On Stability of Randomly Switched Nonlinear Systems

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Abstract—**This note is concerned with stability analysis and stabilization of randomly switched systems. These systems may be regarded as piecewise deterministic stochastic systems: the discrete switches are triggered by a stochastic process which is independent of the state of the system, and between two consecutive switching instants the dynamics are deterministic. Our results provide sufficient conditions for almost sure stability and stability in the mean using Lyapunov-based methods when individual subsystems are stable and a certain "slow switching" condition holds. This slow switching condition takes the form of an asymptotic upper bound on the probability mass function of the number of switches that occur between the initial and current time instants. This condition is shown to hold for switching signals coming from the states of finite-dimensional continuous-time Markov chains; our results, therefore, hold for Markovian jump systems in particular. For systems with control inputs, we provide explicit control schemes for feedback stabilization using the universal formula for stabilization of nonlinear systems.**

Index Terms—**Almost sure and mean stochastic stability, random switching, stabilization, switched systems.**

I. Introduction

Randomly switched systems consist of a family of subsystems, together with a random switching signal which specifies the active subsystem at every instant of time. Since the dynamics are governed by an ordinary differential equation between any two successive switching instants, these systems may be regarded as piecewise deterministic stochastic systems [1]. These systems have variable structure, and can be used as models for systems affected by random structural changes. Applications of randomly switched systems include economic and manufacturing systems, communication and biological systems affected by random delays and component failures, etc. One particularly interesting phenomenon is observed in certain sea snails "Pleurobranchea" and "Tritonia."[1] These organisms have simple neural networks and persistent stimuli cause them to swim. Random changes in stimuli, e.g., scent of random food locations or random noxious environmental conditions, cause them to take orienting turns towards food, or avoidance turns away from the noxious agents, respectively.

A particular class of piecewise deterministic stochastic systems has received widespread attention, namely, Markovian jump linear systems (MJLS). These systems may be realized as a family of linear subsystems, together with a switching signal generated by the state of a continuous-time Markov chain. Stability and stabilization (see [2] for a detailed survey on different notions and results on stochastic stability) of MJLS have been extensively investigated, especially, under the assumption that the parameters of the Markov chain are completely known; see, e.g., [3]–[6] and the references therein. In particular, almost sure stabilization and stabilization in the mean of MJLS is discussed in [5], where the authors also establish precise equivalences between different stability notions for MJLS.

In this note, we neither restrict ourselves to linear subsystems nor to Markovian switching signals. Our results provide sufficient conditions for almost sure stability of randomly switched nonlinear systems when each subsystem is stable, and the switching is "slow" in a certain statistical sense. The slow switching condition takes the form of an upper bound on the probability mass function of the number of switches between the initial and current time instants. This condition is shown to hold in the case of switching signals coming from finite-dimensional continuous-time Markov chains; consequently, our results can be applied to Markovian jump systems under appropriate conditions on the parameters of the generator matrices of the underlying Markov chains. Since almost sure stability implies stability in probability [7], our results also provide sufficient conditions for stability in probability of randomly switched systems; a variant of stability in the mean is also obtained. Based on our analysis, we propose control schemes which achieve almost sure stabilization and stabilization in the mean for systems with control inputs, by employing the universal formula for nonlinear feedback stabilization [8].

A myriad of techniques have been employed to study stability and stabilization of piecewise deterministic stochastic systems. Hamilton–Jacobi–Bellman (HJB) equation-based optimal control schemes for piecewise deterministic stochastic systems are well studied; see, e.g., [1] for a detailed account. Linear control systems admit analytically solvable linear quadratic optimal design methods, and such techniques have been effectively combined with the stochastic nature of structural variations in [4]; stabilization schemes based on Lyapunov exponents are studied in [5]. Game-theoretic techniques [9] in the presence of disturbance inputs and spectral theory of Markov operators [10] have also been employed to study these systems. Stabilization schemes using robust control methods are investigated in [11]; see, also, the references cited there. Stochastic hybrid systems, where the switching signal and its transition probabilities are state dependent, are studied in [12] and [13], using an extended definition of the infinitesimal generator and optimal control strategies, respectively.

In contrast to the aforementioned references, our techniques rely on the method of multiple Lyapunov functions, discussed in the context of stability analysis and stabilization of deterministic switched systems in [14, ch. 3]. Our results conceptually parallel the ones on deterministic switched systems, which require stability of individual subsystems and a slow switching condition; see, e.g., [14, ch. 3]. We employ multiple Lyapunov functions for stability analysis and stabilizing controller design, coupled with suitable assumptions to take care of the stochastic nature of the switching signal. Recently, a method of stabilization in probability of Markovian jump systems, with control and white noise disturbance inputs for each subsystem, has been proposed in [15], which is similar in spirit. We propose stronger results that apply to a wider class of systems and switching signals, although our model is simpler due to the absence of noise.

II. Preliminaries

Let the Euclidean norm be denoted by $\|\cdot\|$, the interval $[0,\infty[$ by $\mathbb{R}_{\geqslant 0}$, and the set of natural numbers $\{1,2,\ldots\}$ by $\mathbb{N}$. Recall that a continuous function $\alpha : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}$ if α is strictly increasing with $\alpha(0)=0$, of class $\mathcal{K}_\infty$ if in addition $\alpha(r)\to\infty$ as $r\to\infty$; we write $\alpha\in\mathcal{K}$ and $\alpha\in\mathcal{K}_\infty$, respectively.

