Rendezvous Without Coordinates

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***Abstract*—We study minimalism in sensing and control by considering a multi-agent system in which each agent moves like a Dubins car and has a limited sensor that reports only the presence of another agent within some sector of its windshield. Using a simple quantized control law with three values, each agent tracks another agent (its target) assigned to it by maintaining that agent within this windshield sector. We use Lyapunov analysis to show that by acting autonomously in this way, the agents will achieve rendezvous given a connected initial assignment graph and the assumption that an agent and its target will merge into a single agent when they are sufficiently close. We then proceed to show that, by making the quantized control law slightly stronger, a connected initial assignment graph is not required and the sensing model can be weakened further. A distinguishing feature of our approach is that it does not involve any estimation procedure aimed at reconstructing coordinate information. Our scenario thus provides an example in which an interesting task is performed with extremely coarse sensing and control, and without state estimation. The system was implemented in computer simulation, accessible through the Web, of which the results are presented in the paper.**

***Index Terms*—Consensus, distributed control, multiagent systems, rendezvous, minimalism.**

I. Introduction

A GENERALLY desirable feature in autonomous systems is to perform tasks with minimal information. For feedback systems, such information is captured in the form of sensing data and a control law: a system completes a loop by taking in sensing data, estimating its state, making a control decision, and executing the control. Often, however, the information being minimized is limited to only sensing or only control. Sacrificing good state estimation usually demands compensation from more precise control and vise versa. The seemingly inescapable trade-off between sensing and control capabilities prompts the question: Is any interesting task achievable with both coarse sensing and quantized control?

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In this paper, we investigate multi-agent rendezvous with an emphasis on minimalism in both sensing and control. The agents are Dubins car vehicles [9], [20] equipped with sensors of limited field-of-view and bounded range. We show that the vehicles, each operating independently under a three-level feedback quantized control law, will rendezvous without ever obtaining coordinate data or performing state estimation. This result is based on three temporary assumptions:

- Every agent is initially assigned at most one target to form a connected assignment graph;
- An agent can track its target with its sensor, even in the presence of occlusion.
- Any agent without a target will stop moving.

and one standing assumption:

- An agent and its target "merge" into a single agent when they are sufficiently close by.

The merging assumption, which can be satisfied with stronger sensing and control in near-range only, is due to the Dubins car dynamics and the simple control law considered. Upon establishing that identical agents can achieve guaranteed rendezvous in finite time, results of similar flavor are obtained for agents with bounded, possibly different forward velocities. We then continue to show that, with a slightly stronger quantized control law, the three temporary assumptions are not necessary for identical agents. As an interesting side note, we also show (in Appendix C) that our problem and results match those of the classic cyclic pursuit when agents can change headings instantly.

The rendezvous problem in the control context catches our attention as an actively pursued problem during the past decade. An early formulation and algorithmic solution of the multi-agent rendezvous problem is introduced in [1], in which agents have limited range sensing capabilities. *Stop-and-go* strategies extending the algorithm in [1] are proposed in [22] and [23], which cover various synchronous and asynchronous formulations. An n-dimensional rendezvous problem was approached via proximity graphs in [6]. A research area that is closely related to rendezvous is *cyclic pursuit* or the n*-bug problem*, in which each agent pursues on a directed cycle its reachable neighboring agent. The mathematical study of pursuit curves and pursuit polygons with differential constraints originated this line of research [2], [5], with the focus being whether, when, where, and how the participating agents meet each other. Stable pursuit polygons in cyclic pursuit are also investigated in [34]. In the past few years, due partly to the realization that state estimation (sensing and computation) and control are both key but distinct components in practical systems, cyclic pursuit problems with feedback control have been further explored [25], [28], [37], [38], with [28] giving a history and review on cyclic pursuit.

In control theory, the problem of controlling a plant using coarse quantized measurements of its state (or output) has re-

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ceived much attention in recent years. Quantized control, an active branch of control theory, focuses particularly on minimal data rate control laws. The motivation for studying such control problems comes from situations in which the rate of information flow between the plant and the controller has to be minimized due to communication bandwidth constraints, shared network resources, security concerns, or other considerations. For some classes of systems, most notably linear ones, precise conditions have been obtained on how much information is needed for control; e.g., see [4], [8], [10], [14], [15], [21], [30], [32], [39], and [42]. However, in minimizing data rate, quantized control bases the control decisions on estimation of state coordinates, which requires significant computational resources as well as sophisticated analysis tools.

Minimalism also appears in robotics research that vies for simple abstract sensors. Some earlier efforts are *bug algorithms* for navigation in environments with obstacles [19], [27] and algorithms for manipulating convex polygonal parts using sensorless robots [11], [13]. Recently, *gap navigation trees* for optimal navigation were introduced in [40]. In assuming weak sensors, these works typically equip the robots with sophisticated control laws. For example, [40] requires the robot to reliably move toward depth-map discontinuities.

In contrast to the aforementioned studies, our work has the distinguishing feature that we engage quantized control without assuming the availability of perfect coordinate information within sensing range. This aspect of our study promotes the understanding of the least amount of information or data rate required for a given task, which in turn offers insights into the task's inherent complexity. From a practical standpoint, minimalism in sensing or control leads to less complicated system design, improved robustness, lower production cost, and reduced energy consumption. Moreover, when physical size of agents is constrained, for instance in swarm robotics, sensing and computation become very limited, requiring a minimal design. To the best of our knowledge, there has been no attempt to reduce sensing and control, to the extremely limited combination considered in our paper, to complete tasks of broad research interest, such as rendezvous.

A central element of our approach is the Lyapunov function that we use, which assumes an unusual linear form (in distance) that simplifies the analysis. Lyapunov analysis has also been applied to study autonomous group coordination over graphs in [16], but that paper's approach requires the agents' full awareness of their local environments. Besides [16], distributed consensus has also been the focus of [7], [12], [29], and [33]. Our study is partly inspired by work on planning for a differential drive with a limited field-of-view [3], which does not, however, consider minimalism. This paper extends the results found in the conference version [43]. In particular, finer rendezvous guarantee is achieved in some cases, sometimes with relaxed requirements on vehicle model and/or sensing capability. The results presented in the paper are also demonstrated with simulation (see Fig. 1 and Section VII).

The paper is organized as follows. Section II gives a detailed formulation of the rendezvous problem that we solve. Section III introduces a graph structure with a connectivity condition as the first ingredient of the sufficient conditions for rendezvous. Section IV defines the candidate Lyapunov function and gives the second ingredient of the sufficient conditions, the windshield angle requirement, for agents moving at a single constant speed. The result is then generalized to agents with bounded, varying speeds in Section V. In Section VI, we provide the remaining condition on the angular velocity requirement, followed by extensions that remove the aforementioned temporary assumptions. In Appendix C, we show how our results specialize to the classic cyclic pursuit problem. Simulation results are presented in Section VII, after which we conclude with Section VIII.

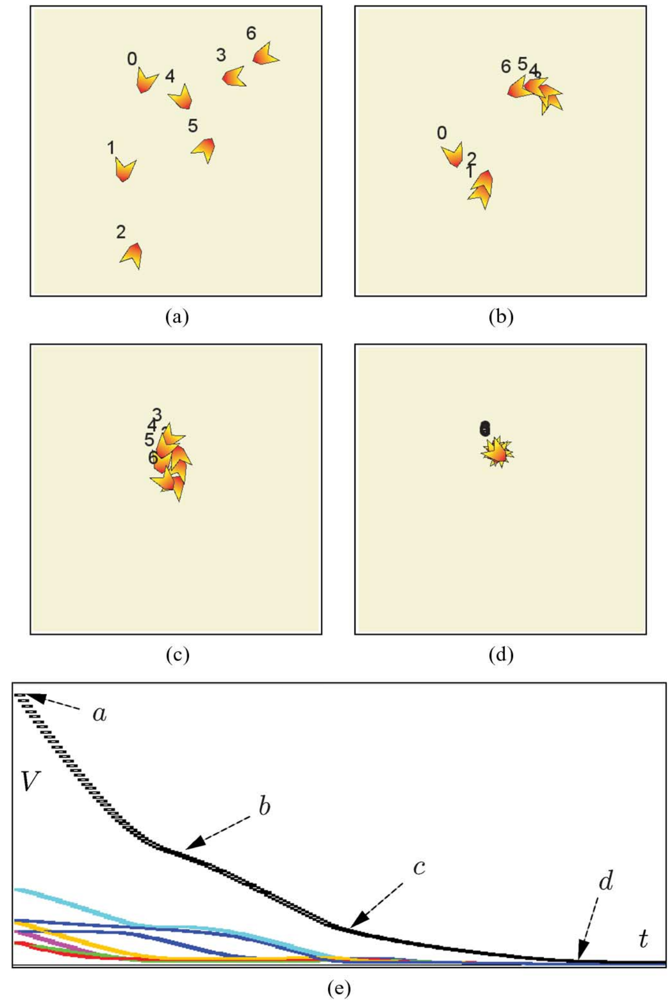

Fig. 1. Screen captures from simulation program of a seven agent cyclic pursuit with one agent running after another. (a)–(d) Four snapshots of the rendezvous process. (e) Plot of the Lyapunov function over time during the rendezvous process. The seven lower valued curves are individual distances between agents and their targets; the higher valued curve is the sum of the seven distances (the Lyapunov function), with (a)–(d) on the plot corresponding to the times when the four snapshots are taken.

