\mathbb{R}$ is the set of real numbers, $|\cdot|$ is the Euclidean norm, and $\|\cdot\|_{\mathcal{I}}$ is the supnorm of a signal over the interval $\mathcal{I} \subseteq [0,\infty)$. Recall that a continuous function $\alpha : [0,\infty) \to [0,\infty)$ is of class $\mathcal{K}_\infty$ if α is strictly increasing, with $\alpha(0) = 0$, and $\alpha(r) \to \infty$ as $r \to \infty$. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is a function of class $\mathcal{K}_\infty$ for every fixed t, and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for every fixed r.

II. Quantized control system

We start with a setting in which the uncertain plant is linear and belongs to a known finite set of plants. We will consider more general settings such as continuum uncertainty sets and nonlinear plants in later sections.

Consider an uncertain linear plant Γ parameterized by a parameter p, and denote by p^* the true but unknown parameter:

$$\Gamma : \begin{cases} \dot{x} = A_{p^*}x + B_{p^*}u \\ y = C_{p^*}x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, and $y \in \mathbb{R}^{n_y}$ is the output. The parameter $p^* \in \mathbb{R}^{n_p}$ belongs to a known finite set $\mathcal{P} := \{p_1, \ldots, p_m\}$, where m is the cardinality of $\mathcal{P}$.

Assumption 1 *(A_p, B_p) is stabilizable, and (A_p, C_p) is detectable for every $p \in \mathcal{P}$.*

A (static) quantizer is a map $\mathrm{Q} : \mathbb{R}^{n_y} \to \{q_1, \ldots, q_N\}$, where $q_1, \ldots, q_N \in \mathbb{R}^{n_y}$ are quantization points, and Q has the following properties: 1) $|y| \leqslant M \Rightarrow |\mathrm{Q}(y) - y| \leqslant \Delta$, and 2) $|y| > M \Rightarrow |\mathrm{Q}(y)| > M - \Delta$. The numbers M and Δ are known as *the range* and the *error bound* of the quantizer Q. A *dynamic quantizer* Q_ν, having an additional parameter ν which can be changed over time, is defined as

$$\mathrm{Q}_\nu(y) := \nu \mathrm{Q}(y/\nu),$$

where Q is a static quantizer with the range M and the error bound Δ. From the property 1) of the quantizer, we have

$$|y| \leqslant \nu M \Rightarrow |\mathrm{Q}(y) - y| \leqslant \nu\Delta. \tag{2}$$

The parameter ν is known as a *zooming variable*: increasing ν corresponds to zooming out and essentially obtaining a new quantizer with larger range and quantization error, whereas decreasing ν corresponds to zooming in and obtaining a quantizer with a smaller range but also a smaller quantization error.

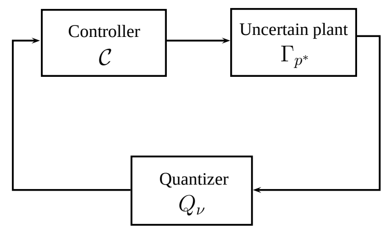

Fig. 1. Quantized closed-loop system

Assuming that the plant is unstable, the objective is to asymptotically stabilize the plant while the information available to the controller is $\mathrm{Q}_\nu(y)$ instead of y. The quantized control system is depicted in Fig. 1, where $\mathcal{C}$ denotes the overall controller for the plant.

III. Supervisory control

A. Without quantization

We recover the supervisory (adaptive) control framework [8] for the case without quantization. In supervisory control, there are multiple controllers, and which controller to connect to the plant is orchestrated by a supervisor (see Fig. 2 for an illustration of the idea); for more detailed background on supervisory control, see, e.g., [14, Chapter 6] or [8] and the references therein.

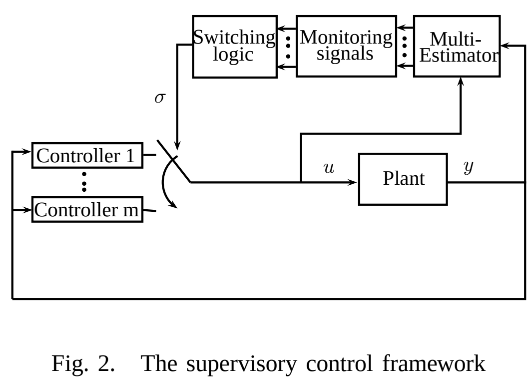

Fig. 2. The supervisory control framework

We present one particular design of supervisory control for linear plants, in which the controllers utilize the multi-estimator's state (more detail below). We can also have more general forms of dynamic controllers which do not use the multiestimator state, provided that the multi-controller and the multi-estimator combination (known as the injected system) satisfies certain conditions; see the general (nonlinear) setting in Section V for detail.

- **Multi-estimator**: A multi-estimator is a collection of dynamics, one for each fixed parameter $p \in \mathcal{P}$. The multi-estimator takes in the input u and produces a bank of outputs $y_p, p \in \mathcal{P}$. The multi-estimator should have the following property: there is $\hat{p} \in \mathcal{P}$ such that

$$|y_{\hat{p}}(t) - y(t)| \leqslant c_e e^{-\lambda_e (t-t_0)} |y_{\hat{p}}(t_0) - y(t_0)| \quad (3)$$

for all $t \geqslant t_0$, for all u, and for some $c_e \geqslant 0$ and $\lambda_e > 0$. This property is known as the matching property in supervisory control.
One such multi-estimator for (1) is the collection of the dynamics

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + L_p(y_p - y), \\ y_p &= C_p x_p, \end{aligned} \qquad p \in \mathcal{P}, \quad (4)$$

where C_p are such that $A_p + L_p C_p$ is Hurwitz for every $p \in \mathcal{P}$. The matching property (3) is satisfied with $\hat{p} = p^*$, $c_e = 1$, and $\lambda_e = 1$ because $y_{p^*} - y = C_{p^*}(x_{p^*} - x)$, and (4) with $p = p^*$, and (1) implies that $(d/dt)(x_{p^*} - x) = (A_{p^*} + L_{p^*} C_{p^*})(x_{p^*} - x)$.

- **Multi-controller**: A family of *candidate feedback gains* $\{K_p\}$ is designed such that $A_p + B_p K_p$ is Hurwitz for every $p \in \mathcal{P}$. Then the *family of controllers* is

$$u_p = K_p x_p \quad p \in \mathcal{P}. \quad (5)$$

- **Monitoring signals**: *Monitoring signals* $\mu_p, p \in \mathcal{P}$ are certain norms of the output estimation errors, $y_p - y$. Here, the monitoring signals are generated as

$$\dot{\hat{\mu}}_p = -\lambda \hat{\mu}_p + \gamma |y_p - y|^2, \; \hat{\mu}_p(0) = 0, \quad (6a)$$
$$\mu_p = \varepsilon + \hat{\mu}_p, \quad (6b)$$

for some $\gamma, \varepsilon, \lambda > 0$. The numbers γ, ε, and λ are design parameters and need to satisfy

$$0 < \lambda < \lambda_0 \quad (7)$$

for some constant λ_0 related to the eigenvalues of $A_p + B_p K_p, \; p \in \mathcal{P}$ (for detail, see the proof in Appendix A).

- **Switching logic**: A switching logic produces a switching signal that indicates at every time the active controller. In this paper, we use the *scale-independent hysteresis switching logic* [10]:

$$\sigma(t) := \begin{cases} \operatorname*{argmin}_{q \in \mathcal{P}} \mu_q(t) & \text{if } \exists\, q \in \mathcal{P} \text{ such that} \\ & (1+h)\mu_q(t) \leqslant \mu_{\sigma(t^-)}(t), \\ \sigma(t^-) & \text{else}, \end{cases} \quad (8)$$

where $h > 0$ is a *hysteresis constant*; h is a design parameter and satisfies the following condition:

$$\frac{\ln(1+h)}{\lambda m} > \frac{\ln \mu_V}{\lambda_0 - \lambda} \quad (9)$$

for some constant μ_V (see the proof in Appendix A for the definition of μ_V). The control signal applied to the plant is $u(t) = u_\sigma(t) := K_{\sigma(t)} x_{\sigma(t)}(t)$.

B. With quantization

With quantization, the multi-estimator (4) becomes

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + L_p(y_p - \mathrm{Q}_\nu(y)), \\ y_p &= C_p x_p, \end{aligned} \qquad p \in \mathcal{P}. \quad (10)$$

The above equation can be rewritten as $\dot{x}_p = A_p x_p + B_p u + L_p(y_p - y + y - \mathrm{Q}_\nu(y))$. Due to the presence of the term $y - \mathrm{Q}_\nu(y)$ in the foregoing equation, the matching condition (3) becomes

$$\begin{aligned} |y_{\hat{p}}(t) - y(t)| &\leqslant c_e e^{-\lambda_e (t-t_0)} |y_{\hat{p}}(t_0) - y(t_0)| \\ &\quad + \gamma_e \|y - \mathrm{Q}_\nu(y)\|_{[t_0,t)} \quad \forall t \geqslant t_0, \forall u \end{aligned} \quad (11)$$

for some $c_e, \gamma_e \geqslant 0$, $\lambda_e > 0$. Similarly as before, the condition (11) is satisfied with $\hat{p} = p^*$, $c_e = 1$, $\lambda_e = 1$, and $\gamma_e = \|C_{p^*} L_{p^*}\|$. The monitoring signal generator (6) becomes

$$\dot{\hat{\mu}}_p = -\lambda \hat{\mu}_p + \gamma |y_p - \mathrm{Q}_\nu(y)|^2, \; \hat{\mu}_p(0) = 0, \quad (12a)$$
$$\mu_p = \varepsilon + \hat{\mu}_p. \quad (12b)$$

IV. Stability of supervisory control with quantization

Denote by $\mathcal{K}(c)$ the class of continuous functions from $\mathbb{R}^\ell$ to $\mathbb{R}$ for some $\ell \in \mathbb{Z}$ and $c > 0$ such that, if $f \in \mathcal{K}(c)$, then $f(z) \to c$ as $|z| \to 0$. Let $x_{\mathbb{E}} := (x_{p_1}, \ldots, x_{p_m})^T$ for some ordering $p_1, \ldots, p_m$ of $\mathcal{P}$, and $k_c := \max_{p \in \mathcal{P}} \|C_p\|$. We have the following result concerning static quantizers (i.e. those with fixed zooming variables ν).