We define the family of systems

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P} \tag{1}$$

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[1]This particular phenomenon was communicated to the authors by P. K. Jha.

where the state $x \in \mathbb{R}^n$, $\mathcal{P}$ is a finite index set of N elements: $\mathcal{P} = \{1, \ldots, N\}$, the functions $f_p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are locally Lipschitz, $f_p(0) = 0$, and $p \in \mathcal{P}$. To define a switched system for the family, we consider a piecewise constant function (continuous from the right by convention) $\sigma : \mathbb{R}_{\geqslant 0} \longrightarrow \mathcal{P}$, which specifies at every instant of time t the index $\sigma(t) = p \in \mathcal{P}$ of the active subsystem. The *switched system* [14] for the family (1) generated by this *switching signal* σ is

$$\dot{x} = f_\sigma(x) \quad x(0) = x_0, \qquad t \geqslant 0. \tag{2}$$

We assume that there are no jumps in the state x at the switching instants.

Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geqslant 0}, \mathsf{P})$ be a complete filtered probability space, where $(\mathfrak{F}_t)_{t\geqslant 0}$ is the natural filtration (satisfying the usual conditions) generated by a càdlág random process σ taking values in $\mathcal{P}$. Let σ be the switching signal to the family (1), generating the *randomly switched system* (2). Let the switching instants of σ be denoted by τ_i, $i = 1, 2, \ldots$, and let $\tau_0 := 0$ by a convention. As a consequence of the hypotheses of our main result, there is no explosion almost surely (see Remark 4 for details); therefore, the sequence $(\tau_i)_{i\geqslant 0}$ is divergent. Finally, we assume that for every compact subset $K \subset \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ there exists an integrable function m_K satisfying $\sup_{p\in\mathcal{P}} \|f_p(x)\| \leqslant m_K(t)$ for all $(t, x) \in K$. Hence, almost surely, there exists a unique solution to (2) in the sense of Carathéodory [17] over a nontrivial time interval containing 0; existence and uniqueness of a global solution will follow from the hypotheses of our main result. We let $x(\cdot)$ denote this solution. For $x_0 = 0$, the solution to (2) is identically 0 for every σ; we will ignore this trivial case in the sequel.

We focus on the notion of stability defined in the following.

Definition 1: The system (2) is said to be ***globally asymptotically stable almost surely*** (GAS a.s.) if the following two properties are simultaneously verified:

SP1) $\mathsf{P}(\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0$ such that $\|x_0\| < \delta(\varepsilon)$ implies $\sup_{t\geqslant 0} \|x(t)\| < \varepsilon) = 1$;

SP2) $\mathsf{P}(\forall r, \varepsilon' > 0\ \exists T(r, \varepsilon') \geqslant 0$ such that $\|x_0\| < r$ implies $\sup_{t\geqslant T(r,\varepsilon')} \|x(t)\| < \varepsilon') = 1$.

$\diamondsuit$

III. Stability Under Random Switching

A. Global Asymptotic Stability Almost Surely

For a switching signal σ, we denote the number of switches on the interval $]0, t]$ by $N_\sigma(t)$. The following main result of this article provides sufficient conditions for GAS a.s. of (2).

Theorem 2: Consider the system (2). Suppose that there exist continuously differentiable functions $V_p : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $p \in \mathcal{P}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and real numbers $\widetilde{\lambda}, \overline{\lambda}, \lambda_\circ > 0$, $\mu > 1$, such that the following hold:

1) $\alpha_1(\|x\|) \leqslant V_p(x) \leqslant \alpha_2(\|x\|)\ \forall x \in \mathbb{R}^n, \forall p \in \mathcal{P}$;
2) $(\partial V_p/\partial x) f_p(x) \leqslant -\lambda_\circ V_p(x)\ \forall x \in \mathbb{R}^n, \forall p \in \mathcal{P}$;
3) $V_{p_1}(x) \leqslant \mu V_{p_2}(x)\ \forall x \in \mathbb{R}^n, \forall p_1, p_2 \in \mathcal{P}$;
4) $\exists M \in \mathbb{N} \cup \{0\}$ such that $\forall k \geqslant M$ we have $\mathsf{P}(N_\sigma(t) = k) \leqslant ((\overline{\lambda}t)^k \mathrm{e}^{-\widetilde{\lambda}t}/k!)$;
5) $\mu < (\lambda_\circ + \widetilde{\lambda})/\overline{\lambda}$.

Then, (2) is GAS a.s.

Before proving Theorem 2, let us make the following observations.

Remark 3: Hypothesis 2) of Theorem 2 implies that each subsystem of the family (1) is globally asymptotically stable; the right-hand side of the inequality being linear in V_p is no loss of generality, see [18, Th. 2.6.10]. Hypothesis 3) first appeared in [19] and has almost become a standard in deterministic switched systems literature. It restricts the class of applicable Lyapunov functions by requiring the existence of a maximal global constant ratio among the functions, but it is not known whether this hypothesis actually incurs a loss of generality. Quadratic Lyapunov functions are universally considered for linear systems, and in this case, the existence of a global constant μ is automatically guaranteed. Since the left-hand side of the inequality in hypothesis 4) is a probability measure of an event, the right-hand side may be replaced by $\min\{(\mathrm{e}^{-\widetilde{\lambda}t}(\overline{\lambda}t)^k/k!), 1\}$. In the special case when $\widetilde{\lambda} = \overline{\lambda}$, this hypothesis states that the number of switches $N_\sigma(t)$ on $[0, t[$ of σ is eventually upper bounded by the probability mass function of a Poisson process of parameter $\overline{\lambda}$. $\triangleleft$