II. Problem Statement

A. Vehicle Model

Consider a set of n *agents*, in which agent i is a point vehicle located at $p_i = (p_{i1}, p_{i2})$ in the plane with orientation ψ_i [see Fig. 2(a)]. Each vehicle moves as a Dubins car:

$$\dot{p}_{i1} = v_i \cos\psi_i \quad \dot{p}_{i2} = v_i \sin\psi_i \quad \dot{\psi}_i = u_i \tag{1}$$

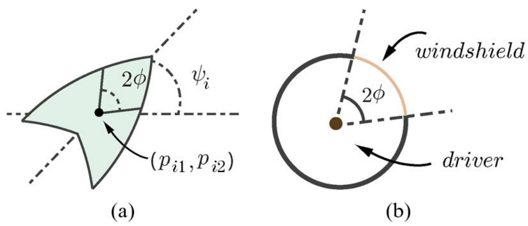

Fig. 2. (a) Agent's vehicle. (b) Agent's windshield, or field-of-view.

in which v_i is the forward speed, and the control is $u_i \in \{-\omega_i, 0, \omega_i\}$ for some fixed $\omega_i > 0$. For fixed v_i, such a vehicle either moves along a straight line ($u_i = 0$) or turns clockwise/counterclockwise ($u_i = \pm\omega_i$) along a circle with fixed radius. We use this fact later without further elaboration. Let $X = SE(2)^n$ denote the *state space*, in which $x \in X$ yields the position and orientation of all agents. Some of our results require that the agents are *identical*, which means $v_i = v_j$ and $\omega_i = \omega_j$ for all pairs, i, j, of agents.

B. Sensing Model and Control Law

The vehicle's sensor is a quantized variant of bearing-only sensors (e.g., see [36]). It has a limited angular field-of-view, centered at ψ_i with a span $(-\phi, \phi)$ for some given $\phi \in (0, \pi]$, which is the same and fixed for all agents in a system [see Fig. 2(b)]. By imagining that one sits in the driver's seat, the field-of-view can be considered as a *windshield*. The sensing range of each agent should be large enough to allow it to track its target until rendezvous occurs, which is certainly guaranteed if the range is unlimited. However, a bounded range is sufficient as a consequence of Proposition 10 and a precise bound can be calculated given the agents' initial configuration. Initially, we assume that the sensor can follow a target in the windshield: An agent cannot occlude another in terms of sensor view. This assumption will be lifted for identical agents in Section VI.

In the previous section we mentioned that agents will try to maintain their targets in the windshield; this appears to require an initial condition in which each agent has its target in the windshield. Assume such an initial condition for the moment; we later show that this requirement is not necessary in Section VI. For an agent i, let its target j initially reside in the $(-\phi, \phi)$ sector of i's windshield. Let ϵ_ϕ be a tiny angle satisfying $0 < \epsilon_\phi \ll \phi$. The introduction of ϵ_ϕ provides a way to maintain j in i's windshield of span $(-\phi, \phi)$: agent i can notice when j is about to disappear from i's windshield and start turning to position j towards the center of i's windshield. As long as i turns fast enough, j will not leave i's windshield. Assuming i has a single target j, the observation space can be restricted as $Y_i = \{-1, 0, 1\}$ and an observation y_i for agent i is obtained as

$$y_i = \begin{cases} 1 & j \text{ appears in } i\text{'s } [\phi - \epsilon_\phi, \phi) \text{ sector,} \\ 0 & j \text{ remains in } i\text{'s } (-\phi + \epsilon_\phi, \phi - \epsilon_\phi) \text{ sector,} \\ -1 & j \text{ appears in } i\text{'s } (-\phi, -\phi + \epsilon_\phi] \text{ sector} \end{cases} \quad (2)$$

which defines a simple instantaneous mapping $h : X \to Y_i$. While the mapping given by (2) is not defined on all of X, the part of X on which h is undefined is safe to ignore because of our temporary assumption that agents can keep targets in their windshield span. For each agent i, the sensor does not provide metric information, but instead indicates one of three simple quantized states with respect to some agent j and the windshield. Several possible implementations of the above sensing model are discussed in Appendix A. Our sensor-feedback control law $k_i : Y_i \to U_i$ is then defined simply as

$$u_i = \omega_i y_i. \quad (3)$$

Additionally, if an agent has no target, we assume that it does not move. We later remove this extra assumption for identical agents.

C. Merging

Since the Dubins car vehicle model has differential constraints which prevent the vehicle from turning arbitrarily fast, when two agents are chasing each other and get very close, they may not be able to keep each other in the windshield anymore. In such cases, they may keep turning to one side to circle each other forever. This phenomenon may also occur in general with n agents chasing one another in a cyclic fashion [28]. To resolve this, we introduce the assumption throughout the paper that once agent i and its target agent j are within a small predetermined distance ρ, they combine into a single agent. We call this operation *merging* and ρ the *merging radius*. A formal definition will be given in the next section after the concept of *assignment graph* is defined. If two agents have each other as targets, then merging is mutual, in which case they as a whole are considered as having no target and both stop moving. In practice, merging can be achieved by synchronizing the control for the involved agents such that all agents in a merged group follow the same control signal. This is possible without co-locating the agents as communication becomes feasible when agents are sufficiently close.

With the Dubins car vehicle model, three possible sensing outputs, three control inputs, and the merging assumption, we want to determine conditions under which agents are guaranteed to rendezvous, involving no state estimation using coordinates.

III. Assignment Graph, Liveness Condition, and Graph-Compatible Lyapunov Function

In this section we define a directed graph G called the *assignment graph*, in which each vertex is associated with an agent i having position p_i and orientation ψ_i. We then derive the connectivity condition that G must satisfy for the agents to rendezvous. Finally, we relate graph property and rendezvous via the notion of graph-compatible Lyapunov function. The several basic graph properties being used in the discussion can be found in [41].

To rendezvous, agents must move relative to each other in some way. We formalize this relationship as *assignment*: we say that agent i is *assigned to* agent j if j is i's target. We define the assignment graph G in an obvious way: G initially has n vertices, one for each agent, and there is a directed edge $e_{i,j}$ from i to j if and only if agent i is assigned to agent j. The set of edges of G is denoted $E(G)$. For an edge $e_{i,j}$, let $\ell_{i,j}$ denote the distance between agents i, j's positions in $\mathbb{R}^2$. With the introduction of the assignment graph, we define *merging* properly as: the merging of agent i into agent j is triggered if

$$e_{i,j} \in E(G), \qquad \ell_{i,j} \leq \rho. \tag{4}$$

After a merge of agent i reaching agent j, vertex i and edge $e_{i,j}$ are deleted from G. Any edge $e_{k,i}$ (or $e_{i,k}$) that existed in $E(G)$ before the merge is replaced by edge $e_{k,j}$ (or $e_{j,k}$), if such an edge does not already exist.

We say that an assignment graph G is *connected* (or weakly connected) if its underlying undirected graph has a single component. We say that G is *live* if it has at least one edge. If a graph only has a single vertex, we call it live by definition. It is desirable to maintain liveness in G at all times. If G is not live, then it has more than one vertex but no edges; it is then not possible for the system to rendezvous without additional assumptions. Since liveness is not preserved under merging, we need a stronger property from the initial graph. Assuming that an agent never loses its target prior to merging, this property is captured in the following lemma.