Theorem 1 *Consider the uncertain system* (1) *and the supervisory control scheme described in Section III with* *the design parameters satisfying* (7) *and* (9). *Let* t_0 *be an arbitrary time, and suppose that* $|x_{\mathbb{E}}(t_0)| \leqslant \bar{x}_0$ *and* $|\mu_{\hat{p}}(t_0)| \leqslant \bar{\mu}_0$ *for some constants* $\bar{x}_0, \bar{\mu}_0 > 0$. *Let* $X_0 > 0$ *and* $\bar{y}_0 := k_c(X_0 + \bar{x}_0)$. *Suppose that the zooming variable* ν *is fixed. There exist*

- *a function* $\chi_{\Delta,\nu} \in \mathcal{K}\big(k_c\gamma(\nu\Delta/\sqrt{\varepsilon})(\sqrt{\varepsilon} + a_1\nu\Delta) + a_2\nu\Delta\big)$ *for some positive constants* a_1, a_2, *and* $\gamma \in \mathcal{K}_\infty$,
- *a function* $\psi^x_{\Delta,\nu} \in \mathcal{K}\big(\gamma(\nu\Delta/\sqrt{\varepsilon})(\sqrt{\varepsilon} + a_1\nu\Delta) + \gamma_e\nu\Delta\big)$, *where* γ_e *is as in* (11)

such that if

$$\chi_{\Delta,\nu}(\bar{x}_0, \bar{\mu}_0, \bar{y}_0) < \nu M, \tag{13}$$

then $\forall |x(t_0)| \leqslant X_0$, *all the closed-loop signals are bounded, and for every* $\epsilon_x > 0$, $\exists T < \infty$ *such that*

$$|x(t)| \leqslant \psi^x_{\Delta,\nu}(\bar{\mu}_0, \bar{y}_0) + \epsilon_x \quad \forall t \geqslant t_0 + T. \tag{14}$$

Remark 1 *To better convey the idea and not get bogged down in complicated details, we do not give the explicit formulae for* $\chi_{\Delta,\nu}$ *and* $\psi_{\Delta,\nu}$ *in the theorem; see Appendix A for details (equations* (61) *and* (65)*). Note from* (65) *that* $\psi_{\Delta,\nu}$ *implicitly depends on* $\bar{x}_0$ *via* $\bar{y}_0$ *in* c_2. *There are two interpretations of the condition* (13)*: 1) for a given* M, Δ, *and* ν *such that* $\nu M > k_c\gamma(\nu\Delta/\sqrt{\varepsilon})(\sqrt{\varepsilon} + a_1\nu\Delta) + a_2\nu\Delta$, *there exist small enough* $\bar{x}_0, \bar{y}_0$, *and* $\bar{\mu}_0$ *such that* (13) *holds (this follows from the property that* $\chi_{\Delta,\nu} \in \mathcal{K}(k_c\gamma(\nu\Delta/\sqrt{\varepsilon})(\sqrt{\varepsilon} + a_1\nu\Delta) + a_2\nu\Delta)$*), and 2) for a given* $\bar{x}_0, \bar{y}_0$, *and* $\bar{\mu}_0$, *the condition* (13) *holds if* M *is large enough (since* $\chi_{\Delta,\nu}$ *does not depend on* M*).*

The proof of Theorem 1 comprises four main stages:

- We establish a bound on the signal $\mu_{\hat{p}}$ in terms of the error bound Δ using the property (11) of the multi-estimator
- We then establish a bound on the state $x_{\mathbb{E}}$ (which is known as the state of the *injected system*; see Appendix A) in terms of the error bound Δ
- We show that the condition (13) on M and Δ ensures that the state x cannot get out of the ball of radius νM (and hence, the quantizer guarantees the error bounded for all time)
- From boundedness of $x_{\mathbb{E}}$, we finally conclude ultimate boundedness of the plant state x.

Technical details of the proof are interesting as it combines the techniques in supervisory control and dynamic quantization. For clarity of the presentation, we choose to leave them in Appendix A.

The importance of Theorem 1 is that it provides a condition on the quantization range M and the quantization error Δ of a static quantizer (this condition depends on the bounds on initial states) that guarantees closed-loop stability. More precisely, we achieve not just boundedness but ultimate boundedness, characterized by (33). Note that the ultimate bound in (33) can be larger than X_0. This non-contraction situation would occur if the quantization error is large comparatively to the initial condition. For state contraction, we will need additional constraints on M, Δ, and X_0 for smallness of quantization error.

If there is state contraction at time T, then one can achieve asymptotic stability by using a dynamic quantizer, varying the zooming variable ν as well as the parameter ε in the supervisory control scheme as $x_{\mathbb{E}}$ gets closer to the origin. Unlike the case of known plants [12] where one only needs to worry about the contraction of the plant state x, here one needs to take into account the asymptotic behavior of other state variables coming from the supervisory control scheme such as μ_p and $|y_p - y|$.

A *logarithmic* scalar variable ξ with a factor ρ and a period T is defined as follows (c.f. [3]):

$$\xi(t) := \begin{cases} \xi(kT) & \text{if } t \in [kT, (k+1)T) \\ \rho\xi(kT) & \text{if } t = (k+1)T, \end{cases} \quad k = 0, 1, \ldots. \tag{15}$$

The following result says that using a dynamic quantizer with a logarithmic zooming variable, we can achieve closed-loop asymptotic stability. For the proof, see Appendix B.

Theorem 2 *Consider the uncertain system* (1) *and the supervisory control scheme described in Section III with the design parameters satisfying* (7) *and* (9). *Let* t_0 *be an arbitrary time, and suppose that* $|x_{\mathbb{E}}(t_0)| \leqslant \bar{x}_0$, *and* $|\mu_{\hat{p}}(t_0)| \leqslant \bar{\mu}_0$ *for some constants* $\bar{x}_0, \bar{\mu}_0 > 0$. *Let* $X_0 > 0$ *and* $\bar{y}_0 := k_c(X_0 + \bar{x}_0)$. *There exist*

- *a function* $\chi_{\Delta,\nu} \in \mathcal{K}\big(k_c\gamma(\nu\Delta/\sqrt{\varepsilon})(\sqrt{\varepsilon} + a_1\nu\Delta) + a_2\nu\Delta\big)$ *for some positive constants* a_1, a_2, *and* $\gamma \in \mathcal{K}_\infty$,
- *a function* $\psi_{\Delta,\nu} \in \mathcal{K}\big(\gamma(\nu\Delta/\sqrt{\varepsilon})(\sqrt{\varepsilon} + a_1\nu\Delta)\big)$
- *positive constants* a_3, a_4, *and* a_5

such that if (13) *holds and*

$$\psi_{\Delta,\nu}(\bar{\mu}_0, \bar{y}_0) < \bar{x}_0, \tag{16a}$$

$$a_3\varepsilon + a_4\nu^2\Delta^2 < \bar{\mu}_0, \tag{16b}$$

$$a_5\nu\Delta < \bar{y}_0, \tag{16c}$$

then one can find $\rho \in (0, 1)$ *and* $0 < T < \infty$ *such that under the logarithmic* ε *with factor* ρ^2 *and period* T, *and the logarithmic zooming variable* μ *with factor* ρ *and period* T, *for all* $|x(0)| \leqslant X_0$, *the plant state*

$|x(t)| \to 0$ *as* $t \to \infty$*, and all the closed-loop signals are bounded.*

Remark 2 *As discussed in Remark 1, the condition* (13) *can always be satisfied for large enough* M *or small enough* $\bar{x}_0, \bar{\mu}$*, and* $\bar{y}_0$*. However,* $\bar{x}_0, \bar{\mu}$*, and* $\bar{y}_0$ *also need to be lower bounded as in* (16a), (16b)*, and* (16c)*. Nevertheless,* $\psi_{\Delta,\nu} \to 0$ *as* $\{\Delta, \varepsilon\} \to 0$ *so for a given* $\bar{x}_0, \bar{\mu}$*, and* $\bar{y}_0$*,* (16a), (16b)*, and* (16c) *hold if* Δ *is small enough. Compared to Theorem 1, the extra conditions* (16a), (16b)*, and* (16c) *place an upper bound on* Δ *for given* $\bar{x}_0, \bar{\mu}$*, and* $\bar{y}_0$ *to ensure that the signals in the supervisory control system are contracting after a certain time. Combining this contraction property with the zooming-in technique, we achieve asymptotic stability. In Theorem 1, this contraction is not needed when one is only concerned with stability, not asymptotic stability.*

Remark 3 *The conditions* (13) *and* (16) *on* M *and* Δ *imply a lower bound on the number of quantization bits. Suppose that each component of* x *has the same range and is equally quantized into* $2^{n_{\mathcal{Q}}}$ *regions using* $n_{\mathcal{Q}}$ *quantization bits. Then* $n_{\mathcal{Q}} = \log_2\lceil M/\Delta \rceil$*. Then the condition* (13) *and* (16) *can be rewritten into the form* $n_Q > \log_2\lceil \chi_{\Delta,\nu}(\bar{x}_0, \bar{\mu}_0, \bar{y}_0)/(\nu\Delta)\rceil$.

Remark 4 *If the bound* X_0 *on the initial state is not available, we can include a zooming-out stage at the beginning (see [12]) so that after a certain time* t_0*, we guarantee* $|x(t_0)| < \nu M$*. This means increasing* ν *faster than the system can blow up (for any value of* $p \in \mathcal{P}$*) until the quantizer no longer saturates.*

V. Nonlinear systems

Our result for adaptive stabilization with quantization obtained so far can be extended to a certain class of nonlinear systems, using the result in [23]. Consider a parameterized nonlinear uncertain plant $\Gamma(p^*)$ where $p^* \in \mathbb{R}^{n_p}$ is the true but known parameter:

$$\Gamma(p^*) : \begin{cases} \dot{x} = f(x, u, p^*) \\ y = h(x, p^*), \end{cases} \tag{17}$$

where f is Lipschitz in x, u, h is continuous in x, and $h(0, p) = 0\ \forall p \in \mathcal{P}$. As in the previous section, for the sake of the presentation, we assume that p^* belongs to a finite set $\mathcal{P}$.

The supervisory control framework for nonlinear plants is similar in spirit to those for linear plants described in Section III, albeit now we work with nonlinear dynamics in general. The control tool to deal with stability of nonlinear systems is the *input-to-state stability* (ISS) framework (see, e.g., [21]).

A. Without quantization

- **Multi-estimator:** The multi-estimator can be written generally as

$$\begin{aligned} \dot{x}_E &= F(x_{\mathbb{E}}, y, u), \\ y_p &= H_p(x_{\mathbb{E}}), \end{aligned} \qquad p \in \mathcal{P} \tag{18}$$

where $x_{\mathbb{E}} := (x_{p_1}, \ldots, x_{p_m})$ is the state of the multi-estimator for some ordering $p_1, \ldots, p_m$ of the set $\mathcal{P}$, and the dynamics of x_p are

$$\begin{aligned} \dot{x}_p &= \hat{f}_p(x_p, y, u), \\ y_p &= h_p(x_p), \end{aligned} \qquad p \in \mathcal{P}. \tag{19}$$

The multi-estimator should have the following property: There exists $\hat{p} \in \mathcal{P}$ such that for all u,

$$|y_{\hat{p}}(t) - y(t)| < \beta_e(|y_{\hat{p}}(t_0) - y(t_0)|, t - t_0)\ \forall\, t \geqslant t_0, \tag{20}$$

for some $\beta_e \in \mathcal{KL}$.