Remark 4: Suppose σ satisfies hypothesis 4) of Theorem 2. Then, the probability of an explosion, i.e., of an accumulation of infinitely many switches over a finite time interval is zero. Indeed, if $\zeta \in \mathbb{R}_{\geqslant 0}$, the event that there is an explosion at $t = \zeta$ is given by $\bigcap_{\nu\in\mathbb{N}}\{N_\sigma(\zeta) \geqslant \nu\}$. However, $\mathsf{P}(\bigcap_{\nu\in\mathbb{N}}\{N_\sigma(\zeta) \geqslant \nu\}) \leqslant \overline{\lim}_{\nu\uparrow\infty} \sum_{k=\nu}^{\infty} \mathsf{P}(N_\sigma(\zeta) = k)$, and using hypothesis 4), we get $\overline{\lim}_{\nu\uparrow\infty} \sum_{k=\nu}^{\infty} \mathsf{P}(N_\sigma(\zeta) = k) \leqslant \overline{\lim}_{\nu\uparrow\infty} \sum_{k=\nu}^{\infty} \mathrm{e}^{-\widetilde{\lambda}\zeta}((\overline{\lambda}\zeta)^k/k!) = 0$. Since $\zeta \in \mathbb{R}_{\geqslant 0}$ is arbitrary, we conclude that there is no explosion almost surely. $\triangleleft$

The proof of Theorem 2 follows the sequence of Lemmas described next. The idea of the proof may be briefly summarized as follows. The property SP2) is proved first by first estimating the expected value of $V_{\sigma(t)}(x(t))$ for an arbitrary $t \geqslant 0$ via the moment generating function of $N_\sigma(t)$, and then proving a.s. asymptotic convergence of $(V_{\sigma(t)}(x(t)))_{t\geqslant 0}$ via Tonelli's theorem and an auxiliary Lemma that proves asymptotic convergence of $\|x(\cdot)\|$ from the finiteness of a certain nonnegative integral. We also observe that since the (finite) family of subsystems is uniformly locally Lipschitz, the maximal temporal growth rate of trajectories is upper bounded by a constant in a neighborhood of 0. The SP1) property can now be established utilizing this fact and the SP2) property that is proved first, thereby completing the proof.

Lemma 5: Suppose that hypotheses 2) and 3) of Theorem 2 hold. Then, we have $\mathsf{E}[V_{\sigma(t)}(x(t))] \leqslant \mathsf{E}[\mathrm{e}^{(\ln\mu)N_\sigma(t)}]V_{\sigma(0)}(x_0)\mathrm{e}^{-\lambda_\circ t}\ \forall t \geqslant 0$.

Proof: Recall that $(\tau_i)_{i\in\mathbb{N}}$ are the switching instants of σ. It follows from hypothesis 2) that for $t \in [\tau_i, \tau_{i+1}[$, we have

$$V_{\sigma(\tau_i)}(x(t)) \leqslant V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_\circ(t-\tau_i)}.$$

In conjunction with hypothesis 3), this yields

$$V_{\sigma(\tau_{i+1})}(x(\tau_{i+1})) \leqslant \mu V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_\circ(\tau_{i+1}-\tau_i)}.$$

Iterating the last inequality from $i = 0$ to $i = N_\sigma(t)$ for an arbitrary time $t > 0$, we arrive at

$$V_{\sigma(t)}(x(t)) \leqslant \mu^{N_\sigma(t)}\mathrm{e}^{-\lambda_\circ t}V_{\sigma(0)}(x_0).$$

Since the initial condition is deterministic, taking expectations on both sides of the previous inequality, we get

$$\mathsf{E}\left[V_{\sigma(t)}(x(t))\right] \leqslant \mathsf{E}\left[\mu^{N_\sigma(t)}\right]\mathrm{e}^{-\lambda_\circ t}V_{\sigma(0)}(x_0)$$

which proves the claim. ■

Lemma 6: Suppose that hypothesis 4) of Theorem 2 holds. Then, $\exists S \geqslant 0$ such that the *moment generating function* $\mathsf{E}[\mathrm{e}^{sN_\sigma(t)}]$ of $N_\sigma(t)$ satisfies $\mathsf{E}[\mathrm{e}^{sN_\sigma(t)}] \leqslant S + \mathrm{e}^{(\mathrm{e}^s\overline{\lambda}-\widetilde{\lambda})t}\ \forall s \geqslant 0$.

Proof: Using hypothesis 4), for $s \geqslant 0$, a little computation leads to

$$\begin{aligned}\mathrm{E}\left[\mathrm{e}^{sN_\sigma(t)}\right] &= \sum_{k=0}^{\infty} \mathrm{e}^{sk}\mathsf{P}\left(N_\sigma(t)=k\right)\\ &\leqslant \sum_{k=0}^{M-1} \mathrm{e}^{sk}\mathsf{P}\left(N_\sigma(t)=k\right) + \sum_{k=M}^{\infty} \mathrm{e}^{sk}\frac{(\overline{\lambda}t)^k \mathrm{e}^{-\widetilde{\lambda}t}}{k!}\\ &\leqslant S + \mathrm{e}^{(\mathrm{e}^s\overline{\lambda}-\widetilde{\lambda})t}\end{aligned}$$

where $S := \sum_{k=0}^{M-1}\mathrm{e}^{sk} \geqslant 0$. Clearly, $\mathsf{E}[\mathrm{e}^{sN_\sigma(t)}]$ is well defined for $t \geqslant 0$.

Lemma 7: If $\alpha_1 \in \mathcal{K}$ and $\int_0^\infty \alpha_1(\|x(t)\|)\mathrm{d}t < \infty$ a.s., then $\lim_{t\to\infty} x(t) = 0$ a.s.