Lemma 1: The assignment graph G is live for all $t \geq 0$, under arbitrary evolution of the agent positions and merging, if and only if it is connected at time $t = 0$.

Proof: (If) A connected graph is live and the connectivity of the graph is preserved under merging. (Indeed, connectivity means that there exists a path from any vertex to any other vertex, and merging vertices does not destroy any paths.)

(Only If) If G is not connected, then it has more than one connected component. By merging we can collapse each connected component to a single point. This results in a graph with more than one vertex and no edges, which is not live. □

By Lemma 1, for a system of n agents to rendezvous, the assignment graph G for the system must have at least $n-1$ edges that connect all the vertices. If a connected graph with n nodes has $n-1$ edges, it is a tree. In particular, in an $n-1$ edge connected assignment graph, all vertices but one have a single outgoing edge, making the graph an *intree*, a directed acyclic graph in which all maximal directed paths point to a single vertex, its *root*. The agent at the root of the tree has no assigned target; hence, it does not move by the control law.

The condition to guarantee rendezvous for an intree assignment graph will turn out to be quite trivial. However, such a formation is asymmetric in the sense that exactly one of the agents (the root) has no assignment. Such an assignment is not possible to achieve in a decentralized manner. The simplest symmetrical assignment graph G has n assignments, one for each agent. We do not investigate the case in which one agent has more than one assignment at a given instant, since a more complicated control protocol will be required. Let us denote the assignment graph of our interest as *single-target* assignment graph. When G has n edges, the extra edge induces a cycle; therefore G has a single cycle on which each agent is assigned to the next one. The assignment also guarantees that any agent not on a cycle has a directed path to agents on the cycle (see Fig. 3). The behavior of the agents on the cycle is not affected by any agent not on the cycle. As we shall see later, this observation plays an important role in obtaining sufficient conditions for guaranteed rendezvous.

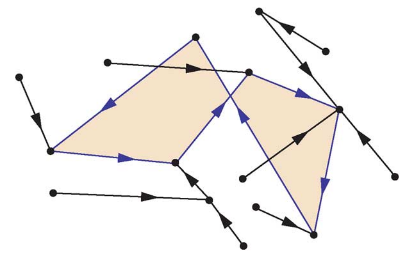

Fig. 3. Assignment graph, G, with n edges of which the underlying undirected graph is connected and contains a cycle (polygon) that encloses the shaded area.

Given an assignment graph G, we may define a function of the form

$$V = \sum_{e_{i,j} \in E(G)} V_{i,j} \tag{5}$$

in which $V_{i,j}$ depends only on the states of agents i and j. We say such a function is a *graph-compatible Lyapunov function* if it has the property that $V_{i,j} \geq 0$ and $V_{i,j} = 0$ if and only if $\ell_{i,j} = 0$ for all pairs of i, j. For such a Lyapunov function V, we say it is *rendezvous positive definite* if and only if $V = 0$ when all agents are in the same (unspecified) location, and $V > 0$ otherwise. This leads to the following

Lemma 2: A graph-compatible Lyapunov function V is rendezvous positive definite if and only if its assignment graph G is connected.

Proof: (If) It is clear that $V \geq 0$. Suppose that G is connected and $V = 0$. Since G contains a path connecting all agents, and since the lengths of all edges in this path must be zero, we have rendezvous.

(Only If) If G is not connected, then it has more than one connected component. If each connected component collapses to a single point, V becomes zero, even though we do not have rendezvous of all agents. □

As an alternative to the above direct proof, we could deduce Lemma 2 from Lemma 1: by liveness, the graph will have at least one edge (and thus V will be positive) as long as rendezvous does not occur.

IV. Guaranteed Rendezvous of Identical Agents

In this section we provide sufficient conditions that guarantee rendezvous for identical agents. Recall that agents are *identical* if all agents have the same speeds. Our results here are given for agents with unit speed ($v_i = 1$), which generalizes easily to agents with arbitrary but identical speeds via scaling. After introducing the candidate Lyapunov function, we first study cyclic and intree assignment graph cases separately before arriving at the most general result that combines the two cases. In the main body of this paper, we only include proofs of theorems that are essential in delivering the ideas of our approach; the rest of the proofs can be found in Appendix B.

A. A Candidate Lyapunov Function

Based on an assignment graph G, we define a candidate Lyapunov function $V : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$V(x) = \sum_{i,j \in E(G)} \ell_{i,j}. \tag{6}$$

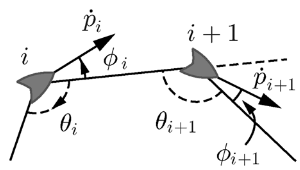

Fig. 4. Two Dubins car agents in a cyclic pursuit.

This definition of V is insensitive to edge directions in G. Unlike more conventional quadratic Lyapunov functions, this specific V takes a form linear in $\ell_{i,j}$, which means that the corresponding time derivative is not linear in $\ell_{i,j}$ or p_i. The reason for the choice of the Lyapunov function will become clear in the next subsection. When a merge happens, we assume that the distance between the two merged agents is frozen, which is achievable by synchronizing their control. This gives us a continuous V. Alternatively, we could also let the two merged agents have zero distance, which would induce a jump in V. Since a system with n agents only has up to $(n-1)$ merges, these discontinuities would not affect the overall rendezvous behavior of the system.

The function V defined in (6) is clearly a graph-compatible Lyapunov function. By Lemma 2, it is rendezvous positive definite if and only if G is connected. We may then study the behavior of this Lyapunov function over single-target assignment graphs to derive sufficient conditions for rendezvous of the system. For moving agents, V can be considered as a function of time; its time derivative is then

$$\dot{V} = \sum_{i,j \in E(G)} \dot{\ell}_{i,j}. \tag{7}$$

B. Cyclic Case

We start the analysis with cyclic pursuit: the n edges of G form a single polygon. Given the Dubins car model (1) and our control law (3), for two consecutive agents i and $i+1$ on the polygon, let θ_i be either of the two angles of the polygon at vertex i. For example, we may use the internal angles for simple (non-self-intersecting) polygons. In the special case of $n=2$, $\theta_i = 0$. Let ϕ_i be the angle between $\dot{p}_i$ and the line segment from p_i to p_{i+1} (see Fig. 4). Note that ϕ_i is not the same as ϕ, half of the windshield span, which is fixed and the same for all i. The angle ϕ_i is positive if θ_i and ϕ_i start from different sides with respect to the ray $\overrightarrow{p_i p_{i+1}}$. The directions in which θ_i, ϕ_i increase are marked with arrows in Fig. 4.

For a specific $\ell_{i,i+1}$, agent i's movement will cause it to shorten at a rate of $v_i \cos\phi_i$; agent $i+1$'s movement will cause it to lengthen at a rate of $v_{i+1}\cos(\pi - (\theta_{i+1} + \phi_{i+1})) = -v_{i+1}\cos(\theta_{i+1} + \phi_{i+1})$. The derivative of $\ell_{i,i+1}$ is then

$$\begin{aligned} \dot{\ell}_{i,i+1} &= \left\langle \dot{p}_{i+1} - \dot{p}_i, \frac{p_{i+1} - p_i}{|p_{i+1} - p_i|} \right\rangle \\ &= -\, v_i \cos\phi_i - v_{i+1}\cos(\theta_{i+1} + \phi_{i+1}). \end{aligned} \tag{8}$$

After summing up (8) for all i and rearranging, we have for the cyclic case

$$\dot{V} = \sum_i -v_i \left(\cos\phi_i + \cos(\theta_i + \phi_i)\right). \tag{9}$$

For identical agents with unit speed $(v_i = 1)$, (9) becomes

$$\dot{V} = \sum_i -\left(\cos\phi_i + \cos(\theta_i + \phi_i)\right). \tag{10}$$

We want to keep $\dot{V}$ negative at all times prior to rendezvous. From (10) we get the following.

Lemma 3: For any integer $n \geq 2$, the windshield angle $\phi > \pi/n$ permits trajectories for which $\dot{V} > 0$.

By Lemma 3, for any $n \geq 2$ and $\phi > \pi/n$, pursuit cycles exist for which $\dot{V} \geq 0$. We are now ready to give a sufficient condition for rendezvous.