- **Multicontroller:** A family of *candidate controllers*

$$\begin{aligned} \dot{x}_{\mathbb{C}} &= g_p(x_{\mathbb{C}}, y, u), \\ u_p &= r_p(x_{\mathbb{C}}, y), \end{aligned} \qquad q \in \mathcal{P}, \tag{21}$$

are designed such that the controller indexed by p stabilizes the plant with the same index. Moreover, $r_p(0, 0) = 0$ for all $p \in \mathcal{P}$.

The (switched) injected system is the combination of the multi-estimator and the multi-controller, and is a switched system. The injected system with the controller indexed by $p \in \mathcal{P}$ is

$$\begin{aligned} \dot{x}_{CE} &= \begin{bmatrix} g_p(x_{\mathbb{C}}, H_p(x_{\mathbb{E}}) - \tilde{y}_p, r_p(x_{\mathbb{C}}, H_p(x_{\mathbb{E}}) - \tilde{y}_p)) \\ F(x_{\mathbb{E}}, H_p(x_{\mathbb{E}}) - \tilde{y}_p, r_p(x_{\mathbb{C}}, H_p(x_{\mathbb{E}}) - \tilde{y}_p)) \end{bmatrix} \\ &=: f_p(x_{\mathbb{CE}}, \tilde{y}_p), \end{aligned} \tag{22}$$

where $x_{\mathbb{CE}} := \begin{pmatrix} x_{\mathbb{C}} \\ x_{\mathbb{E}} \end{pmatrix}$ is the state of the injected system. The following is an assumption on the injected systems (22) (see also Remark 5 below).

Assumption 2 *There exist continuously differentiable functions* $V_p : \mathbb{R}^n \to [0, \infty),\ p \in \mathcal{P}$*, class* $\mathcal{K}_\infty$ *functions* $\alpha_1, \alpha_2, \gamma$*, and numbers* $\lambda_0 > 0$ *such that* $\forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^\ell$*, and* $\forall p, q \in \mathcal{P}$*, we have*

$$\alpha_1(|\xi|) \leqslant V_p(\xi) \leqslant \alpha_2(|\xi|), \tag{23}$$

$$\frac{\partial V_p}{\partial \xi} f_p(\xi, \eta) \leqslant -\lambda_0 V_p(\xi) + \gamma(|\eta|), \tag{24}$$

$$V_p(\xi) \leqslant \mu_V V_q(\xi). \tag{25}$$

- **Monitoring signals and switching logic:** The monitoring signals $\mu_p,\ p \in \mathcal{P}$, are generated as follows:

$$\begin{aligned} \dot{\bar{\mu}}_p &= -\lambda\bar{\mu}_p + \gamma(|y_{\hat{p}} - y|), \quad z_p(0) = 0, \\ \mu_p(t) &= \varepsilon + \bar{\mu}_p(t), \end{aligned} \tag{26}$$

for some $\varepsilon > 0$, $\lambda \in (0, \lambda_0)$, where λ_0 and γ are as in (24).

The switching logic is the scale-independent hysteresis switching logic defined as in (8). At every switching time τ, we make $x_{\mathbb{C}}(\tau^-) = x_{\mathbb{C}}(\tau)$. The control signal is

$$u(t) = r_{\sigma(t)}(x_{\mathbb{C}}, y).$$

Remark 5 *If every subsystem is ISS, then for every $p \in \mathcal{P}$ there exist class $\mathcal{K}_\infty$ functions $\alpha_{1,p}, \alpha_{2,p}, \gamma_p$, numbers $\lambda_{\circ,p} > 0$, and ISS-Lyapunov functions V_p, satisfying*

$$\begin{aligned} \alpha_{1,p}(|\xi|) \leqslant V_p(\xi) \leqslant \alpha_{2,p}(|\xi|), \\ \frac{\partial V_p}{\partial \xi} f_p(\xi) \leqslant -\lambda_{\circ,p} V_p(\xi) + \alpha_2\gamma_p(|\eta|), \end{aligned}$$

$\forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^\ell$; see [20], [22]. If the set $\mathcal{P}$ is finite, then (23) *and* (24) *are trivially satisfied. Also, if the set $\mathcal{P}$ is compact, and suitable continuity assumptions on $\{\alpha_{1,p}, \alpha_{2,p}, \alpha_2\gamma_p\}_{p\in\mathcal{P}}$ and $\{\lambda_{\circ,p}\}_{p\in\mathcal{P}}$ with respect to p hold,* (23) *and* (24) *follow. We shall henceforth stipulate that our collection of ISS-Lyapunov functions $\{V_p\}_{p\in\mathcal{P}}$ satisfies* (23) *and* (24).

The set of possible ISS-Lyapunov functions is restricted by the condition (25). *This inequality does not hold, for example, if V_p is quadratic for one value of p and quartic for another. If $\mu = 1$, the relation* (25) *implies that $V = V_p,\ p \in \mathcal{P}$ is a common ISS-Lyapunov function for the family of the subsystems. In this case, the switched system is ISS for* arbitrary switching *(also called* uniformly input-to-state stable *[16]).*

B. With quantization

In the case with quantization, the multi-estimator (19) becomes

$$\begin{aligned} \dot{x}_E &= F(x_{\mathbb{E}}, \mathrm{Q}_\nu(y), u), \\ y_p &= H_p(x_{\mathbb{E}}), \end{aligned} \qquad p \in \mathcal{P}. \tag{27}$$

The matching property (20) becomes

$$\begin{aligned} |y_{\hat{p}}(t) - y(t)| < \beta_e(|y_{\hat{p}}(t_0) - y(t_0)|, t - t_0) \\ + \gamma_e(\|y - \mathrm{Q}_\nu(y)\|_{[t_0,t)}) \quad \forall\, t \geqslant t_0 \end{aligned} \tag{28}$$

for some $\beta_e \in \mathcal{KL}$ and $\gamma_e \in \mathcal{K}_\infty$ (cf. the linear case (11), where $\beta_e(r,t) = c_e e^{-\lambda_e t} r$, and γ_e is a constant).

The monitoring signal generator becomes

$$\begin{aligned} \dot{\bar{\mu}}_p &= -\lambda\bar{\mu}_p + \gamma(|y_p - \mathrm{Q}_\nu(y)|), \quad z_p(0) = 0, \\ \mu_p(t) &= \varepsilon + \bar{\mu}_p(t). \end{aligned} \tag{29}$$

C. Stability with quantization

Recall that a plant is input-output-to-state (IOSS) (see, e.g., [21]) if the state x of the (open-loop) plant satisfies the following property

$$|x(t)| \leqslant \beta(|x(t_0)|, t - t_0) + \gamma_u(\|u\|_{[t_0,t)}) + \gamma_y(\|y\|_{[t_0,t)}) \tag{30}$$

for all $t \geqslant t_0$ for some $\beta \in \mathcal{KL}, \gamma_u, \gamma_y \in \mathcal{K}_\infty$. Define a function $k_c \in \mathcal{K}$ as

$$k_c(z) := \max_{p\in\mathcal{P}} \sup_{|x|\leqslant z} h_p(x).$$

Define $\phi(t, t_0, z) := \int_{t_0}^{t} e^{-\lambda(t-s)}\gamma(2\beta_e(z, s - t_0))ds$, where γ is as in (24), and β_e is as in (28). The function ϕ is positive definite, bounded above (by $\gamma(2c_e z)/\lambda$). We further assume that ϕ and γ have the following properties:

$$\phi(t, t_0, z) \to 0 \text{ as } (t - t_0) \to \infty\ \forall z \geqslant 0 \tag{31a}$$

$$\phi(t, t_0, \alpha z) \leqslant \alpha\phi(t, t_0, z) \quad \forall \alpha \in [0, 1), \forall z \geqslant 0 \tag{31b}$$

Note that the condition (31b) is required for $\alpha \in [0, 1)$ only, and it holds for γ other than quadratic as in the linear setting (for example, (31b) holds for $\gamma(z) = z^3$ and $\beta_e(r,t) = c_e e^{-\lambda_e r} r$).

The following theorems are nonlinear counterparts of Theorem 1 and Theorem 2 for linear systems (the proofs are in Appendix C and Appendix D).

Theorem 3 *Consider the uncertain system* (1) *and the supervisory control scheme described in Section V-B with the design parameters satisfying* (7) *and* (9). *Suppose that the plant is IOSS, and* (31) *holds. Let t_0 be an arbitrary time, and suppose that $|x_{\mathbb{E}}(t_0)| \leqslant \bar{x}_0$ and $|\mu_{\hat{p}}(t_0)| \leqslant \bar{\mu}_0$ for some constants $\bar{x}_0, \bar{\mu}_0 > 0$. Let $X_0 > 0$ and $\bar{y}_0 =: k_c(X_0) + k_c(\bar{x}_0)$. Suppose that $|y_{\hat{p}}(t_0) - y(t_0)| \leqslant \bar{y}_0$. There exist*

- *a function $\chi_{\Delta,\nu} \in \mathcal{K}(\gamma_1(\bar{\gamma}_0(\gamma_0(\nu\Delta)/\varepsilon))(\varepsilon + \gamma_2(\nu\Delta))) + \gamma_3(\nu\Delta)$ for some $\gamma_1 \in \mathcal{K}$, and $\bar{\gamma}_0, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$,*
- *a function $\psi^x_{\Delta,\nu} \in \mathcal{K}(\bar{\gamma}_x(\gamma_1(\bar{\gamma}_0(\gamma_0(\nu\Delta)/\varepsilon)(\varepsilon + \gamma_2(\nu\Delta))) + \gamma_e(\nu\Delta)))$ for some $\bar{\gamma}_x \in \mathcal{K}_\infty$ where γ_e is as in* (28)

such that if

$$\chi_{\Delta,\nu}(\bar{x}_0, \bar{\mu}_0, \bar{y}_0) < \nu M, \tag{32}$$

then $\forall|x(t_0)| \leqslant X_0$, *all the closed-loop signals are bounded, and for every* $\epsilon_x > 0$, $\exists T < \infty$ *such that*

$$|x(t)| \leqslant \psi^x_{\Delta,\nu}(\bar{\mu}_0, \bar{y}_0) + \epsilon_x \quad \forall t \geqslant t_0 + T. \tag{33}$$

Theorem 4 *Consider the uncertain system* (1) *and the supervisory control scheme described in Section V-B with the design parameters satisfying* (7) *and* (9)*. Suppose that the plant is IOSS, and* (31) *holds. Let* t_0 *be an arbitrary time, and suppose that* $|x_{\mathbb{E}}(t_0)| \leqslant \bar{x}_0$ *and* $|\mu_{\hat{p}}(t_0)| \leqslant \bar{\mu}_0$ *for some constants* $\bar{x}_0, \bar{\mu}_0 > 0$*. Let* $X_0 > 0$ *and* $\bar{y}_0 =: k_c(X_0) + k_c(\bar{x}_0)$*. Suppose that* $|y_{\hat{p}}(t_0) - y(t_0)| \leqslant \bar{y}_0$*. There exist*