Sketch of Proof: Suppose that $\alpha_1(\|x(t)\|)$ does not converge to 0 on a set of positive probability. Then, for every event in this set, there is some $\varepsilon > 0$ such that after any time T there exists a time at which $\|x(\cdot)\|$ will be larger than ε. $\|x(\cdot)\|$ always stays above ε after that, in which case the integral cannot be finite, or there exists a time s at which $\|x(s)\| = \varepsilon$. Since f_p is locally Lipschitz for every $p \in \mathcal{P}$ and $\mathcal{P}$ is a finite set, we know that $\|x(\cdot)\|$ cannot converge to 0 faster than a certain exponential function of time $\varepsilon\mathrm{e}^{-L(t-s)}$, where L is the uniform Lipschitz constant for $\{f_p\}_{p\in\mathcal{P}}$ on the ε-ball around 0. Since $\alpha_1 \in \mathcal{K}$, $\alpha_1(\|x(t)\|)$ is lower bounded by $\alpha_1(\varepsilon\mathrm{e}^{-L(t-s)})$. However, since the integral $\int_0^\infty \alpha_1(\|x(t)\|)\mathrm{d}t < \infty$, we can find a time T after which the tail of the integral is less than η for an arbitrary preassigned $\eta > 0$. Picking $\eta = \int_s^\infty \alpha_1(\varepsilon\mathrm{e}^{-L(t-s)})\mathrm{d}t$, we reach a contradiction. ■

Lemma 8: The system (2) has the following property: for every $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that $\|x(t)\| \leqslant \|x_0\|\mathrm{e}^{L_\varepsilon t}$ $\forall t \geqslant 0$ as long as $\|x(t)\| < \varepsilon$.

This is a standard calculation that employs the locally Lipschitz condition on the set of vector fields $\{f_p\}_{p\in\mathcal{P}}$.

Proof of Theorem 2: To prove that (2) is GAS a.s., we need to verify the SP1) and SP2) properties in Definition 1.

From Lemmas 5 and 6, it follows that $\int_0^\infty \mathsf{E}[V_{\sigma(t)}(x(t))]\mathrm{d}t < \infty$, and by Tonelli's theorem, we have

$$\int_0^\infty \mathsf{E}\left[V_{\sigma(t)}(x(t))\right]\mathrm{d}t = \mathsf{E}\left[\int_0^\infty V_{\sigma(t)}\left(x(t)\right)\mathrm{d}t\right] < \infty.$$

By hypothesis 1), we get $\int_0^\infty \alpha_1(\|x(t)\|)\mathrm{d}t < \infty$ a.s., and Lemma 7 shows that $\lim_{t\to\infty}\|x(t)\| = 0$ a.s., which proves SP2)

Now, we verify SP1). Fix $\varepsilon > 0$. We know from the SP2) property proved previously that almost surely there exists $T(1,\varepsilon) > 0$ such that $\|x_0\| < 1$ implies $\sup_{t\geqslant T(1,\varepsilon)}\|x(t)\| < \varepsilon$. Select $\delta(\varepsilon) = \min\{\varepsilon\mathrm{e}^{-L_\varepsilon T(1,\varepsilon)}, 1\}$. By Lemma 8, $\|x_0\| < \delta(\varepsilon)$ implies

$$\|x(t)\| \leqslant \|x_0\|\mathrm{e}^{L_\varepsilon t} < \delta(\varepsilon)\mathrm{e}^{L_\varepsilon T(1,\varepsilon)} < \varepsilon \qquad \forall t \in [0, T(1,\varepsilon)].$$

Further, the SP2) property guarantees that with the previous choice of δ and x_0, we have $\sup_{t\geqslant T(1,\varepsilon)}\|x(t)\| < \varepsilon$ on a set of full measure. Thus, $\|x_0\| < \delta(\varepsilon)$ implies $\sup_{t\geqslant 0}\|x(t)\| < \varepsilon$ a.s. Since ε is arbitrary, the SP1) property of (2) follows.

We conclude that (2) is GAS a.s. ■

Remark 9: Besides GAS a.s., global asymptotic stability in the mean is another important stability concept. The system (2) is said to be *globally asymptotically stable in the mean* (GAS-M) if the following two properties are simultaneously verified:

SM1) $\forall\varepsilon > 0\ \exists\widetilde{\delta}(\varepsilon) > 0$ such that $\|x_0\| < \widetilde{\delta}(\varepsilon)$ implies $\sup_{t\geqslant 0}\mathsf{E}[\|x(t)\|] < \varepsilon$;

SM2) $\forall r, \varepsilon' > 0\ \exists\widetilde{T}(r,\varepsilon') \geqslant 0$ such that $\|x_0\| < r$ implies $\sup_{t\geqslant\widetilde{T}(r,\varepsilon')}\mathsf{E}[\|x(t)\|] < \varepsilon'$.

We have seen that the hypotheses of Theorem 2 imply that $\lim_{t\to\infty}\mathsf{E}[\alpha_1(\|x(t)\|)] = 0$. If α_1 is convex, then this immediately gives $\lim_{t\to\infty}\mathsf{E}[\|x(t)\|] = 0$ via Jensen's inequality,[2] which is SM2). Lemmas 5 and 6 show that $\sup_{t\geqslant 0}\mathsf{E}[\alpha_1(\|x(t)\|)] \leqslant \alpha_2(\|x_0\|)(S+1)$, which under the assumption that α_1 is convex, implies SM1). In practice, the Lyapunov functions are usually taken to be polynomial powers of the state x and α_1 is convex. Otherwise, a property analogous to GAS-M still holds, with $\alpha_1(\|x(t)\|)$ replacing $\|x(t)\|$ everywhere, and therefore, depends on α_1. ◁