Theorem 4: Unit speed cyclic pursuit of n Dubins car agents will rendezvous if the agents maintain their targets in the windshields of span $(-\phi, \phi)$ with

$$0 < \phi \leq \begin{cases} \frac{\pi}{2}, & n = 2 \\ \cos^{-1}\frac{n-1}{n}, & n = 3, 4 \\ \frac{\pi}{n}, & n \geq 5. \end{cases} \tag{11}$$

The problem of how the agents can maintain their targets in a given windshield span, which is addressed with Proposition 16, is not part of this theorem. We proceed to prove Theorem 4 by first introducing several lemmas; essentially we want to show that if ϕ satisfies (11), then $\dot{V} < 0$. We may then apply a Lyapunov theorem to conclude. To facilitate the discussion, define the first and second terms of $\dot{V}$ in (10) as

$$\begin{aligned} h(\Phi) &= h(\phi_1, \ldots, \phi_n) \\ &:= \sum_i -\cos\phi_i, \\ f(\Theta, \Phi) &= f(\theta_1, \ldots, \theta_n, \phi_1, \ldots, \phi_n) \\ &:= \sum_i -\cos(\theta_i + \phi_i) \end{aligned} \tag{12}$$

and we have

$$\dot{V} = h(\Phi) + f(\Theta, \Phi). \tag{13}$$

For notational convenience, we use f in place of $f(\Theta, \Phi)$ and h in place of $h(\Phi)$ when it is appropriate to do so. By boundedness of the cosine function, $-n \leq h, f \leq n$. Since $0 \leq |\phi_i| < \phi$ for all i, h can be made arbitrarily close to $-n$ by lowering ϕ. Therefore, if for every fixed n, there exists some $\delta_n > 0$ such that $f < n - \delta_n$, some small ϕ can be chosen to make $h < -n + \delta_n$ to obtain $\dot{V} = h + f < 0$. The bound $\phi \leq \pi/2$ suffices for $n = 2$, which is straightforward to verify. Hence, we work with some $n \geq 3$ and first consider the case in which the pursuit cycle of the agents is a simple (non-self-intersecting) polygon; the self-intersecting polygon case then follows similarly. Recall that Lemma 3 suggests that we need ϕ to be no more than π/n. As we assume that $|\phi_i| < \phi$ for all i, we need

$$-\pi/n < -\phi < \phi_i < \phi < \pi/n \tag{14}$$

adding $0 < \theta_i < 2\pi$ gives

$$-\frac{\pi}{n} < (\theta_i + \phi_i) < 2\pi + \frac{\pi}{n}. \tag{15}$$

We partition the θ_i's satisfying (15) into two disjoint sets

$$\Theta_o=\left\{(\theta_1,\ldots,\theta_n)|\exists i, \theta_i+\phi_i \in\left(-\frac{\pi}{n},\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi+\frac{\pi}{n}\right)\right\}$$
$$\Theta_i=\left\{(\theta_1,\ldots,\theta_n)|\forall i, \theta_i+\phi_i \in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right\}. \quad (16)$$

On Θ_o (the subscript o stands for "outside"), for at least one i, $-\cos(\theta_i+\phi_i)<0$. We immediately have $f<n-1$ on Θ_o. The following lemma tells us how f behaves on Θ_i (the subscript i stands for "inside").

Lemma 5: Unit speed cyclic pursuit of n Dubins car agents with simple polygon pursuit cycle and satisfying (14) has the property that $f(\Theta,\Phi)$ has a single stationary point on the interior of Θ_i.

Proof: If we fix ϕ_i's, then f, as a linear combination of cosine functions of θ_i's, is bounded and continuous. Recall that for simple polygons, we may use the internal angles as θ_i's. A simple polygon has the property that its internal angles sum up to $(n-2)\pi$ [35], which gives us

$$g(\Theta)=g(\theta_1,\ldots,\theta_n):=\sum_i \theta_i=(n-2)\pi. \quad (17)$$

Again, we use g in place of $g(\Theta)$ for notational convenience. Both f and g are analytic functions over the θ_i's. We may use $g-(n-2)\pi=0$ as the equality constraint to apply the method of Lagrange multipliers [26] over f. The method produces the Lagrangian

$$\begin{aligned}\Lambda(\theta_1,\ldots,\theta_n,\lambda)&=f(\theta_1,\ldots,\theta_n,\phi_1,\ldots,\phi_n)\\&\quad-\lambda\left(g(\theta_1,\ldots,\theta_n)-(n-2)\pi\right)\\&=-\sum_i\cos(\theta_i+\phi_i)\\&\quad-\lambda\left(\sum_i\theta_i-(n-2)\pi\right). \quad (18)\end{aligned}$$

The possible stationary points of f satisfy $\nabla_{\theta_1,\ldots,\theta_n,\lambda}\Lambda=0$, or equivalently

$$\sin(\theta_i+\phi_i)=\lambda, \quad \text{for all } i. \quad (19)$$

From (14) and (17), we have

$$(n-3)\pi<\sum_i(\theta_i+\phi_i)<(n-1)\pi. \quad (20)$$

As a side note, the Θ_i slice may be empty when $n\le 5$ by the first inequality of (20). For any $n\ge 3$, by the pigeonhole principle [41], for at least one i, $(\theta_i+\phi_i)$ must be in the range $(0,((n-1)/n)\pi)$, which means that for that i, $\sin(\theta_i+\phi_i)=\lambda>0$. By (19), for all i, $\sin(\theta_i+\phi_i)=\lambda>0$. This forces f to have a single stationary point on the interior of Θ_i. □

Lemma 6: At the stationary point in Lemma 5,we have

$$f(\Theta,\Phi)\le -n\cos\frac{(n-2)\pi+n\phi}{n}. \quad (21)$$

Lemma 7: Unit speed cyclic pursuit of n Dubins car agents with simple polygon pursuit cycle has the property

$$f(\Theta,\Phi)\le\max\left\{-n\cos\frac{(n-2)\pi+n\phi}{n},n-1\right\} \quad (22)$$

and $\dot V<0$, if the agents maintain their targets in the windshields of span $(-\phi,\phi)$ with ϕ satisfying (11).

The key property making the proof of Lemma 7 work is that the internal angles of any simple polygon in the plane sum up to $(n-2)\pi$, which is less than $n\pi$. We can then choose ϕ to make $\sum(\theta_i+\phi_i)$ less than $n\pi$ and the pigeonhole principle guarantees that some $(\theta_i+\phi_i)$ will be less than π. In turn, it is guaranteed by the method of Lagrange multipliers that for all i, $(\theta_i+\phi_i)$ must take the same value λ and must be less than π for f to take extreme values on the Θ_i slice. Combining the Θ_i, Θ_o slices then gives the result. The same technique can be applied when the polygon is not a simple one.

Lemma 8: Unit speed cyclic pursuit of n Dubins car agents with self-intersecting polygon pursuit cycle has the property $\dot V<0$ if the agents maintain their targets in the windshields of span $(-\phi,\phi)$ with ϕ satisfying (11).

Proof of Theorem 4: Having proved that the agents may choose a windshield span satisfying (11) to ensure $\dot V<0$, by the standard Lyapunov theorem on asymptotic stability with respect to a set (e.g., see [24]), all agents will rendezvous. The attractive set here is the "diagonal" in $\mathbb{R}^{2n}$, in fact, its ρ-neighborhood in which ρ is the merging radius. Note that the introduction of ρ also addresses the issue that our V is not differentiable when some agents are in the same location; however, the result is valid even without this regularization (because a Lyapunov function with respect to an invariant set need not be differentiable on that set itself [24]). □

Once ϕ is fixed, the right side of (22) is determined, which leads easily to the existence of some $\delta_\phi>0$ for which $\dot V<-\delta_\phi$ for all time $t>0$. This yields the following.

Corollary 9: The system in Theorem 4 achieves rendezvous in finite time.

It is natural to ask whether the system is stable in the sense of Lyapunov: will some agents get arbitrarily far away from the rest during the converging process? The answer is no. Denoting V_0 as the value of V at $t=0$, we formalize the notion with the following proposition.

Proposition 10: For the system in Theorem 4, $\dot V\le 0$ ensures that all agents are inside a bounding disc with radius at most $V_0/2\sqrt{3}$ for all time $t\ge 0$.