- *a function* $\chi_{\Delta,\nu} \in \mathcal{K}(\gamma_1(\bar{\gamma}_0(\gamma_0(\nu\Delta)/\varepsilon))(\varepsilon + \gamma_2(\nu\Delta))) + \gamma_3(\nu\Delta)$ *for some* $\gamma_1 \in \mathcal{K}$, *and* $\bar{\gamma}_0, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$,
- *a function* $\psi_{\Delta,\nu,\varepsilon} \in \mathcal{K}(\bar{\gamma}_1(\bar{\gamma}_0(\gamma_0(\nu\Delta)/\varepsilon))(\varepsilon + \gamma_2(\nu\Delta))))$ *for some* $\bar{\gamma}_1 \in \mathcal{K}_\infty$
- *class* $\mathcal{K}_\infty$ *functions* $\gamma_4, \gamma_5, \overline{\alpha}_1, \overline{\alpha}_2$

such that if (32) *holds and*

$$\overline{\alpha}_1(\psi_{\Delta,\nu,\varepsilon}(\bar{\mu}_0, \bar{y}_0)) < \rho\overline{\alpha}_1(\bar{x}_0), \tag{34a}$$

$$\varepsilon + \gamma_4(\nu\Delta) < \rho\bar{\mu}_0, \tag{34b}$$

$$\gamma_5(\nu\Delta) < \rho\bar{y}_0, \tag{34c}$$

$$k_\varepsilon\varepsilon + \gamma_4(k_\nu\nu\Delta) \leqslant \rho(\varepsilon + \gamma_4(\nu\Delta)) \ \forall\nu, \Delta \tag{34d}$$

$$\gamma_5(k_\nu\nu\Delta) \leqslant \rho\gamma_5(\nu\Delta) \ \forall\nu, \Delta \tag{34e}$$

$$\overline{\alpha}_1(\psi_{\Delta,k_\nu\nu,k_\varepsilon\varepsilon}(\bar{\mu}_0, \bar{y}_0)) \leqslant \rho\overline{\alpha}_1(\psi_{\Delta,\nu,\varepsilon}(\bar{\mu}_0, \bar{y}_0))) \ \forall\nu, \Delta, \varepsilon \tag{34f}$$

$$\overline{\alpha}_2(\overline{\alpha}_1^{-1}(\rho\overline{\alpha}_1(\bar{x}_0))) \leqslant \rho(\overline{\alpha}_2(\bar{x}_0)) \tag{34g}$$

$$\gamma_0(k_\nu\nu\Delta) \leqslant k_\varepsilon\varepsilon \quad \forall\nu, \Delta, \varepsilon \tag{34h}$$

for some constants $\rho, k_\nu, k_\varepsilon \in (0,1)$*, then there exists* $0 < T < \infty$ *such that under the logarithmic* ε *with factor* k_ε *and period* T*, and the logarithmic zooming variable* ν *with factor* k_ν *and period* T*, for all* $|x(0)| \leqslant X_0$*, we have* $|x(t)| \to 0$ *as* $t \to \infty$*, and all the closed-loop signals are bounded.*

Remark 6 *When* γ_4 *and* γ_5 *are linear, then* (34d) *and* (34e) *are true for all* $0 < \{k_\varepsilon, k_\nu\} < \rho$*. When* $\psi_{\Delta,k_\nu\nu,k_\varepsilon\varepsilon}$ *is linear in* ν, ε*, then* (34f) *is true for all* $0 < \{k_\varepsilon, k_\nu\} < \rho$*. The condition* (34g) *is true, for example when* $\overline{\alpha}_2 = c_\alpha\overline{\alpha}_1$ *for some constant* c *(which is more general than the linear case in which* $\overline{\alpha}_1$ *and* $\overline{\alpha}_2$ *are quadratic).*

The condition (32) *places a constraint on the quantizer parameters* M, Δ *for stability (Theorem 3). The additional conditions* (34a)-(34c) *place an upper bound on the quantization error bound* Δ *(see also the discussion for the linear case in Remark 2), and the conditions* (34d)-(34h) *place further restriction on the structure of the nonlinear plant in order to guarantee asymptotic stability with dynamic quantization.*

VI. Continuum uncertainty set

So far, we have assumed that the set $\mathcal{P}$ is finite. For the case of continuum uncertainty sets, under a certain robustness assumption, we can still achieve asymptotic stability. To utilize notations in the previous sections, denote a continuum uncertainty set by $\Omega \subseteq \mathbb{R}^{n_p}$ and denote by $\mathcal{P}$ a finite index set such that $\bigcup_{i\in\mathcal{P}} \Omega_i = \Omega$ for some Ω_i, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. How to divide Ω into Ω_i and what the number of subsets is are interesting research questions of their own and are not pursued here (see [1]). For every subset Ω_i, pick a nominal value p_i. By this procedure, we obtain a finite family of nominal plants, $\{P(p_1), \ldots, P(p_m)\}$. The difference between the case with a continuum uncertain set Ω and the case with a finite uncertainty set $\mathcal{P}$ in Section II is that we may not have exact matching i.e., $p^* \notin \{p_1, \ldots, p_m\}$.

A. Linear plants

Assumption 3 *There exists an index* $\hat{p} \in \mathcal{P}$ *such that for the plant* $P(p^*)$ *with the observer*

$$\begin{cases} \dot{\hat{x}}_{\hat{p}} = A_{\hat{p}}\hat{x}_{\hat{p}} + B_{\hat{p}}u + L_{\hat{p}}(y_{\hat{p}} - Q_\nu(y)) \\ y_{\hat{p}} = C_{\hat{p}}\hat{x}_{\hat{p}}, \end{cases} \tag{35}$$

we have

$$\begin{aligned} |x(t) - x_{\hat{p}}(t)| \leqslant \bar{c}_e e^{-\lambda_e(t-t_0)}|y(t_0) - y_{\hat{p}}(t_0)| \\ + \gamma_e\|y - Q_\nu(y)\|_{[t_0,t)} \end{aligned} \tag{36}$$

for some $\bar{c}_e, \lambda_e, \gamma_e > 0$ *for all inputs* u.

Basically, the assumption says that there is a robust state estimator in the set $\mathcal{P}$ for the original plant, even if $p^* \notin \mathcal{P}$ (note that for the case $p^* \in \mathcal{P}$, the assumption is exactly the same as (11)). If the system matrices are continuous with respect to p, then the assumption above is true if $|p^* - \hat{p}|$ is small enough (due to robustness and structural stability property of LTI systems), and the set $\mathcal{P}$ is finite if, for example, Ω is compact. If Assumption 3 holds, then all the reasonings and results for linear systems in Section IV hold for a continuum uncertainty set Ω without any modification.

B. Nonlinear plants

For nonlinear plants, Theorem 3 and Theorem 4 hold for continuum uncertainty sets if the following assumption is true.

Assumption 4 *There exists an index $\hat{p} \in \mathcal{P}$ such that for the plant $P(p^*)$ with the observer*

$$\begin{cases} \dot{\hat{x}}_{\hat{p}} = \hat{f}_p(\hat{x}_{\hat{p}}, \mathrm{Q}_\nu(y), u), \\ y_p = h_p(x_{\hat{p}}), \end{cases} \tag{37}$$

we have

$$\begin{aligned} |y(t) - y_{\hat{p}}(t)| \leqslant \beta_e(|y(t_0) - y_{\hat{p}}(t_0)|, t - t_0) \\ + \gamma_e(\|y - \mathrm{Q}_\nu(y)\|_{[t_0,t)}) \end{aligned} \tag{38}$$

for some $\beta_e \in \mathcal{KL}$ and $\gamma_e \in \mathcal{K}$ for all inputs u.

For more general continuum uncertainty sets Ω, Assumptions 3 and 4 above can be relaxed further to include unmodeled dynamics resulting from parameter mismatching. For example, we may want to require the estimator (35) to be robust with respect to small unmodeled dynamics such that $|x(t) - x_{\hat{p}}(t)| \leqslant \bar{c}_e e^{-\lambda_e(t-t_0)}|y(t_0) - y_{\hat{p}}(t_0)| + \gamma_e\|y - \mathrm{Q}_\nu(y)\|_{[t_0,t)} + \Delta_u\|u\|_{[t_0,t)} + \Delta_x\|\hat{x}_p\|_{[t_0,t)}$ where Δ_u, Δ_x are such that $\{\Delta_u, \Delta_x\} \to 0$ as $\hat{p} \to p^*$; see [9]. Another approach is to use the so-called hierarchical hysteresis switching logic as in [7].

VII. Conclusions

In this paper, we treated the problem of stabilizing uncertain systems with quantization. We used the supervisory control framework to deal with plant uncertainty. For a static quantizer, we provided a condition between the quantization range and the quantization error bound to guarantee closed loop stability. With a dynamic quantizer, we provided a zooming strategy on the quantization zooming variable ν and on the parameter ε of the supervisory control scheme to achieve asymptotic stability for the closed loop.

Future research can extend this work in several directions. One direction is to consider other types of limited information, such as sampling, delay, or package loss, or a combination of those with quantization. In this direction, it may be fruitful to combine the approach in this paper with the result in [13]. Yet another direction could be relaxing the matching condition and treating the case of supervisory control of uncertain plants with unmodeled dynamics using dynamic quantization.

References

[1] B. D. O. Anderson, T. S. Brinsmead, F. De Bruyne, J. P. Hespanha, D. Liberzon, and A. S. Morse. Multiple model adaptive control, part 1: Finite controller coverings. *Int. J. Robust Nonlinear Control*, 10:909–929, 2000.

[2] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Automat. Control*, 45:1279–1289, 2000.

[3] N. Elia and S.K. Mitter. Stabilization of linear systems with limited information. *IEEE Trans. Automat. Control*, 46:1384–1400, 2001.

[4] M. Fu and L. Xie. Finite-level quantized feedback control for linear systems. In *Proc. 45th IEEE Conference on Decision and Control*, pages 1117–1122, 2006.

[5] T. Hayakawa, H. Ishii, and K. Tsumura. Adaptive quantized control for nonlinear uncertain systems. In *Proc. American Control Conference*, 2006.

[6] J. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proc. of IEEE Special Issue on Technology of Networked Control Systems*, 95:138–162, 2007.

[7] J. P. Hespanha, D. Liberzon, and A. S. Morse. Hysteresis-based switching algorithms for supervisory control of uncertain systems. *Automatica*, 39(2):263–272, 2003.

[8] J. P. Hespanha, D. Liberzon, and A. S. Morse. Overcoming the limitations of adaptive control by means of logic-based switching. *Systems and Control Lett.*, 49(1):49–65, 2003.

[9] J. P. Hespanha, D. Liberzon, A. S. Morse, B. D. O. Anderson, T. S. Brinsmead, and F. Bruyne. Multiple model adaptive control, part 2: Switching. *Int. J. Robust Nonlinear Control*, 11:479–496, 2001.

[10] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. In *Proc. 38th IEEE Conf. on Decision and Control*, pages 2655–2660, 1999.

[11] M.D. Lemmon and R. Sun. Performance-rate functions for dynamically quantized feedback systems. In *Proc. 45th IEEE Conference on Decision and Control*, pages 5513–5518, 2006.