Remark 10: With a slight modification in the hypotheses of Theorem 2, we can employ standard results in martingale theory to conclude almost sure global asymptotic convergence of $x(\cdot)$. Indeed, if the condition 4) is strengthened to the conditional version $\mathsf{P}(N_\sigma(t) - N_\sigma(s) = k|\mathfrak{F}_s) \leqslant \mathrm{e}^{-\widetilde{\lambda}(t-s)}(\overline{\lambda}(t-s))^k/k!$ for all $k \in \mathbb{N}\cup\{0\}$, $0 \leqslant s < t < \infty$, then a calculation in the spirit of Lemma 5 shows that $(V_{\sigma(t)}(x(t)))_{t\geqslant 0}$ is a supermartingale. Lemma 5 also shows that $\lim_{t\to\infty}\mathsf{E}[V_{\sigma(t)}(x(t))] = 0$, which implies that the aforesaid process is a potential. A standard result in martingale theory (e.g., [16, p. 18, Problem 3.16]) now implies that the process $(V_{\sigma(t)}(x(t)))_{t\geqslant 0}$ converges to 0 a.s. Considering hypothesis 1) of Theorem 2, we conclude that $(\|x(t)\|)_{t\geqslant 0}$ converges to 0 a.s. ◁

B. Markovian Jump Systems

We note that hypothesis 4) of Theorem 2 stipulates that $\forall t \in \mathbb{R}_{\geqslant 0}$ the tail of the probability mass function of the random variable $N_\sigma(t)$ is majorized (i.e., stochastically dominated) by the probability mass function of a "maximally" switching jump-stochastic process. This hypothesis can be verified, in particular, if σ is the state of a continuous-time Markov chain, with a given generator matrix $Q = [q_{ij}]_{\mathrm{N}\times\mathrm{N}}$ and a given initial probability distribution π° (recall that N is the number of elements of $\mathcal{P}$); we denote this by $\sigma \sim (\pi^\circ, Q)$. Lemma 11 and Corollary 12 make this statement precise.

Let us recall some basic facts about continuous-time Markov chains; see, e.g., [21] for further details. Associated with the Markov chain $\sigma \sim (\pi^\circ, Q)$ is the Kolmogorov forward equation

$$\dot{P}(t) = P(t)Q \quad P(0) = I_{\mathrm{N}\times\mathrm{N}}, \qquad t \geqslant 0$$

where $I_{\mathrm{N}\times\mathrm{N}}$ is the N-dimensional identity matrix; the probability (row) vector at any time $t \geqslant 0$ is given by $\pi(t) = \pi^\circ P(t)$. We need the following two facts.

MC1) The generator matrix $Q = [q_{ij}]_{\mathrm{N}\times\mathrm{N}}$ satisfies $(q_{ij} \geqslant 0\ i \neq j)$, and $(\sum_{j\in\mathcal{P}\setminus\{i\}} q_{ij} = -q_{ii})$ for $i, j \in \mathcal{P}$.

MC2) $\mathsf{P}(\sigma(t+h) = j|\sigma(t) = i) = \delta_{ij} + q_{ij}h + o(h)$ for $h > 0$, and δ_{ij} is the Kronecker delta. This is known as the infinitesimal description of a continuous-time Markov chain.

We define

$$\overline{q} := \max\{|q_{ii}||i \in \mathcal{P}\},\ \widetilde{q} := \max\{q_{ij}|i, j \in \mathcal{P}\}. \tag{3}$$

Lemma 11: Suppose that $\sigma \sim (\pi^\circ, Q)$ is a Markov chain. Then, $\forall t \in \mathbb{R}_{\geqslant 0}$, we have $\mathsf{P}(N_\sigma(t) = k) \leqslant \mathrm{e}^{-\widetilde{q}t}(\overline{q}t)^k/k!\ \forall k \in \mathbb{N}\cup\{0\}$.

Proof: For $t \in \mathbb{R}_{\geqslant 0}$ and $k \in \mathbb{N}\cup\{0\}$, define $\eta_k(t) := \mathsf{P}[N_\sigma(t) = k]$. For $h > 0$ sufficiently small, $\forall k \in \mathbb{N}\cup\{0\}$

$$\eta_k(t+h) = \sum_{i=0}^{k}\mathsf{P}\left(N_\sigma(t+h) - N_\sigma(t) = i\right)\mathsf{P}\left(N_\sigma(t) = k-i\right). \tag{4}$$

By the infinitesimal description of a Markov chain MC2)

$$\mathsf{P}\left(N_\sigma(t+h) - N_\sigma(t) = 0\right) \leqslant 1 - \widetilde{q}h + o(h) \tag{5}$$

and

$$\mathsf{P}\left(N_\sigma(t+h) - N_\sigma(t) = 1\right) \leqslant \overline{q}h + o(h). \tag{6}$$

[2]We recall Jensen's inequality [20, p. 80]: if X is an integrable real-valued random variable on $(\Omega, \mathfrak{F}, \mathsf{P})$, and if $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function, then $\phi(\mathsf{E}[|X|]) \leqslant \mathsf{E}[\phi(|X|)]$.

For all natural numbers $k \geqslant 2$, MC2) shows that

$$\mathsf{P}\left(N_\sigma(t+h) - N_\sigma(t) = k\right) = o(h). \tag{7}$$

Using (5)–(7), we continue the calculation in (4)

$$\eta_k(t+h) \leqslant (1-\widetilde{q}h+o(h))\,\eta_k(t)+(\overline{q}h+o(h))\,\eta_{k-1}(t)+o(h)$$

which leads to

$$\frac{\eta_k(t+h)-\eta_k(t)}{h} \leqslant -\widetilde{q}\eta_k(t) + \overline{q}\eta_{k-1}(t) + O(h).$$

Taking limits with $h \downarrow 0$, the following differential inequality is obtained:

$$\dot{\eta}_k(t) \leqslant -\widetilde{q}\eta_k(t) + \overline{q}\eta_{k-1}(t), \qquad \eta_k(0) = 0 \qquad \forall k \in \mathbb{N}.$$