Proof: In a cyclic pursuit, the candidate Lyapunov function, V, is exactly the circumference of the pursuit cycle. Given any two different agents i,j on the pursuit cycle, there are two disjoint, undirected paths from i to j. One of these two paths must be no more than $V/2$ in total length and by repeated application of the triangle inequality, the straight line distance between i,j cannot exceed $V/2$. The $2D$ version of Jung's theorem [17], [18] then tells us that there exists a bounding circle of the point set of all agents with radius no more than $V/2\sqrt{3}$. Since $\dot V\le 0$, $V\le V_0$ for all $t\ge 0$. Hence, there is no blowup prior to convergence (i.e., the system is stable in the sense of Lyapunov). □

C. Intree Case

When identical agents engage in a pursuit with an intree assignment graph, since there is a stationary agent, it is relatively simple to guarantee that $\dot V<0$ by choosing an appropriate ϕ.

Lemma 11: Unit speed pursuit of n Dubins car agents with an intree assignment graph has the property $\dot{V} < 0$ and will rendezvous in finite time if the agents maintain their targets in the windshields of span $(-\phi, \phi)$ with $0 < \phi < \cos^{-1}((n-2)/(n-1))$.

Proof: When $n = 2$, one agent is stationary and one agent is moving; $\phi < \pi/2$ guarantees $\dot{V} < 0$. When $n \geq 3$, for an edge $e_{i,j}$ in an intree, we may express $\dot{\ell}_{i,j}$ similarly as (8) (note the speeds are all 1)

$$\dot{\ell}_{i,j} = -\cos\phi_i - \cos(\theta_j + \phi_j). \tag{23}$$

When the target agent j is the root of the intree, the second term in (23) is zero since the root has no target and does not move, i.e., $\dot{\ell}_{i,j} = -\cos\phi_i$ for at least one edge $e_{i,j}$ of the assignment graph. Let this edge be $e_{k,m}$. Summing over all assignment edges yields

$$\dot{V} = -\cos\phi_k + \sum_{e_{i,j}\in E(G), e_{i,j}\neq e_{k,m}} -\left(\cos\phi_i + \cos(\theta_j + \phi_j)\right). \tag{24}$$

$\dot{V} < 0$ is equal to

$$\cos\phi_k > \sum_{e_{i,j}\in E(G), e_{i,j}\neq e_{k,m}} -\left(\cos\phi_i + \cos(\theta_j + \phi_j)\right). \tag{25}$$

Assuming $0 < \phi < \pi/2$, $\cos\phi_k > \cos\phi$ and $-\cos\phi + 1 > -(\cos\phi_i + \cos(\theta_j + \phi_j))$, (25) is guaranteed by

$$\cos\phi > (n-2)(1 - \cos\phi) \tag{26}$$

which is equivalent to $0 < \phi < \cos^{-1}((n-2)/(n-1))$. The finite time guarantee follows the argument from Corollary 9. □

D. Cycle Plus Branches

Given an arbitrary connected, single-target assignment graph with a cycle plus some branches, Theorem 4 and Lemma 11 ensure that the whole system will rendezvous in a *sequential* manner, First, the agents on the cycle will rendezvous and merge into a single stationary agent. The rest of the agents will then merge into the stationary agent. We call this type of rendezvous *sequential rendezvous*. So far, however, there is no guarantee that the entire system has $\dot{V} < 0$ all the time. Therefore, no equivalent of Proposition 10 can be stated for an arbitrary connected, single-target assignment graph yet, which means that such a system may not be stable in the sense of Lyapunov. In this subsection we show that it is possible to guarantee $\dot{V} < 0$ at all times for identical agents.

Theorem 12: In unit speed pursuit of n Dubins car agents with arbitrary connected, single-target assignment graph, if the agents maintain their targets in the windshields of span $(-\phi, \phi)$ with fixed ϕ satisfying

$$0 < \phi < \min\left\{\frac{\pi}{5}, \frac{\pi}{n}\right\} \tag{27}$$

then all agents remain in a disc dependent on V_0 and rendezvous in finite time.

Proof: We only need to show that $\dot{V} < 0$ always holds under the condition (27), the same arguments from Corollary 9 and Proposition 10 then give us the rest. Lemma 8 and Lemma 11 cover the single cycle and intree assignment graphs, which leaves only the case of a cycle plus some branches (see Fig. 3). For any connected, single-target assignment graph with n agents, there is a single cycle. Assume that there are k agents on that cycle. Let V_k denote the corresponding Lyapunov function for the k-cycle and let $\phi < \pi/n$. Then for the agents on the cycle

$$\dot{V}_k = h_k + f_k < -\sum_{i=0}^{k}\cos\frac{\pi}{n} + \sum_{i=0}^{k}\cos\frac{\pi}{k} = k\left(\cos\frac{\pi}{k} - \cos\frac{\pi}{n}\right) \tag{28}$$

holds for $k \geq 5$. Here h_k, f_k are defined similarly as h, f before. For the agents not on the cycle, we have

$$\dot{\ell}_{i,j} = -\cos\phi_i - \cos(\theta_j + \phi_j) \leq 1 - \cos\phi_i \leq 1 - \cos\frac{\pi}{n}. \tag{29}$$

Summing up all the terms for the cycle plus $(n-k)$ other agents not on that cycle, we have

$$\begin{aligned}\dot{V} &< k\left(\cos\frac{\pi}{k} - \cos\frac{\pi}{n}\right) + (n-k)\left(1 - \cos\frac{\pi}{n}\right)\\ &= -k\left(1 - \cos\frac{\pi}{k}\right) + n\left(1 - \cos\frac{\pi}{n}\right).\end{aligned} \tag{30}$$

We want to show that the right-hand side above is negative at all times. We may do this by showing that the function

$$f_1(x) = x\left(1 - \cos\frac{\pi}{x}\right) \tag{31}$$

decreases monotonically for $5 \leq x < \infty$. This can be verified by showing that $\dot{f}_1 < 0$ on the domain. When $k = 2, 3, 4$, $\dot{V}_k$ is less than -1 by direct computation; hand checking shows that (27) guarantees $(n-k)(1 - \cos\phi_i) < 1$, which in turn guarantees $\dot{V} < 0$. □

V. Guaranteed Rendezvous of Agents With Bounded, Varying Speeds

In this section, we generalize the results for identical agents by removing the restriction that requires all v_i's to be identical. We say that the speed v_i of agent i is *bounded* if $0 < v_{\min} \leq v_i \leq v_{\max} < +\infty$ for some constants $v_{\min}$ and $v_{\max}$. The velocity v_i may change over time. When all agents' speeds in a pursuit are bounded, we say the pursuit is a *bounded speed pursuit*.

Theorem 13: Bounded speed cyclic pursuit of n Dubins car agents will rendezvous in finite time if the agents maintain their targets in the windshields of span $(-\phi, \phi)$ with

$$\phi < \cos^{-1}\frac{nv_{\max} - v_{\min}\left(1 - \cos\frac{\pi}{n}\right)}{nv_{\max}}. \tag{32}$$

Proof: For the proof we work with (9) and use the approach in the proof of Lemma 7. The simple polygon case is covered here; the proof for the self-intersecting polygon case then follows that of Lemma 8 similarly, which we do not repeat.

Let $\mathcal{V}$ represent the agents speeds $v_1, \ldots, v_n$ and define f, h, and g as

$$\begin{aligned} f(\Theta, \Phi, \mathcal{V}) &:= \sum_i -v_i \cos(\theta_i + \phi_i) \\ h(\Phi, \mathcal{V}) &:= \sum_i -v_i \cos\phi_i \\ g(\Theta) &:= \sum_i \theta_i. \end{aligned} \tag{33}$$

The structures, Θ_o, Θ_i, and the hyperplane $g - (n-2)\pi = 0$ from Lemma 7 remain the same. For the Θ_i slice by the hyperplane, applying the method of Lagrange multipliers to f as a function of θ_i's with constraint $g - (n-2)\pi = 0$ yields that for all i

$$v_i \sin(\theta_i + \phi_i) = \lambda. \tag{34}$$

Once again, holding ϕ_i's fixed, for f to take maximum on the Θ_i slice, for all $i v_i \sin(\theta_i + \phi_i)$ must take the same value and therefore, must be positive if we keep $\phi < 2\pi/n$. Let us assume that we pick some $\phi < \pi/n$, then

$$\sum_i (\theta_i + \phi_i) < (n-2)\pi + n\frac{\pi}{n} = (n-1)\pi. \tag{35}$$

By the pigeonhole principle, for at least one k, $(\theta_k + \phi_k) < (n-1)\pi/n$. Therefore

$$\begin{aligned} f(\Theta, \Phi) &= \sum_i -v_i \cos(\theta_i + \phi_i) \\ &< \sum_{i \neq k} v_i - v_k \cos\frac{(n-1)\pi}{n} \\ &= \sum_i v_i - v_k \left(1 - \cos\frac{\pi}{n}\right) \\ &\le \sum_i v_i - v_{\min}\left(1 - \cos\frac{\pi}{n}\right). \end{aligned} \tag{36}$$