[12] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39:1543–1554, 2003.

[13] D. Liberzon. On stabilization of linear systems with limited information. *IEEE Trans. Automat. Control*, 48(2):304–307, 2003.

[14] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003.

[15] Q. Ling and M.D. Lemmon. Stability of quantized control systems under dynamic bit assignment. In *Proc. of American Control Conference*, pages 4915–4920, 2004.

[16] J. L. Mancilla-Aguilar and R. A. Garcia. On the existence of common Lyapunov triples for ISS and iISS switched systems. In *Proc. 39th IEEE Conf. on Decision and Control*, pages 3507–3512, 2000.

[17] D. E. Miller. A new approach to model reference adaptive control. *IEEE Trans. Automat. Control*, 48(5):743–757, 2003.

[18] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans. Feedback control under data rate constraints: an overview. *Proc. of the IEEE*, 95(1):108–137, 2007.

[19] D. Nesic and D. Liberzon. A unified approach to controller design for systems with quantization and time scheduling. In *Proc. 46th IEEE Conf. Decision and Control*, pages 3939–3944, 2007.

[20] L. Praly and Y. Wang. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input to state stability. *Mathematics of Control, Signals and Systems*, 9:1–33, 1996.

[21] E. D. Sontag. Input to state stability: Basic concepts and results. In P. Nistri and G. Stefani, editors, *Nonlinear and Optimal Control Theory*, pages 166–220. Springer-Verlag, Berlin, 2007.

[22] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems and Control Lett.*, 24:351–359, 1995.

[23] L. Vu, D. Chatterjee, and D. Liberzon. Input-to-state stability of switched systems and switching adaptive control. *Automatica*, 43(4):639–646, 2007.

[24] W. S. Wong and R.W. Brockett. Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback. *IEEE Trans. Automat. Control*, 44:1049–1053, 1999.

APPENDIX

A. Proof of Theorem 1

Notations

Let $\tilde{y}_p := y_p - \mathrm{Q}_\nu(y)$ be the difference between the estimated output with index p and the actual received data—the quantized output $\mathrm{Q}_\nu(y)$. If $|x(t_0)| \leqslant X_0$, then

$$|y_{\hat{p}}(t_0) - y(t_0)| \leqslant k_c(X_0 + \bar{x}_0) = \bar{y}_0 \qquad (39)$$

in view of $y = C_{p^*}x$, $y_{\hat{p}} = C_{\hat{p}}x_{\hat{p}}$, and $|x_{\hat{p}}(t_0)| \leqslant \bar{x}_0$.

Let $T_{max} := \sup\{t \in [t_0, \infty) : |y(t)| \leqslant \nu M\}$. At time t_0, we have $|y(t_0)| \leqslant k_c X_0 \leqslant \bar{y}_0$. If $\nu M > \chi_{\Delta,\nu}$ where $\chi_{\Delta,\nu}$ is defined as in (61) below, then $\bar{y}_0 < \nu M$ because $\chi_{\Delta,\nu} \geqslant \bar{y}_0$ by virtue of $c_e \geqslant 1$. This implies that $T_{max} > t_0$.

Boundedness of a monitoring signal

From the definition of μ_p in (12), we have

$$\mu_p(t) = \varepsilon + e^{-\lambda(t-t_0)}\hat{\mu}_p(t_0) + \gamma\int_{t_0}^{t} e^{-\lambda(t-s)}|\tilde{y}_p(s)|^2 ds \quad \forall p \in \mathcal{P}. \qquad (40)$$

Note that in (12), we set $\hat{\mu}_p(0) = 0$ at the starting time $t = 0$. The formula (40) above uses arbitrary initial time t_0 and this is important for the analysis of dynamic quantization later. From (2), we have

$$|y(s) - \mathrm{Q}_\nu(y(s))| \leqslant \nu\Delta \quad \forall s \in [t_0, T_{max}). \qquad (41)$$

From (11), (41), and the fact $\tilde{y}_p(s) = y_p(s) - \mathrm{Q}_\nu(y(s)) = y_p(s) - y(s) + y(s) - \mathrm{Q}_\nu(y(s))$, we have

$$|\tilde{y}_{\hat{p}}(s)| \leqslant c_e e^{-\lambda_e(s-t_0)}|y_{\hat{p}}(t_0) - y(t_0)| + (1+\gamma_e)\nu\Delta \quad \forall s \in [t_0, T_{max}). \qquad (42)$$

From (40) and (42), in view of the inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$, we get $\forall t \in [t_0, T_{max})$,

$$\mu_{\hat{p}}(t) \leqslant \varepsilon + e^{-\lambda(t-t_0)}\mu_p(t_0) + 2\gamma c_e^2 f(t-t_0)|y_{\hat{p}}(t_0) - y(t_0)|^2 + (2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2, \qquad (43)$$

where

$$f(t-t_0) := \int_{t_0}^{t} e^{-\lambda(t-s)}e^{-2\lambda_e(s-t_0)}ds = \begin{cases} \dfrac{e^{-2\lambda_e(t-t_0)} - e^{-\lambda(t-t_0)}}{\lambda - 2\lambda_e} & \text{if } \lambda \neq 2\lambda_e \\ e^{-\lambda(t-t_0)}(t-t_0) & \text{if } \lambda = 2\lambda_e. \end{cases} \qquad (44)$$

The function f is positive definite and bounded above by 1. The reason we want to use the function f instead of bounding it by 1 in (43) is because $f(t, t_0) \to 0$ as $t - t_0 \to \infty$ and this captures the decaying contribution of the initial output difference $|y_{\hat{p}}(t_0) - y(t_0)|$.

The switched injected system

An *injected system* is obtained by combining the multi-estimator and a candidate controller. Due to switching among the controllers, what we have is the so-called switched injected system. For a fixed controller u_q, $q \in \mathcal{P}$, from (5) and (10), the injected system is

$$\dot{x}_p = A_p x_p + B_p K_q x_q + L_p(y_p - \mathrm{Q}_\nu(y)), \quad \forall\, p \in \mathcal{P}.$$

We can rewrite the above equation as

$$\dot{x}_p = (A_p + L_p C_p)x_p + L_{p,q}x_q + L_p(y_q - \mathrm{Q}_\nu(y)),$$

where $L_{p,q} := B_p K_q - L_p C_q$. The injected system with a controller with index q can be written explicitly as

$$\dot{x}_q = (A_q + B_q K_q)x_q + L_q(y_q - \mathrm{Q}_\nu(y)), \qquad (45a)$$

$$\dot{x}_p = (A_p + L_p C_p)x_p + L_{p,q}x_q + L_p(y_q - \mathrm{Q}_\nu(y)), \quad p \neq q. \qquad (45b)$$

The foregoing dynamics take the form

$$\dot{x_{\mathbb{E}}} = \mathbf{A}_q x_{\mathbb{E}} + \mathbf{B}_q(x_q - \mathrm{Q}_\nu(x)), \qquad (46)$$

where the definitions of $\mathbf{A}_q$ and $\mathbf{B}_q$ are obvious. It is clear that if $x_q - \mathrm{Q}_\nu(x) = 0$ then $x_q \to 0$ by (45a) and then all $x_p \to 0$ by (45b), which means that $\mathbf{A}_q$ is Hurwitz (since the system is linear). Explicit formulae for $\mathbf{A}_p$ and $\mathbf{B}_p$ are

$$\mathbf{A}_p = \otimes_i(A_i + L_i C_i) + \begin{bmatrix} 0_{(j-1)} & L_{p_1,p} & 0_{n-j-1} \\ & \vdots & \\ 0_{(j-1)} & L_{p_m,p} & 0_{n-j-1} \end{bmatrix}, \qquad (47a)$$

$$\mathbf{B}_q = \begin{bmatrix} L_{p_1} \\ \vdots \\ L_{p_m} \end{bmatrix}, \qquad (47b)$$

where $\otimes$ is the Kronecker product, $p_i \equiv p$, and $0_{(j)}$ is the $n_y \times n_y(j-1)$ zero matrix.

Since $\mathbf{A}_p$ are Hurwitz for all p, there exists a family of quadratic Lyapunov functions $V_p(x_{\mathbb{E}}) = x_{\mathbb{E}}^T P_p x_{\mathbb{E}}, P_p^T = P_p > 0$ such that

$$\underline{a}|x_{\mathbb{E}}|^2 \leqslant V_p(x_{\mathbb{E}}) \leqslant \overline{a}|x_{\mathbb{E}}|^2 \tag{48a}$$

$$\frac{\partial V_p(x_{\mathbb{E}})}{\partial x}(\mathbf{A}_p x_{\mathbb{E}} + \mathbf{B}_p \tilde{y}_p) \leqslant -\lambda_0 V_p(x_{\mathbb{E}}) + \gamma|\tilde{x}_p|^2 \tag{48b}$$

for some constants $\underline{a}, \overline{a}, \lambda_0, \gamma_0 > 0$ (the existence of such common constants for the family of Lyapunov functions is guaranteed since $\mathcal{P}$ is finite). There exists a number $\mu_V \geqslant 1$ such that

$$V_q(x) \leqslant \mu_V V_p(x) \quad \forall x \in \mathbb{R}^n, \forall p, q \in \mathcal{P}. \tag{49}$$

We can always pick $\mu_V = \overline{a}/\underline{a}$ but there may be other smaller μ_V satisfying (49) (for example, $\mu_V = 1$ if V_p are the same for all p even though $\overline{a}/\underline{a} > 1$).

1) The switching signal σ: The hysteresis switching lemma (see, e.g., [7, Lemma 1] with the scaled signals $\bar{\mu}_p(t) = e^{\lambda t}\mu_p(t)$, which are nondecreasing) give

$$N_\sigma(t, t_0) \leqslant N_0 + \frac{t - t_0}{\tau_a}, \tag{50}$$

where $N_\sigma(t, t_0)$ is the number of switches in (t_0, t),

$$N_0 := 1 + m + \frac{m}{\ln(1+h)} \ln(\mu_q(t)/\varepsilon) \tag{51a}$$

$$\tau_a := \ln(1+h)/(m\lambda). \tag{51b}$$

From (39), (43), and (51a), we obtain

$$N_0 \leqslant 1 + m + \frac{m}{\ln(1+h)} \times \ln((\varepsilon + \bar{\mu}_0 + 2\gamma c_e^2 \bar{y}_0^2 + 2(\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2)/\varepsilon).$$

Since N_0 is bounded, the switching signal σ on the interval (t_0, T_{max}) is an average dwell-time switching signal with the average dwell-time τ_a .

2) The exponentially weighted integral norm of $\tilde{y}_\sigma$: Also, from the hysteresis switching lemma, we have

$$\sum_{k=0}^{N_\sigma(t,t_0)} \bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k) \leqslant m((1+h)\bar{\mu}_\ell(t) - \min_{p \in \mathcal{P}} \bar{\mu}_p(t_0)), \tag{52}$$

where t_k are the switching times in (t_0, t). From (40), we get $\bar{\mu}_p(t) = \varepsilon e^{\lambda t} + e^{\lambda t_0}\mu_p(t_0) + \int_{t_0}^t e^{\lambda s}\gamma|\tilde{y}_p(s)|^2 ds$.