(We have identical differential inequalities starting with $t > 0$ and $h < 0$ sufficiently small.) A similar analysis yields

$$\dot{\eta}_0(t) \leqslant -\widetilde{q}\eta_0(t), \qquad \eta_0(0) = 1.$$

In matrix notation, the set of differential inequalities involving $\dot{\eta}_k$, $k \in \mathbb{N} \cup \{0\}$, stands as

$$\begin{bmatrix} \dot{\eta}_0 \\ \dot{\eta}_1 \\ \dot{\eta}_2 \\ \vdots \end{bmatrix} \leqslant \begin{bmatrix} -\widetilde{q} & 0 & 0 & \cdots \\ \overline{q} & -\widetilde{q} & 0 & \cdots \\ 0 & \overline{q} & -\widetilde{q} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix} \qquad \begin{bmatrix} \eta_0(0) \\ \eta_1(0) \\ \eta_2(0) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{8}$$

where the "$\leqslant$" is interpreted componentwise. Clearly, $\eta_0(t) \leqslant \mathrm{e}^{-\widetilde{q}t}$, $t \geqslant 0$, satisfies the first differential inequality. We claim that

$$\eta_k(t) \leqslant \mathrm{e}^{-\widetilde{q}t}(\overline{q}t)^k/k! \qquad \forall t \geqslant 0 \qquad \forall k \in \mathbb{N} \tag{9}$$

is a solution to (8). Indeed, for $k = 1$, we have $\dot{\eta}_1 \leqslant \overline{q}\eta_0 - \widetilde{q}\eta_1 \leqslant \overline{q}\mathrm{e}^{-\widetilde{q}t} - \widetilde{q}\eta_1$, which leads to

$$\mathrm{e}^{\widetilde{q}t}\eta_1(t) \leqslant \mathrm{e}^{\widetilde{q}t}\eta_1(0) + \overline{q}\int_0^t \mathrm{d}s$$

hence, $\eta_1\mathrm{e}^{\widetilde{q}t} \leqslant (\overline{q}t)$ (in view of $\eta_1(0) = 0$), yielding $\eta_1(t) \leqslant (\overline{q}t)\mathrm{e}^{-\widetilde{q}t}$, $t \geqslant 0$. Having verified the claim for $k = 1$, an induction argument shows that for arbitrary j

$$\mathrm{e}^{\widetilde{q}t}\eta_{j+1}(t) \leqslant \mathrm{e}^{\widetilde{q}t}\eta_{j+1}(0) + \overline{q}\int_0^t \frac{(\overline{q}s)^j}{j!}\mathrm{d}s$$

hence, $\eta_{j+1}\mathrm{e}^{\widetilde{q}t} \leqslant (\overline{q}t)^{j+1}/(j+1)!$ (in view of $\eta_{j+1}(0) = 0$), yielding $\eta_{j+1}(t) \leqslant \mathrm{e}^{-\widetilde{q}t}(\overline{q}t)^{j+1}/(j+1)!$, $t \geqslant 0$. In view of the definition of $\eta_k(t)$, the thesis of the Lemma follows. ■

Corollary 12: Consider the system (2), and let $\overline{q}$ and $\widetilde{q}$ be defined by (3). Suppose that $\sigma \sim (\pi^\circ, Q)$ is a Markov chain, and that there exist continuously differentiable functions $V_p : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $p \in \mathcal{P}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a real number $\mu > 1$, such that the following hold:

1) $\alpha_1(\|x\|) \leqslant V_p(x) \leqslant \alpha_2(\|x\|)\ \forall x \in \mathbb{R}^n, \forall p \in \mathcal{P}$;
2) $(\partial V_p/\partial x) f_p(x) \leqslant -\lambda_\circ V_p(x)\ \forall x \in \mathbb{R}^n, \forall p \in \mathcal{P}$;
3) $V_{p_1}(x) \leqslant \mu V_{p_2}(x)\ \forall x \in \mathbb{R}^n, \forall p_1, p_2 \in \mathcal{P}$;
4) $\mu < (\lambda_\circ + \widetilde{q})/\overline{q}$.

Then, (2) is GAS a.s.

Proof: It follows directly from Lemma 11 and Theorem 2 with $M = 0$ in hypothesis 4).

IV. Stabilization of Randomly Switched Control Systems

In this section, we establish a method for designing controllers that ensure almost sure global asymptotic stability of control-affine randomly switched systems in the closed loop. The method may be viewed as an application of our results in Section III. We assume that at each instant of time t, the state $\sigma(t) \in \mathcal{P}$ of the random switching signal is perfectly known to the controller.

Consider the affine in control switched system

$$\dot{x} = f_\sigma(x) + \sum_{i=1}^m g_{\sigma,i}(x)u_i, \qquad x(0) = x_0, \qquad t \geqslant 0 \tag{10}$$

where $x \in \mathbb{R}^n$ is the state, u_i, $i = 1,\ldots,m$, are the control inputs, $u_i \in \mathbb{R}$, f_p, and $g_{p,i}$ are smooth vector fields on $\mathbb{R}^n$, with $f_p(0) = 0$ and $g_{p,i}(0) = 0$, for each $p \in \mathcal{P}, i \in \{1,\ldots,m\}$. With a feedback control function $\overline{u}_\sigma(x) = [u_{\sigma,1}(x),\ldots,u_{\sigma,m}(x)]^{\mathrm{T}}$, the closed loop system stands as

$$\dot{x} = f_\sigma(x) + \sum_{i=1}^m g_{\sigma,i}(x)\overline{u}_{\sigma,i}(x), \qquad x(0) = x_0, \qquad t \geqslant 0. \tag{11}$$

Our objective is to select the control function $\overline{u}_\sigma$ so that (11) is GAS a.s. Let the switching signal σ be a stochastic process as defined in Section II, and let $x_0 \neq 0$.