To make $f + h < 0$, we need $-h > f$, which is true if

$$\sum_i v_i \cos\phi_i > \sum_i v_i - v_{\min}\left(1 - \cos\frac{\pi}{n}\right). \tag{37}$$

One way to satisfy this is to make sure that for each i

$$v_i \cos\phi > v_i - \frac{v_{\min}}{n}\left(1 - \cos\frac{\pi}{n}\right) \tag{38}$$

or equivalently

$$\phi < \cos^{-1} \frac{v_i - \frac{v_{\min}}{n}\left(1 - \cos\frac{\pi}{n}\right)}{v_i}. \tag{39}$$

The right side of (39) achieves the global minimum when $v_i = v_{\max}$, which gives us (32) as a sufficient condition for $\dot{V} < 0$ on the Θ_i slice. On the Θ_o slice, we have that $f \le \sum v_i - \min\{v_i\}$, which is less than the last expression in (36); therefore, (32) also works for the Θ_o slice. The finite time guarantee follows the argument from Corollary 9. □

Moving to the intree case, when agents have different speeds, no equivalent of Lemma 11 can be stated since $\dot{V} < 0$ can no longer be guaranteed. A simple example is illustrated in Fig. 5. Agent r is at the root and does not move. Suppose agent i moves very fast and agents (j, k in the figure) following i barely move. Also assume that all agents are almost collinear. It is straightforward to see that, after a short period of time (the second drawing), the sum of the length of all edges, or V, increases. This suggests that $\dot{V}$ must be positive at some point. However, such a system will still rendezvous. Supposing that the stationary agent is r, at least one agent, say i, is assigned to r. Thus, $\dot{\ell}_{i,r} = -v_i \cos\phi_i < 0$ whenever $\phi < \pi/2$. Hence, agent i will merge into agent r in finite time, and all other agents will eventually follow. We have proved:

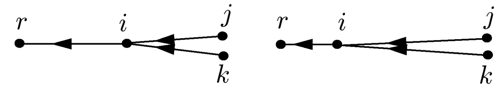

Fig. 5. Bounded speed intree pursuit may not have $\dot{V} < 0$ at all times.

Lemma 14: Bounded speed pursuit of n Dubins car agents with an intree assignment graph will merge into the stationary agent in finite time if the agents maintain their targets in the windshields of span $(-\phi, \phi)$ with $\phi < \pi/2$.

Since it is not possible to guarantee $\dot{V} < 0$ at all times for the bounded speed case, a result like Theorem 12 is out of the question. However, since Theorem 13 and Lemma 14 parallel Theorem 4 and Lemma 11, the argument giving us sequential rendezvous in the identical agents case continues to hold:

Theorem 15: Bounded speed pursuit of n Dubins car agents with arbitrary connected, single-target assignment graph will rendezvous in finite time if the agents maintain their targets in the windshields of span $(-\phi, \phi)$ with fixed ϕ satisfying (32).

VI. Condition on Angular Velocity and Extensions

A. Angular Velocity Condition

So far, all of our results assume that agents must maintain their targets in some appropriate $(-\phi, \phi)$ range. We now show that agents do not need to rotate arbitrarily fast to achieve this by providing an upper bound on the rotational speed for the bounded speed pursuit case; the result readily generalizes to other cases.

Proposition 16: In bounded speed pursuit of n Dubins car agents with windshield span $(-\phi, \phi)$, for any $\rho > 0$, selecting

$$\omega_i > \frac{v_{\max}}{\rho} \tag{40}$$

is sufficient for an agent to either keep its target within its windshield or merge with its target.

B. Rendezvous Without Connectivity or Distinguishability

In the set of sufficient conditions stated in our earlier theorems that guarantees rendezvous, we have assumed that the assignment graph G has a single connected component initially. We now show that the assumption of single connected assignment graph and agents stopping are not necessary by: 1. Allowing agents to perform *reassignment*. 2. Allowing agents without targets to merge with other agents within distance ρ. Initially, the assignment graph G has n components: each component contains a single agent. Look at one such component G_i with a single agent i. If there are other agents within distance ρ of i, they will merge with i, therefore, we may assume that there are no other agents within distance ρ of agent i. To rendezvous with other agents, let agent i pick any target in its windshield and if it cannot find any, let it start turning. We will show that with two full turns, it can find agents at least ρ away from it or confirm that there are no such agents to conclude rendezvous is achieved. We call this procedure of finding new targets *reassignment*. We have the following:

Proposition 17: By allowing merging and reassignment for agents that have no targets, bounded speed pursuit of n Dubins car agents will achieve rendezvous without the initial connectivity requirement, provided that the system will rendezvous if given a connected initial assignment graph and

$$\omega_i \geq \frac{3v_{max}}{\rho}. \tag{41}$$

Our sensing model (see Appendix A) may do without agents being distinguishable if agents are capable to take continuous videos of their targets in windshield. For agents moving at unit speed, such capability is not necessary provided that when an agent is confused by two possible targets that are collinear with it, it can follow the target that is closer. Such switching is a natural one since a closer target will block a more distant one, effectively removing the assumption requiring no visibility occlusion. Alternatively, if all agents simply send out a beacon signal, a closer agent will have a stronger signal than a more distant one. We have:

Proposition 18: By allowing merging and reassignment for agents that have no targets, as well as the ability for an agent to choose a closer agent as target when two possible targets become collinear with it, unit speed speed pursuit of n Dubins car agents will achieve rendezvous with $\omega_i \geq 3/\rho$ and windshield angle satisfying (27), without initial connectivity or agent distinguishability.

Proof: Denoting agent i and its two possible targets t_{near} and t_{far}, when such switching of targets happens, V of the system cannot get larger since the affected term in V during a target switching will have $l_{i,t_{near}} \leq l_{i,t_{far}}$ and $l_{i,t_{near}}$ is always chosen. At the same time, Theorem 12 guarantees that $\dot{V}$ remains negative. □

Proposition 18 also implies that agents will no longer need to stop before they all rendezvous.

VII. Implementation and Simulation Results

A Java implementation was written to verify the theoretical developments, which focuses on the two key scenarios of rendezvous: assignment graph with a single cycle and assignment graph with a cycle plus branches. The program[1] was developed adhering to the Java 5 language standard under the Eclipse environment.

Discretization is necessary to implement the continuously evolving system. Sensing needs to be discretized because the Dubins car model allows jumps in angular velocity; therefore, there could be a large number of control switches within a short time interval. In our implementation, we divide the windshield into a left sector and a right sector. The triggering event is simulated by remembering the latest observation of the target agent j's relative position in pursuing agent i's windshield at time step k

$$y_i[k] = \begin{cases} -1 & \text{agent } j \text{ is in the left sector} \\ 0 & \text{agent } j \text{ is out of view} \\ 1 & \text{agent } j \text{ is in the right sector} \end{cases} \tag{42}$$

and comparing that with the next observation. If the agent is near the center of the windshield, we can give it 1 or -1. The quantized control can be encoded as

$$u_i[k] = \begin{cases} \omega & y_i[k] = 0, y_i[k-1] = -1 \\ 0 & y_i[k] \in \{-1, 1\}, y_i[k-1] = 0 \\ -\omega & y_i[k] = 0, y_i[k-1] = 1 \\ u_i[k-1] & \text{otherwise} \end{cases} \tag{43}$$

with $u_i[0] = 0$ by assumption. This control law can be expressed as an equivalent four state automaton. The implementation of an agent is therefore also quite simple, which enables the simulation to scale well with respect to the number of agents in the system. Plotting the Lyapunov function for systems with more than 10 000 agents can be handled quickly on our computer with a small memory footprint; smooth animation is possible for 100 agents, with the Lyapunov function and each $\ell_{i,i+1}$ plotted simultaneously in a separate window.