We then have

$$\begin{aligned}
&\sum_{k=0}^{N_\sigma(t,t_0)} \bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k) \\
&= \sum_{k=0}^{N_\sigma(t,t_0)} \int_{t_0}^{t_{k+1}} e^{\lambda s}\gamma|\tilde{y}_{\sigma(t_k)}(s)|^2 ds - \int_{t_0}^{t_k} e^{\lambda s}\gamma|\tilde{y}_{\sigma(t_k)}(s)|^2 ds \\
&\quad + e^{\lambda t_{k+1}}\mu_{\sigma(t_k)}(t_0) - e^{\lambda t_k}\mu_{\sigma(t_k)}(t_0) \\
&\geqslant \sum_{k=0}^{N_\sigma(t,t_0)} \int_{t_0}^{t_{k+1}} e^{\lambda s}\gamma|\tilde{y}_{\sigma(t_k)}(s)|^2 ds - \int_{t_0}^{t_k} e^{\lambda s}\gamma|\tilde{y}_{\sigma(t_k)}(s)|^2 ds \\
&= \sum_{k=0}^{N_\sigma(t,t_0)} \int_{t_k}^{t_{k+1}} e^{\lambda s}\gamma|\tilde{y}_{\sigma(t_k)}(s)|^2 ds = \gamma \int_{t_0}^{t} e^{\lambda s}|\tilde{y}_\sigma(s)|^2 ds.
\end{aligned}$$

Dividing both sides of the foregoing inequality by $\gamma e^{\lambda t}$ and then combining with (52), we obtain the following inequality for the exponentially weighted integral norm of $\tilde{y}_\sigma$:

$$\int_{t_0}^{t} e^{-\lambda(t-\tau)}|\tilde{y}_\sigma(\tau)|^2 d\tau \leqslant \frac{m(1+h)}{\gamma}\mu_q(t) \quad \forall q \in \mathcal{P}. \tag{53}$$

Since the subsystems of the switched injected system are stable, we have [7, Corollary 4]

$$\begin{aligned}
|x_{\mathbb{E}}(t)|^2 &\leqslant (\overline{a}/\underline{a})\mu_V^{1+N_0}|x_{\mathbb{E}}(t_0)|^2 e^{-\lambda(t-t_0)} \\
&\quad + \frac{1}{\underline{a}}\mu_V^{1+N_0}\gamma \int_{t_0}^{t} e^{-\lambda(t-\tau)}|\tilde{y}_\sigma(\tau)|^2 d\tau \ \forall t \in [t_0, T_{max})
\end{aligned} \tag{54}$$

for all $q \in \mathcal{P}$ if

$$0 < \lambda < \lambda_0 \tag{55}$$

and

$$\frac{\ln(1+h)}{\lambda m} > \frac{\ln \mu_V}{\lambda_0 - \lambda}. \tag{56}$$

Suppose that h is chosen such that (56) holds so we have (54). From (43) and (54), we get

$$\begin{aligned}
|x_{\mathbb{E}}(t)|^2 \leqslant & c_1|x_{\mathbb{E}}(t_0)|^2 e^{-\lambda(t-t_0)} + c_2\varepsilon + c_2 e^{-\lambda(t-t_0)}\mu_p(t_0) \\
& + 2c_2\gamma c_e^2 f(t-t_0)|y_{\hat{p}}(t_0) - y(t_0)|^2 \qquad (57) \\
& + (2c_2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2 \quad \forall t \in [t_0, T_{max}), \qquad (58)
\end{aligned}$$

where $c_1 := \mu_V^{1+N_0}\overline{a}/\underline{a}$ and $c_2 := \mu_V^{1+N_0} m(1+h)/\underline{a}$.

A condition on M and Δ

From (57), we have

$$|x_{\mathbb{E}}(t)|^2 \leqslant c_1\bar{x}_0^2 + c_2\varepsilon + c_2\bar{\mu}_0 + 2c_2\gamma c_e^2\bar{y}_0^2 + (2c_2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2 =: \bar{x}^2 \quad \forall t \in [t_0, T_{max}). \tag{59}$$

Because $y = y_{\hat{p}} + y - y_{\hat{p}}$, we have

$$|y(t)| \leqslant \|C_{p^*}\||x_{\mathbb{E}}(t)| + |y_{\hat{p}} - y| \leqslant k_c\bar{x} + c_e\bar{y}_0 + \gamma_e\nu\Delta \tag{60}$$

in view of $|x_{\hat{p}} - x| \leqslant c_e\bar{y}_0 + \gamma_e\nu\Delta$ from (11). Let

$$\begin{aligned}\chi_{\Delta,\nu} &:= k_c\bar{x} + c_e\bar{y}_0 + \gamma_e\nu\Delta \\ &= k_c\big(c_1\bar{x}_0^2 + c_2\varepsilon + c_2\bar{\mu}_0 + 2c_2\gamma c_e^2\bar{y}_0^2 \\ &\quad + (2c_2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2\big)^{1/2}\big) \\ &\quad + c_e\bar{y}_0 + \gamma_e\nu\Delta.\end{aligned} \tag{61}$$

Note that c_1, c_2 actually depend on Δ and ν as follows:

$$\begin{aligned}c_1 &= (\bar{a}/\underline{a})\mu_V^{1+m} \times \\ &\left(1+\frac{\bar{\mu}_0}{\varepsilon}+2\gamma c_e^2\frac{\bar{y}_0^2}{\varepsilon}+2\frac{\gamma(1+\gamma_e)^2}{\lambda}\Delta^2\frac{\nu^2}{\varepsilon}\right)^{\frac{m\ln\mu_V}{\ln(1+h)}} \\ &=: c_1(\Delta,\nu,\varepsilon)\end{aligned} \tag{62}$$

and $c_2 = c_1 m(1+h)/(\gamma\bar{a})$. Because $\nu M > \chi_{\Delta,\nu}$ by the hypothesis of the theorem, from (60) and (61), we have $|y(t)| < \nu M \quad \forall t \in [t_0, T_{max})$. From the definition of T_{max}, we must have $T_{max} = \infty$.

If $\{\bar{\mu}_0, \bar{y}_0, \bar{x}_0\} \to 0$, then c_2 converges to a constant of the form $b_0^2(1+b_1\nu^2\Delta^2/\varepsilon)^\kappa$ for some constants b_0, b_1. In view of the inequality $\sqrt{a^2+b^2} < (a+b)$ for $a, b > 0$, we have that as $\{\bar{\mu}_0, \bar{y}_0, \bar{x}_0\} \to 0$, $\chi_{\Delta,M}$ converges to $k_c b_0(1+b_1\nu^2\Delta^2/\varepsilon)^{\kappa/2}(\sqrt{\varepsilon}+a_1\nu\Delta)+a_2\nu\Delta$ for some positive constants a_1, a_2. The function γ in the theorem is $\gamma(z) = b_0(1+b_1z^2)^{\kappa/2}$.

Ultimate boundedness of the plant state

We have $|x_p - x| \leqslant |x_p| + |x| \leqslant |x_{\mathbb{E}}| + |x|$ for all p, so if $|x(t_0)| \leqslant X_0$, then $|y_{\hat{p}}(t_0) - y(t_0)| \leqslant k_c(X_0 + \bar{x}_0) = \bar{y}_0$. Let

$$c_2\varepsilon + (2c_2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2 =: \underline{x}^2. \tag{63}$$

The inequality (57) tells us that whenever $|x_{\mathbb{E}}(t)|^2 > \underline{x}^2$, $x_{\mathbb{E}}$ will decrease to $\underline{x}$ asymptotically (recall that $f(t-t_0) \to 0$ as $t - t_0 \to \infty$). Let $\epsilon_0 > 0$. From (57), we have that

$$|x_{\mathbb{E}}(t)|^2 \leqslant \underline{x}^2 + \epsilon_0 \quad \forall t \geqslant t_0 + T_1, \tag{64}$$

where T_1 is such that

$$c_1\bar{x}_0^2 e^{-\lambda T_1} + c_2 e^{-\lambda T_1}\bar{\mu}_0 + 2c_2\gamma c_e^2 f(T_1)\bar{y}_0^2 \leqslant \epsilon_0.$$

There also exists T_2 such that

$$c_e\bar{y}_0 e^{-\lambda_e T_2} \leqslant \epsilon_0.$$

Let $T := \max\{T_1, T_2\}$. The inequality (64) implies that $|x_{\hat{p}}(t)| \leqslant (\underline{x}^2 + \epsilon_0)^{1/2}$ for all $t \geqslant t_0 + T$. Since $|x_{\hat{p}}(t) - x(t)| \leqslant c_e\bar{y}_0 e^{-\lambda_e(t-t_0)} + \gamma_e\nu\Delta$, it follows that $|x(t)| \leqslant \epsilon_0 + \gamma_e\nu\Delta + (\underline{x}^2+\epsilon_0)^{1/2}$ for all $t \geqslant t_0 + T$; note that for every $\epsilon_x > 0$, there always exists $\epsilon_0 > 0$ such that $\epsilon_0 + (\underline{x}^2+\epsilon_0)^{1/2} = \underline{x} + \epsilon_x$. Since x is bounded and x_p are bounded for all p, it follows that μ_p are also bounded for all p. The ultimate bound on x in the theorem is

$$\psi_{\Delta,\nu} := \gamma_e\nu\Delta + (c_2\varepsilon + (2c_2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2)^{1/2}. \tag{65}$$

As $\{\bar{\mu}_0, \bar{y}_0, \bar{x}_0\} \to 0$, $c_2 \to a_0^2(1+a_1\nu^2\Delta^2/\varepsilon)^\kappa$. Hence, $\psi_{\Delta,\nu}$ converges to $a_3a_0(1+a_1\nu^2\Delta^2/\varepsilon)^{\kappa/2}(\sqrt{\varepsilon}+a_4\nu\Delta) + a_5\nu\Delta$ for some constants a_3, a_4, a_5 as $\{\bar{\mu}_0, \bar{y}_0, \bar{x}_0\} \to 0$.