A universal formula for stabilization of control-affine nonlinear systems was first constructed in [8], for the control taking values in $\mathbb{R}^m$.

Theorem 13: Consider the system (10). Suppose that σ satisfies hypothesis 4) of Theorem 2, and there exists a family of continuously differentiable functions $V_p : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $p \in \mathcal{P}$, such that the following hold:

C1) hypothesis 1) of Theorem 2 holds;
C2) hypothesis 3) of Theorem 2 holds;
C3) $\exists \lambda_\circ > 0$ such that $\forall x \in \mathbb{R}^n \setminus \{0\}$ and $\forall p \in \mathcal{P}$
$$\inf_{u \in \mathbb{R}^m}\left\{\mathrm{L}_{f_p}V_p(x) + \lambda_\circ V_p(x) + \sum_{i=1}^m u_i \mathrm{L}_{g_{p,i}}V_p(x)\right\} < 0;$$
C4) $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $x(\neq 0)$ satisfies $\|x\| < \delta$, then $\exists u \in \mathbb{R}^m$, $\|u\| < \varepsilon$, such that $\forall p \in \mathcal{P}$
$$\mathrm{L}_{f_p}V_p + \sum_{i=1}^m u_i \cdot \mathrm{L}_{g_{p,i}}V_p \leqslant -\lambda_\circ V_p;$$
C5) hypothesis 5) of Theorem 2 holds.

Then, the feedback control

$$\overline{u}_\sigma(x) = [k_{\sigma,1}(x),\ldots,k_{\sigma,m}(x)]^{\mathrm{T}}$$

where

$$k_{p,i}(x) := -\,\mathrm{L}_{g_{p,i}}V_p(x)\cdot\varphi\left(\overline{W}_p(x), \widetilde{W}_p(x)\right) \tag{12a}$$
$$\overline{W}_p(x) := \mathrm{L}_{f_p}V_p(x) + \lambda_\circ V_p(x), \tag{12b}$$
$$\widetilde{W}_p(x) := \sum_{i=1}^m \left(\mathrm{L}_{g_{p,i}}V_p(x)\right)^2 \tag{12c}$$

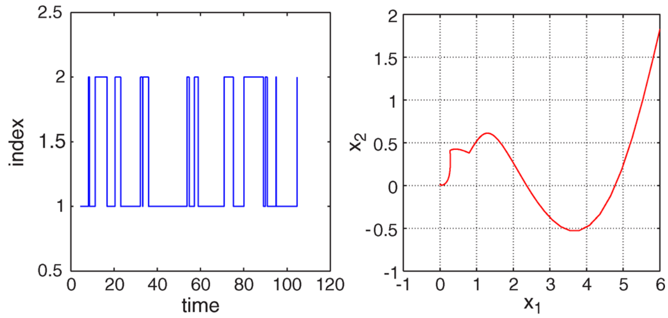
and

$$\varphi(a,b) := \begin{cases} \frac{a+\sqrt{a^2+b^2}}{b}, & \text{if } b \neq 0, \\ 0, & \text{otherwise} \end{cases} \tag{12d}$$

renders (11) GAS a.s.

Proof: The proof relies on the construction of the universal formula in [8]. Fix $t \in \mathbb{R}_{\geqslant 0}$. If $x \neq 0$, applying the definition ofφ, we get

$$\mathrm{L}_{f_{\sigma(t)}}V_{\sigma(t)}(x) + \sum_{i=1}^m k_{\sigma(t),i}(x)\mathrm{L}_{g_{\sigma(t),i}}V_{\sigma(t)}(x)$$
$$= \mathrm{L}_{f_{\sigma(t)}}V_{\sigma(t)}(x) - \widetilde{W}_{\sigma(t)}(x)\cdot\varphi\left(\overline{W}_{\sigma(t)}(x), \left(\widetilde{W}_{\sigma(t)}(x)\right)^2\right)$$



(a) (b)

Fig. 1. Portion of a typical execution of σ defined in Section V, and the corresponding evolution of (13). (a) States of σ against time. (b) Portion of the state trajectory.

$$= -\lambda_\circ V_{\sigma(t)}(x) - \sqrt{\left(\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x)\right)^2 + \left(\widetilde{W}_{\sigma(t)}(x)\right)^2}$$
$$< -\lambda_\circ V_{\sigma(t)}(x).$$

Since t is arbitrary, we conclude that the previous inequality holds for all $t \in \mathbb{R}_{\geqslant 0}$. Note that by C3), if for any $p \in \mathcal{P}$, $x \in \bigcap_{i=1}^{m} \ker(\mathrm{L}_{g_{p,i}} V_p)$, we automatically have $\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) + \lambda_\circ V_{\sigma(t)}(x) < 0$. C4) is the small control property, ensuring continuity of the control function at 0 for each fixed index p; this guarantees the existence of a unique local solution to the switched system.

The previous arguments, in conjunction with C1) and C2), enable us to conclude that the family $(V_p)_{p\in\mathcal{P}}$ satisfies hypotheses 1)–3) of Theorem 2 for the closed-loop system (11). C5) ensures that hypothesis 5) of Theorem 2 holds for (11). Since σ satisfies hypothesis 4) of Theorem 2, it follows from Theorem 2 applied to (11), that (11) is GAS a.s. ■

V. Numerical Example

In this section, we study an illustrative example. Let $\mathcal{P} = \{1, 2\}$, $x = [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$, and let the two subsystems be given by

$$f_1(x) = \begin{bmatrix} -\dfrac{3}{2}x_1 + x_2 \\ (x_1 + x_2)\sin x_1 - 3x_2 \end{bmatrix} \quad f_2(x) = \begin{bmatrix} -2x_1 - x_1^3 \\ x_1 - x_2 \end{bmatrix}.$$

Let σ generate the switched system

$$\dot{x} = f_\sigma(x), \qquad x(0) = x_0, \qquad t \geqslant 0 \tag{13}$$

where σ is a jump stochastic process specified in terms of the holding times $S_k := \tau_{k+1} - \tau_k$ as follows: the sequence $(S_k)_{k\in\mathbb{N}\cup\{0\}}$ is an independent sequence of exponential random variables of parameter $\lambda = 0.2$. An easy calculation shows that σ satisfies hypothesis 4) of Theorem 2 with $\overline{\lambda} = \widetilde{\lambda} = \lambda$.