To verify the cyclic pursuit theorems for the Dubins car model, each agent is assigned a unique integer identifier. For the single cycle case, we initialize the system by positioning the agents randomly and assigning each agent to the one with a preceding identifier; the first agent is assigned to the last one. The behavior of the cyclic pursuit is then observed by watching the agents' trajectories and a simultaneously drawn plot of the Lyapunov function, under different parameter settings. Since we want to observe the behavior of the cyclic pursuit with all agents, the merging radius ρ is set to near zero to inhibit merging. For every attempted set of n agents, the simulation indicates that $\phi = \pi/n$ is a tight rendezvous bound. As predicted, randomly positioned agents rendezvous when $\phi < \pi/n$, but they will eventually diverge if ϕ is only 1% above π/n. In the latter case, the agents get closer quickly in the beginning as random arrangements appear to induce more negative $\dot{V}$. Interestingly, the agents then tend to arrange themselves on a circle and eventually form a regular polygon, making $\dot{V}$ positive. We speculate that the continuous system has the same behavior, which is shown in [28] and [31] with different motion primitive and/or control law. Fig. 6(a)–(d) shows both a converging and a diverging case with $n = 8$. In the rendezvous case, V does not go to zero because ρ is set to be very small. The case with cycle plus branches is implemented similarly: we make a random cycle (with less than n agents) and then attach the rest of the agents to the cycle in a random way. Its simulation behavior [a converging example is shown in Fig. 6(e) and (f)] is very similar to the single cycle case.

VIII. Conclusion and Future Directions

We have pursued a minimalist approach to a multi-agent rendezvous problem. Using a simple agent model with a ternary output sensor, a three level quantized control, and a Dubins car model, we have shown that a group of these constrained agents

[1]The simulation program is fully accessible as a Java applet through Java enabled web browsers at http://msl.cs.uiuc.edu/~jyu18/pe/rendezvous.html.

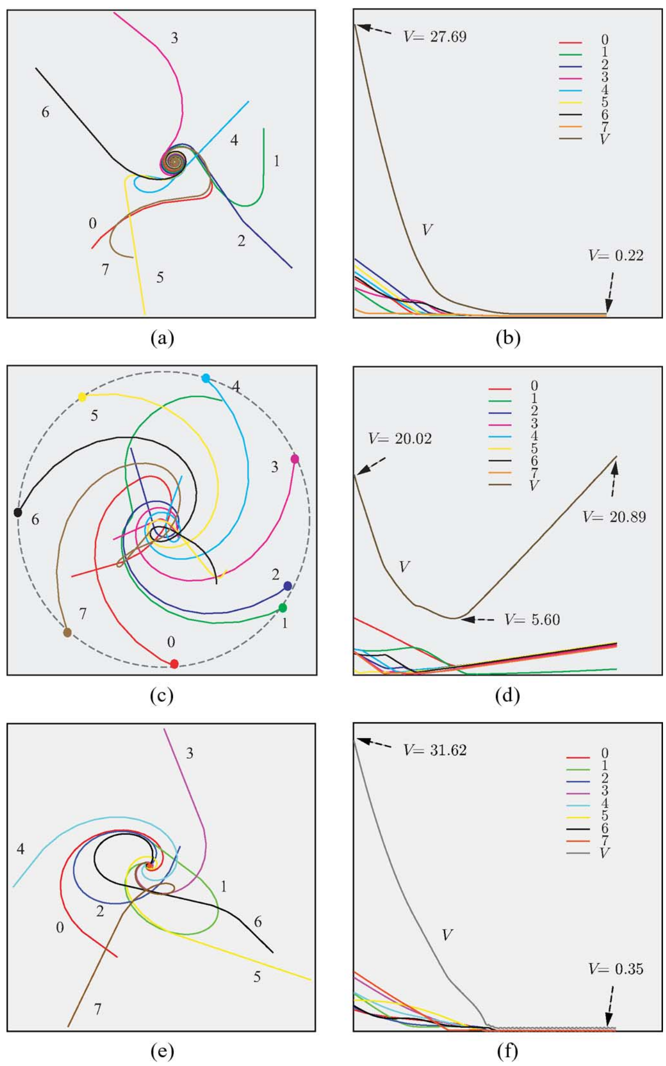

Fig. 6. (a) Eight-agent cyclic pursuit, which converges, with $\phi = 0.35$ and $\omega = 2.0$ (b) Plot of the Lyapunov function for (a), with numbered lines being individual $\ell_{i,i+1}$ terms. (c) Diverging case with ϕ changed to 0.6 holding other conditions the same. The agents arrange themselves on a cycle quickly as they spin away from each other. Dots are the agents' positions as of the last snapshot. (d) Plot of the Lyapunov function for (c). (e) Eight-agent converging pursuit with agents 0, 4, 3 and 5 on a cycle; the branches are: $6 \to 2 \to 0, 7 \to 3, 1 \to 5$. (f) Plot of the Lyapunov function for (e).

can still achieve guaranteed rendezvous behavior without coordinates, communication, or even distinguishability. Furthermore, by exploiting the geometric constraints on internal angles of polygons, we have shown that the convergence behavior of the system is precisely predictable based on the parameters of the individual agents. Our simulation indicates that the bounds derived in the theorems may be close to tight. Moreover, when simulated sensing error is introduced by providing the vehicles with randomly wrong feedback 20% of the time, the simplistic system still behaves as predicted in simulation.

An interesting problem arises from our investigation of the rendezvous problem: For agents on a pursuit cycle, a regular polygon appears to be the "preferred" formation (the agents seem to form it without much effort). Although we are able to avoid global dynamics arguments used in [28], a better understanding of how our system evolves over time will help explain why this is the case. This understanding could also lead to more accurate lower bounds on the time that rendezvous takes for a given arrangement of agents. Another related open problem is prescribing the location of rendezvous, which is theoretically appealing and useful for practical purposes.

Going beyond the paper, we want to approach the following questions. 1) Is it possible for an even simpler agent model to rendezvous? By simpler we mean that one or more of sensing and control are strictly less powerful, holding the rest of the agent model unchanged. 2) Are there any other tasks achievable with similar simple agents? For example, we see that it is possible for the agents to get into clusters; can they form a regular lattice structure? Can we get them to follow prescribed paths up to homotopy?

Even though we focus on the rendezvous task in this paper, our motivation in this work lies with a more general goal: Investigating what task classes are possible with minimal amount of information. For a given task, there seems to be an intrinsic relation among the required strengths of the sensors and the controller of an agent. For example, an agent can move to and touch an object to learn its shape; alternatively, it can take a picture and extract the same information. Thus, there must be some equivalence between those two agent models. A firm grasp of this relation will not only help pin down the most basic requirements for a given task, but also offer powerful design guidance for better autonomous systems.

Appendix A
Possible Sensor Implementations

To implement the sensor based strictly on the instantaneous mapping, we may take continuous video of the entire $(-\phi, \phi)$ windshield sector with markers at $-\phi + \epsilon_\phi$ and $\phi - \epsilon_\phi$. It is then possible to directly locate agent j in the windshield from each snapshot. This implementation also enables i to track j via j's continuous motion trajectory since j is always in the video sequences, even if all agents are indistinguishable based on appearance. Moreover, a sensor in reality does not need to decide ϵ_ϕ: a physical vehicle has an actual size hence will not disappear from the windshield at a single instant in time. Suppose agent j starts to move outside of i's field-of-view; j will actually "stay" on i's windshield boundary for some Δt time before it disappears from i's view, which automatically provides ϵ_ϕ for agent i. This sensor implementation is illustrated in Fig. 7(a): arcs A_l, A_c, A_r correspond to the $(-\phi, -\phi+\epsilon_\phi), (-\phi+\epsilon_\phi, \phi-\epsilon_\phi), (\phi-\epsilon_\phi, \phi)$ sectors of the windshield, respectively. In fact, the left boundary of A_l and the right boundary of A_r in the sensor implementation are not essential; they can be extended and in the extreme case meet at the back of the vehicle [point B in Fig. 7(b)]. In this case, another instantaneous sensor is obtained, with its 2π angular field-of-view partitioned into three slices.