B. Proof of Theorem 2

Let

$$\psi_{\Delta,\nu} := \underline{x}^2 = c_2\varepsilon + (2c_2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2, \tag{66a}$$

$$\underline{\mu} := \varepsilon + (2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2, \tag{66b}$$

$$\underline{y} := (1+\gamma_e)\nu\Delta, \tag{66c}$$

where a_3, a_4, and a_5 in the theorem are $a_3 := 1$, $a_4 = (2\gamma(1+\gamma_e)^2/\lambda)$, and $a_5 := (1+\gamma_e)$. Let $\epsilon_0 > 0$ be a number such that

$$\rho := \max\left\{\frac{(\underline{x}(1+\epsilon_0))^2}{\bar{x}_0^2}, \frac{\underline{\mu}(1+\epsilon_0)}{\bar{\mu}_0}, \frac{\underline{y}(1+\epsilon_0)}{\bar{y}_0}\right\} < 1. \tag{67}$$

Such ϵ_0 always exists since $\bar{x}_0 > \chi_x(\nu,\Delta)$, $\bar{\mu}_0 > \underline{\mu}$, and $\bar{y}_0 > \underline{y}$. Let T_1, T_2, T_3 be such that

$$c_1\bar{x}_0^2 e^{-\lambda T_1} + c_2 e^{-\lambda T_1}\bar{\mu}_0 + 2c_2\gamma c_e^2 f(T_1)\bar{y}_0^2 = (\epsilon_0^2 + 2\epsilon_0)\underline{x}^2 \tag{68a}$$

$$c_e\bar{y}_0 e^{-\lambda_e T_2} = \epsilon_0\underline{y} \tag{68b}$$

$$e^{-\lambda T_3}\bar{\mu}_0 + 2\gamma c_e^2 f(T_3)\bar{y}_0^2 + (2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2 = \epsilon_0\underline{\mu}. \tag{68c}$$

Let $T := \max\{T_1, T_2, T_3\}$. Then from (57) and (68a), we have

$$|x_{\mathbb{E}}(t)| \leqslant (1+\epsilon_0)\underline{x} \leqslant \rho\bar{x}_0 \quad \forall t \geqslant t_0 + T,$$

where the last inequality follows from (67). From (43), (68a), and (67), we have $\mu_{\hat{p}}(t) \leqslant (1+\epsilon_0)\underline{\mu} \leqslant \rho^2\bar{\mu}_0$. Similarly, $|y_{\hat{p}}(t) - y(t)| \leqslant (1+\epsilon_0)\underline{y} \leqslant \rho\bar{y}_0 \quad \forall t \geqslant t_0 + T$.

Consider the logarithmic ε with factor ρ^2 and period T and the logarithmic ν with factor ρ and period T. By the construction of ρ and T, after T time, all the bounds on $x_{\mathbb{E}}$, $\mu_{\hat{p}}$, and $|y_{\hat{p}}(t) - y(t)|$ are reduced by a factor of ρ. From (62), because $\bar{y}_0$, ν are reduced by a factor of ρ, and $\bar{\mu}_0$ and ε are reduced by a factor of ρ^2, the formula for c_1 is unchanged. Thus, c_1 is unchanged, and also, c_2 is unchanged. Then $\chi_{\Delta,\nu}$ in (61) is reduced by a factor of ρ. Therefore, the inequality $\nu M > \chi_{\Delta,\nu}$ still holds for all $t \geqslant T + t_0$ even after we change ν and ε. Similarly, the inequalities $\bar{x}_0 > \underline{x}$, $\bar{\mu}_0 > \underline{\mu}$, and $\bar{y}_0 > \underline{y}$ remain true for all $t \geqslant T$. Because c_1, c_2 are unchanged, from (85), we have that T_1, T_2, T_3 do not change if $\bar{y}_0$, $\bar{x}_0$, ν, $\underline{x}$, and $\underline{y}$ are reduced by a factor ρ and $\bar{\mu}$, $\underline{\mu}$, and ε are reduced by a factor ρ^2. This implies that at time $2T$, we can repeat reducing ν and ε in the same manner, and so on. It follows that

$$|x_{\mathbb{E}}(kT)| \leqslant \rho^k |x_{\mathbb{E}}(0)| \quad k = 1, 2, \ldots.$$

As $k \to \infty$, we have $|x_{\mathbb{E}}| \to 0$, which implies that $|x_{\hat{p}}(t)| \to 0$ as $t \to \infty$. Also, $|x_{\hat{p}}(kT) - x(kT)| \leqslant \rho^k \bar{y}_0$ and so $|x_{\hat{p}}(kT) - x(kT)| \to 0$ as $k \to \infty$. Because $x = x_{\hat{p}} - (x_{\hat{p}} - x)$, we will have $|x(t)| \to 0$ as $t \to \infty$.

C. Proof of Theorem 3

Let $T_{max} := \sup\{t \in [t_0, \infty) : |y(t)| < \nu M\}$. We have $|y(t_0)| \leqslant \max_{p\in\mathcal{P}} \sup_{|z|\leqslant X_0} h_p(z) < \bar{y}_0$. We also have $\bar{y}_0 < \chi_{M,\Delta}$ in view of the definition of $\chi_{M,\Delta}$ as in (79). By the hypothesis of the theorem, we have $|y(t_0)| < \nu M$, and hence, $T_{max} > t_0$.

If $|x(t_0)| \leqslant X_0$, then

$$|y_{\hat{p}}(t_0) - y(t_0)| \leqslant k_c(\bar{x}_0) + k_c(X_0) = \bar{y}_0.$$

Boundedness of a monitoring signal

From the definition of μ_p in (29), we have

$$\begin{aligned}\mu_p(t) &= \varepsilon + e^{-\lambda(t-t_0)}\mu_p(t_0)\\ &\quad + \int_{t_0}^{t} e^{-\lambda(t-s)}\gamma(|y_p(s) - \mathrm{Q}_\nu(y(s))|)ds \quad \forall p \in \mathcal{P}.\end{aligned} \tag{69}$$

From $|y(s) - \mathrm{Q}_\nu(y(s))| \leqslant \nu\Delta$ $\forall s \in [t_0, T_{max})$, (11), (41), and the fact $\tilde{y}_p(s) = y_p(s) - \mathrm{Q}_\nu(y(s)) = y_p(s) - y(s) + y(s) - \mathrm{Q}_\nu(y(s))$, we get

$$\begin{aligned}|\tilde{y}_{\hat{p}}(s)| &\leqslant \beta_e(|y_{\hat{p}}(t_0) - y(t_0)|, s - t_0) + \bar{\gamma}_e(\nu\Delta)\\ &\qquad \forall s \in [t_0, T_{max}),\end{aligned} \tag{70}$$

where $\bar{\gamma}_e$ is the class $\mathcal{K}_\infty$ function defined as $\bar{\gamma}_e(s) := s + \gamma_e(s)$. For a class $\mathcal{K}_\infty$ function γ, we have $\gamma(a+b) \leqslant \gamma(2a) + \gamma(2b)$ for all $a, b > 0$. Therefore, from (70) and (70), we obtain

$$\begin{aligned}\mu_{\hat{p}}(t) &\leqslant \varepsilon + e^{-\lambda(t-t_0)}\mu_p(t_0) + \phi(t, t_0, |y_{\hat{p}}(t_0) - y(t_0)|)\\ &\quad + \gamma(2\bar{\gamma}_e(\nu\Delta))/\lambda,\end{aligned} \tag{71}$$

where ϕ is as in (31).

The switching signal

The properties of the switching signal σ is the same as in the the proof in Appendix A. From (51a) and (71), we get

$$\begin{aligned}N_0 &\leqslant 1 + m + \frac{m}{\ln(1+h)} \times\\ &\quad \ln((\varepsilon + \bar{\mu}_0 + \bar{\phi}(\bar{y}_0) + \gamma(2\bar{\gamma}_e(\nu\Delta))/\lambda),\end{aligned}$$

where $\bar{\phi}(\bar{y}_0) := \sup_{t\geqslant t_0} \phi(t, t_0, \bar{y}_0)$. Since N_0 is bounded, σ is an average dwell-time switching signal on the interval $[t_0, T_{max})$.

In view of (53), we also have

$$\int_{t_0}^{t} e^{-\lambda(t-\tau)}\gamma(|\tilde{y}_\sigma(\tau)|)d\tau \leqslant m(1+h)\mu_q(t) \quad \forall q \in \mathcal{P}. \tag{72}$$

The switched injected system

Under Assumption 2, every subsystem of the switched injected system is ISS with respect to $y_p - \mathrm{Q}_\nu(y) = \tilde{y}_p$. It has been proved [23, Lemma 4.2] that under average dwell-time switching, the switched injected system has an exponentially-weighted ISS property with respect to $\tilde{y}_p$:

$$\begin{aligned}\overline{\alpha}_1(|x_{\mathbb{CE}}(t)|) &\leqslant c_1\overline{\alpha}_2(|x_{\mathbb{CE}}(t_0)|)e^{-\lambda(t-t_0)}\\ &\quad + c_1\int_{t_0}^{t} e^{-\lambda(t-t_0)}\gamma(|\tilde{y}_p(s)|)ds \quad \forall t \in [t_0, T_{max})\end{aligned} \tag{73}$$

for some $\overline{\alpha}_1, \overline{\alpha}_2, \gamma_c \in \mathcal{K}_\infty$, and $c_1 = \mu_V^{1+N_0}$ if

$$\frac{\ln(1+h)}{\lambda m} > \frac{\ln \mu_V}{\lambda_0 - \lambda}. \tag{74}$$

From (73) and (72), we get

$$\overline{\alpha}_1(|x_{\mathbb{CE}}(t)|) \leqslant c_1\overline{\alpha}_2(|x_{\mathbb{CE}}(t_0)|)e^{-\lambda(t-t_0)} + c_2\mu_q(t) \tag{75}$$

for all $t \in [t_0, T_{max})$, where $c_2 := c_1 m(1+h)$. From (71) and (75), we have

$$\begin{aligned}\overline{\alpha}_1(|x_{\mathbb{CE}}(t)|) &\leqslant \overline{\alpha}_2(|x_{\mathbb{CE}}(t_0)|)e^{-\lambda(t-t_0)} + c_2\varepsilon\\ &\quad + c_2 e^{-\lambda(t-t_0)}\mu_{\hat{p}}(t_0) + c_2\phi(t, t_0, \bar{y}_0)\\ &\quad + c_2\gamma(2\bar{\gamma}_e(\nu\Delta))/\lambda \quad \forall t \in [t_0, T_{max}).\end{aligned} \tag{76}$$

A condition on M and Δ

Let

$$\bar{x} := \overline{\alpha}_1^{-1}\big(\overline{\alpha}_2(\bar{x}_0) + c_2\varepsilon + c_2\bar{\mu}_0 + c_2\bar{\phi}(\bar{y}_0) + c_2\gamma(2\bar{\gamma}_e(\nu\Delta))/\lambda\big). \quad (77)$$

Note that c_1, c_2 actually depend on Δ, M, ν as follows:

$$c_1 = \mu_V^{1+m}\big(1 + \frac{\bar{\mu}_0}{\varepsilon} + \bar{\phi}(\bar{y}_0)/\varepsilon + \gamma(2\bar{\gamma}_e(\nu\Delta))/(\lambda\varepsilon)\big)^{\frac{m\ln\mu_V}{\ln(1+h)}} =: c_1(\Delta,\nu,\varepsilon) \quad (78)$$

and $c_2 = (m(1+h)c_1$. We have $|x_{\mathbb{CE}}(t)| \leqslant \bar{x}$ for all $t \in [t_0, T_{max})$. In view of $y = y_{p^\star} + (y - y_{p^\star})$ and (28), we have

$$|y(t)| \leqslant k_c(\bar{x}) + \beta_e(\bar{y}_0, 0) + \gamma_e(\nu\Delta).$$

Define

$$\chi_{\Delta,\nu} := k_c(\bar{x}) + \beta_e(\bar{y}_0, 0) + \gamma_e(\nu\Delta). \quad (79)$$

By the assumption (32) of the theorem, $\chi_{\Delta,\nu} < \nu M$. From the definition of T_{max}, it follows that we have $T_{max} = \infty$.