Consider the following two candidate Lyapunov functions:

$$V_1(x) = \frac{1}{2}(x_1^2 + x_2^2) \quad V_2(x) = \frac{1}{2}x_1^2 + x_2^2$$

corresponding to the previous subsystems f_1 and f_2. Clearly, $V_1 \leqslant 2V_2$ and $V_2 \leqslant V_1$; therefore, $V_i \leqslant 2V_j$ for $i, j \in \mathcal{P}$, which means $\mu = 2$. A quick calculation shows that

$$\mathrm{L}_{f_1} V_1(x) \leqslant -V_1(x) \quad \mathrm{L}_{f_2} V_2(x) \leqslant -V_2(x)$$

which means $\lambda_0 = 1$. It follows easily that hypothesis 5) of Theorem 2 holds and, hence, the system (13) is GAS a.s. A typical execution fragment of (13), initialized at (15, 15), is given in Fig. 1.

VI. Conclusion and Further Work

We have provided sufficient conditions for almost sure stability of randomly switched systems, together with control strategies for almost sure stabilization for systems with control inputs. It may be possible to improve upon the proposed results by utilizing the jump destinations of the switching signal, and in the case of Markov chains, its graph and the associated transition probability matrix. Stabilization of randomly switched systems with control inputs without perfect knowledge of σ is a nontrivial and important issue. Input-to-state stability properties, existence and uniqueness of invariant measures, and other asymptotic properties of randomly switched systems are interesting avenues for future research. Results on these will be reported elsewhere.

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References

[1] M. H. A. Davis, *Markov Models and Optimization*. London, U.K.: Chapman & Hall, 1993.
[2] F. Kozin, “A survey of stability of stochastic systems,” *Automatica*, vol. 5, pp. 95–112, 1969.
[3] P. Bolzern, P. Colaneri, and G. De Nicolao, “On almost sure stability of discrete-time Markov jump linear systems,” in *Proc. 43rd Conf. Decision Control*, 2004, pp. 3204–3208.
[4] Y. Ji and H. J. Chizeck, “Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control,” *IEEE Trans. Autom. Control*, vol. 35, no. 7, pp. 777–788, Jul. 1990.
[5] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, “Stochastic stability properties of jump linear systems,” *IEEE Trans. Autom. Control*, vol. 37, no. 1, pp. 38–53, Jan. 1992.
[6] X. Mao, “Exponential stability of stochastic delay interval systems with Markovian switching,” *IEEE Trans. Autom. Control*, vol. 47, no. 10, pp. 1604–1612, Oct. 2002.
[7] R. Z. Haśminskii, *Stochastic Stability of Differential Equations*. Alphen ann den Rijn, The Netherlands: Sijthoff & Noordhoff, 1980.
[8] E. D. Sontag, “A universal construction of Artstein’s theorem on nonlinear stabilization,” *Syst. Control Lett.*, vol. 13, pp. 117–123, 1989.
[9] T. Başar, “Minimax control of switching systems under sampling,” *Syst. Control Lett.*, vol. 25, no. 5, pp. 315–325, 1995.
[10] J. Huang, I. Kontoyiannis, and S. P. Meyn, “The ODE method and spectral theory of Markov operators,” in *Stochastic Theory and Control (Lawrence, KS, 2001)*, ser. Lecture Notes in Control and Information Sciences. Berlin, Germany: Springer-Verlag, 2002, vol. 280, pp. 205–221.
[11] J. Xiong, J. Lam, H. Gao, and D. W. C. Ho, “On robust stabilization of Markovian jump systems with uncertain switching probabilities,” *Automatica*, vol. 41, no. 5, pp. 897–903, 2005.
[12] M. L. Bujorianu and J. Lygeros, “General stochastic hybrid systems: Modeling and optimal control,” in *Proc. 43rd IEEE Conf. Decision Control*, 2004, pp. 1872–1877.
[13] J. P. Hespanha, “A model for stochastic hybrid systems with application to communication networks,” [Online]. Available: http://www.ece.ucsb.edu/~hespanha/published.html 2004
[14] D. Liberzon, *Switching in Systems and Control*, ser. Systems & Control: Foundations & Applications. Boston, MA: Birkhäuser, 2003.
[15] S. Battilotti and A. De Santis, “Dwell time controllers for stochastic systems with switching Markov chain,” *Automatica*, vol. 41, no. 6, pp. 923–934, Jun. 2005.
[16] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, ser. Graduate Texts in Mathematics, 2nd ed. New York: Springer-Verlag, 1991, vol. 113.
[17] A. F. Filippov, *Differential Equations With Discontinuous Righthand Sides*, ser. Mathematics and Its Applications. Norwell, MA: Kluwer, 1988, vol. 18.
[18] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Application*. New York: Academic, 1969, vol. 1.
[19] J. P. Hespanha and A. S. Morse, “Stability of switched systems with average dwell-time,” in *Proc. 38th IEEE Conf. Decision Control*, 1999, vol. 3, pp. 2655–2660.
[20] P. Billingsley, *Probability and Measure*, 3rd ed. New York: Wiley, 1995.
[21] J. R. Norris, *Markov Chains*. Cambridge, U.K.: Cambridge Univ. Press, 1997.