If $\{\bar{\mu}_0, \bar{y}_0, \bar{x}_0\} \to 0$, c_1 and c_2 converge to a constant of the form $a_0(1+\gamma_0(\nu\Delta)/\varepsilon)^\kappa$ for some positive constant a_0 and $\gamma_0 \in \mathcal{K}_\infty$. From the formulae of $\chi_{M,\Delta}$, c_1, and c_2, it follows that

$$\chi_{M,\Delta} \to \gamma_1(\bar{\gamma}_0(\gamma_0(\nu\Delta)/\varepsilon))(\varepsilon + \gamma_2(\nu\Delta))) + \gamma_3(\nu\Delta) \quad (80)$$

as $\{\bar{\mu}_0, \bar{y}_0, \bar{x}_0\} \to 0$ for some $\gamma_1 \in \mathcal{K}$ and $\gamma_2, \gamma_3 \in \mathcal{K}_\infty$, and $\bar{\gamma}_0(z) := a_0(1+z)^\kappa$.

Ultimate boundedness

Let

$$\underline{x} := \overline{\alpha}_1^{-1}(c_2\varepsilon + c_2\gamma(2\bar{\gamma}_e(\nu\Delta))/\lambda). \quad (81)$$

Let $\epsilon_0 > 0$ and let T_1 be such that

$$\overline{\alpha}_2(\bar{x}_0)e^{-\lambda T_1} + c_2\varepsilon + c_2e^{-\lambda T_1}\bar{\mu}_0 + c_2\phi(t_0+T_1, t_0, \bar{y}_0) + \gamma(2\gamma_e(\nu\Delta))/\lambda \leqslant \overline{\alpha}_1(\underline{x} + \epsilon_0). \quad (82)$$

The lefthand side of the foregoing equation is strictly decreasing in T_1 to the value of the righthand side. Therefore, there is a unique such $0 < T_1 < \infty$ (existence follows from the fact that the lefthand side is greater than the righthand side at $T_1 = 0$). There also exists T_2, $0 < T_2 < \infty$ such that $\beta_e(\bar{y}_0, T_2) \leqslant \epsilon_0$. Let $T_3 := \max\{T_1, T_2\}$. From (76) and (82), $|x_{\hat{p}}(t)| \leqslant \underline{x}+\epsilon_0$ for all $t \geqslant t_0 + T$. Then $|y_{\hat{p}}(t)| \leqslant k_c(\underline{x} + \epsilon_0)$. Because $|y_{\hat{p}}(t) - y(t)| \leqslant \beta_e(\bar{y}_0, t - t_0) + \gamma_e(\nu\Delta)$ in view of (28), it follows that $|y(t)| \leqslant \epsilon_0 + \gamma_e(\nu\Delta) + k_c(\underline{x} + \epsilon_0) =: \underline{y}(\epsilon_0)$ for all $t \geqslant t_0 + T$. We also have $|u_p(t)| \leqslant \max_{p\in\mathcal{P}} \sup_{|z_1|\leqslant\underline{x}+\epsilon_0, |z_2|\leqslant\underline{y}} r_p(z_1, z_2) =: \underline{u}(\epsilon_0)$ $\forall t \geqslant t_0 + T$. From (30), we get $|x(t)| \leqslant \beta(|x(T)|, t-t_0-T) + \gamma_u(\underline{u}(\epsilon_0)) + \gamma_y(\underline{y}(\epsilon_0))$ $\forall t \geqslant t_0+T_2$. Let T_3, $0 < T_3 < \infty$, be such that $\beta(\underline{x}+\epsilon_0, T_3) = \epsilon_0$ and $T := \max\{T_1, T_2, T_3\}$. Then $|x(t)| \leqslant \epsilon_0 + \gamma_u(\underline{u}(\epsilon_0)) + \gamma_y(\underline{y}(\epsilon_0))$ $\forall t \geqslant t_0 + T$. Define

$$\underline{\psi}_{\Delta,\nu} := \gamma_u(\underline{u}(0)) + \gamma_y(\underline{y}(0)). \quad (83)$$

For every $\epsilon_x > 0$, there exists $\epsilon_0 > 0$ such that $\epsilon_0 + \gamma_u(\underline{u}(\epsilon_0)) + \gamma_y(\underline{y}(\epsilon_0)) = \underline{\psi}_{\Delta,\nu} + \epsilon_x$. Therefore, for every $\epsilon_x > 0$, there exists $0 < T < \infty$ such that $|x(t)| \leqslant \underline{\psi}_{\Delta,\nu} + \epsilon_x$ $\forall t \geqslant t_0 + T$. In view of (80), we have that $k_c(\underline{x}) \to \gamma_1(\bar{\gamma}_0(\gamma_0(\nu\Delta)/\varepsilon))(\varepsilon + \gamma_2(\nu\Delta)))$ as $\{\bar{\mu}_0, \bar{y}_0, \bar{x}_0\} \to 0$, where $\gamma_0, \gamma_1, \gamma_2$ are as in (80). It follows that $\underline{\psi}_{\Delta,\nu}$ can always be bounded by a function $\psi^x_{\Delta,\nu}$ of the form $\bar{\gamma}_x(\gamma_1(\bar{\gamma}_0(\gamma_0(\nu\Delta)/\varepsilon)(\varepsilon + \gamma_2(\nu\Delta))) + \gamma_e(\nu\Delta))$ for some $\bar{\gamma}_x \in \mathcal{K}_\infty$.

D. Proof of Theorem 4

Let

$$\underline{\mu} := \varepsilon + \gamma(2\bar{\gamma}_e(\nu\Delta))/\lambda, \quad (84a)$$

$$\underline{y} := \bar{\gamma}_e(\nu\Delta). \quad (84b)$$

The function γ_4 and γ_5 in the theorem are $\gamma_4(z) := \gamma(2\bar{\gamma}_e(z))/\lambda$ and $\gamma_5 = \bar{\gamma}_e$. The function $\psi_{\Delta,\nu,\varepsilon}$ in the theorem is $\underline{x}$ as in (81). Let T_1, T_2, T_3 be such that

$$c_1\overline{\alpha}_2(\bar{x}_0)e^{-\lambda T_1} + c_2e^{-\lambda T_1}\bar{\mu}_0 + 2c_2\phi(t_0+T_1, t_0, \bar{y}_0) \leqslant \rho\overline{\alpha}_1(\bar{x}_0) - \overline{\alpha}_1(\underline{x}) \quad (85a)$$

$$\phi(t_0+T_1, t_0, \rho\bar{y}_0) \leqslant \rho\phi(t_0+T_1, t_0, \bar{y}_0) \quad (85b)$$

$$\beta_e(\bar{y}_0, T_2) \leqslant \rho\bar{y}_0 - \underline{y} \quad (85c)$$

$$e^{-\lambda T_3}\bar{\mu}_0 + \phi(t_0+T_3, t_0, \bar{y}_0) \leqslant \rho\bar{\mu}_0 - \underline{\mu}. \quad (85d)$$

The existence of such $0 < T_1 < \infty$ follows from the hypothesis (34a), and $0 < T_3 < \infty$ follows from the hypothesis (34c).

Let $T := \max\{T_1, T_2, T_3\}$. Then we have

$$\overline{\alpha}_1(|x_{\mathbb{CE}}(t)|) \leqslant \rho\overline{\alpha}_1(\bar{x}_0), \quad (86)$$

$$\mu_{\hat{p}}(t) \leqslant \rho\bar{\mu}_0, \quad (87)$$

$$|y_{\hat{p}}(t) - y(t)| \leqslant \rho\bar{y}_0 \quad \forall t \geqslant t_0 + T. \quad (88)$$

Suppose that at time T, we reduce ν by factor k_ν and ε by factor k_ε. From (78), because of (34h), we have that c_1 does not increase at time T_1 i.e. $c_1(T_1) \leqslant c_1(t_0)$. Then also, $c_2(T_1) \leqslant c_2(t_0)$. From (85c),

$$|y_{\hat{p}}(T) - y(T)| \leqslant \rho\bar{y}_0, \quad (89)$$

in view of $T \geqslant T_2$. From (34d) and (34f),

$$\overline{\alpha}_1(|x_{\mathbb{CE}}(T)|) \leqslant \rho\overline{\alpha}_1(|x_{\mathbb{CE}}(t_0)|). \tag{90}$$

From (34g), (34d), (34e), (85b), (31b), and (89), the lefthand side of (85a) is reduced by a factor of at least ρ at time T. From (90), the righthand side of (85a) is reduced by a factor of at most ρ when we replace $\bar{x}_0$ by $\overline{\alpha}_1^{-1}(\rho\overline{\alpha}_1(\bar{x}_0))$. Thus, (85a) holds true at time T. The inequality (85b) also holds true at time T because of (31b). The inequality (85c) holds true when we replace $\bar{y}_0$ by $\rho\bar{y}_0$ because of (34e) and (89). The inequality (85d) holds true when we replace $\bar{\mu}_0$ by $\rho\bar{\mu}_0$ because of (34d) and (85b). Thus, at time T, all the sub-equations in (85) hold true when we replace the old bounds $\bar{x}_0, \bar{\mu}_0$, and $\bar{y}_0$ by the new bounds which are of factor ρ of the old bounds. It follows that we can repeat the procedure at time $2T$ by reducing ν by factor k_ν and ε by factor k_ε, and so on. We then have

$$|x_{\mathbb{CE}}(t_0 + kT)| \leqslant \rho^k \bar{x}_0 \; k = 1, 2, \ldots.$$

This implies that $|x_{\mathbb{CE}}(t)| \to 0$ as $t \to \infty$. Hence, $|x_p(t)| \to 0$ as $t \to \infty$ for all p, and so $|y_p(t)| \to 0$ as $t \to \infty$ for all p. Also, $|y_{\hat{p}}(kT) - y(kT)| \leqslant \rho^k \bar{y}_0$, and so $|y_{\hat{p}}(t) - y(t)| \to 0$ as $t \to \infty$. We then have $|y(t)| \to 0$ as $t \to \infty$. We also have $|u_p| \to 0$ as $t \to \infty$ for all p because u_p is a function of $x_{\mathbb{C}}$ and y and the control signal is zero when $x_{\mathbb{C}} = 0, y = 0$. From the IOSS property (30), we have that $|x(t)| \to 0$ as $t \to \infty$. From bounded of $x_{\mathbb{CE}}$, it is clear that $x_{\mathbb{E}}$ and $x_{\mathbb{C}}$ are bounded. From boundedness of $y_p - \mathrm{Q}(y)$ (which follows from boundedness of y_p and y), μ_p are bounded for all $p \in \mathcal{P}$.