Alternatively, the sensor may be implemented as simple beam detectors placed at the two windshield boundaries, $-\phi + \epsilon_\phi$ and $\phi - \epsilon_\phi$; we may simply take ϵ_ϕ to be a fraction of ϕ [see Fig. 7(c)]. Except at the two boundaries, the agent is otherwise "blind." Such a sensor, even simpler than previous ones, generates time based events and thus needs to have some immediate history of which sector the target is located in to produce the

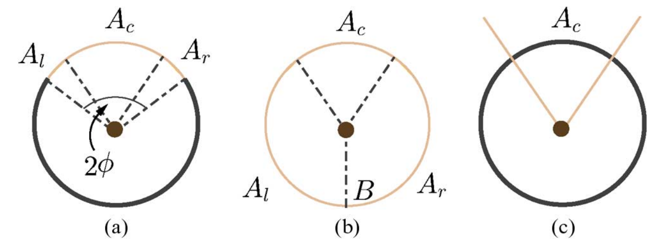

Fig. 7. Three possible implementations of the sensing model.

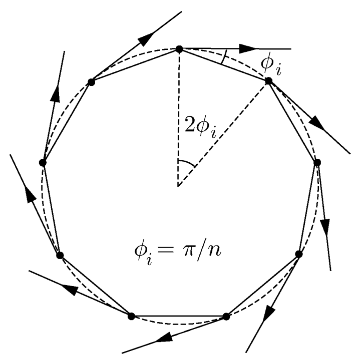

Fig. 8. Regular 9-gon with agents on the vertices and moving along the lines tangent to the enclosing circle of the 9-gon.

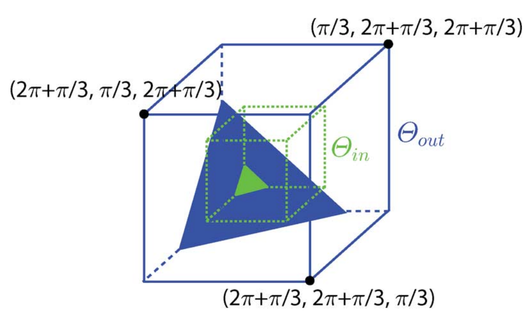

Fig. 9. Θ_i, Θ_o and the slices cut by the $g(\Theta) - (n-2)\pi = 0$ hyperplane, for the case of $n = 3$ and $\sum \phi_i > \pi/2$.

output. It also needs to have some additional mechanism to distinguish the agents that pass the beam detector, since the sensor no longer continuously tracks a target agent.

Appendix B
Remaining Proofs

Proof of Lemma 3: For $n = 2, \theta_1 = \theta_2 = 0$ and it is possible that $\phi_1 = \phi_2 = \pi/2$ by assumption; substituting these into (9) gives $\dot{V} = 0$. For $n \geq 3$, since agents may take arbitrary initial formation, they may lie exactly on the vertices of a regular polygon and have the agents on the next vertex (say clockwise direction, see Fig. 8) as their targets; all agents can also have $\phi_i = \pi/n < \phi$. In this case, since $\theta_i = ((n-2)/n)\pi$ for all i, $\theta_i + \phi_i = ((n-1)/n)\pi$. Substituting these into (9) again gives

$$\dot{V} = \sum_i -v_i \left(\cos\frac{\pi}{n} + \cos\frac{n-1}{n}\pi \right) = 0. \qquad (44)$$

$\dot{V}$ can be easily made strictly positive by letting $\phi_i > \pi/n$ for all i. □

Proof of Lemma 6: When all $(\theta_i + \phi_i)$ are equal and in the interior of Θ_i, we have $(\theta_i + \phi_i) = \sum(\theta_i + \phi_i)/n$ for all i. Hence, at the stationary point

$$\begin{aligned} f(\Theta, \Phi) &= \sum_i -\cos(\theta_i + \phi_i) \\ &= -n\cos\frac{\sum_i(\theta_i + \phi_i)}{n} \\ &= -n\cos\frac{(n-2)\pi + \sum_i \phi_i}{n}. \end{aligned} \qquad (45)$$

By (45), this unique stationary point of f only depends on $\sum \phi_i$ but not on the individual ϕ_i's. Also, fixing θ_i's, the maximum of the last expression in (45) increases as $\sum \phi_i$ increases; hence, the upper bound of f is given by (21). □

Proof of Lemma 7: We may fix ϕ_i's, in which case Θ_i forms a n-dimensional cube (including the boundary) in the n-dimensional Euclidean space with $\theta_i, \ldots, \theta_n$ as the coordinates. Θ_i is fully surrounded by Θ_o. Together, $\Theta_o \cup \Theta_i$ forms a larger n-dimensional cube. Since the θ_i's must also satisfy the constraint $\sum \theta_i = (n-2)\pi$, the valid θ_i's for simple polygon formation live on the slice of the $\Theta_o \cup \Theta_i$ cube cut by the hyperplane $\sum \theta_i - (n-2)\pi = 0$. Geometrically, the slice through Θ_o forms an annulus on the hyperplane $\sum \theta_i - (n-2)\pi = 0$, and the slice through Θ_i is the inside of that annulus. Fig. 9 illustrates these for the case of $n = 3$. By continuity of f, $f \leq n - 1$ on the boundary of the Θ_i slice since the boundary is the inner part of the closure of the Θ_o slice. On the other hand, the Θ_i slice is closed and bounded hence compact. By the extreme value theorem, f must take both maximum and minimum value on the Θ_i slice. Therefore, f takes value no more than the larger of $n-1$ and (21) from Lemma 6. Combining the Θ_o and Θ_i slices, we obtain for any simple n-gon that (22) holds. To ensure $\dot{V} = h + f < 0$, we need $-h$ to be more than the right hand side of (22) and rearranging the resulting inequality yields (11). We note that the easily verified fact that for all $n \geq 5, \pi/n < \cos^{-1}((n-1)/n)$ is used. □

Proof of Lemma 8: When the polygon is self-intersecting, the internal angles are no longer well defined; therefore, the constraint, g, will change. However, for our purpose of calculating $\dot{\ell}_{i,j}$, we can always pick the smaller of the two angles at any vertex of the polygon, since such choice does not affect the value of the cosine function. We define these angles as θ_i and look at $\sum(\pi - \theta_i)$. If we start from any edge of a polygon, self-intersecting or not, and walk along the edges to get back to the starting edge, all the $(\pi - \theta_i)$'s must add up to at least 2π [see Fig. 10(a)]. This is true since the walker must turn at least a cycle to get back and $\sum(\pi - \theta_i)$ is exactly the sum of the angles turned. Thus $\sum \theta_i \leq (n-2)\pi$. The constraint function $g(\Theta)$ can again be defined as that in (17). For self-intersecting polygon, g can be arbitrarily close to 0 [see Fig. 10(b)]. Therefore, $0 < g \leq (n-2)\pi$ and (20) becomes

$$-\pi < \sum_i (\theta_i + \phi_i) < (n-1)\pi. \qquad (46)$$

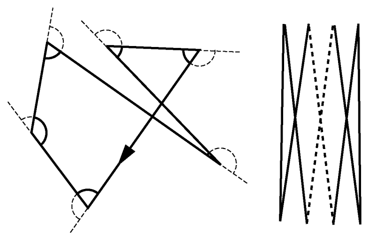

Fig. 10. (a) θ_i's and $(\pi-\theta_i)$'s for a self-intersecting polygon, as marked by the solid arcs and dashed arcs, respectively. The arrow marks the walker's walking direction and starting edge. (b) θ_i's can be made arbitrarily small by "pressing" this polygon from the left and the right.

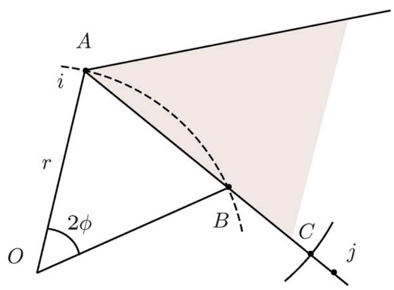

Fig. 11. Dubins car agent i that just loses its target j at point A.

As stated in Lemma 7, we only need to worry about the part of $\{\theta_i+\phi_i\}$ that belongs to Θ_i. That is, the interesting θ_i's must comply to

$$\frac{n\pi}{2} \le \sum_i(\theta_i+\phi_i) < (n-1)\pi. \tag{47}$$

We may fix g satisfying (47) as a constraint and apply the method of Lagrange multipliers similarly; bounds from Lemma 7 clearly hold. □

Proof of Proposition 16: We look at the instant t when agent i just loses sight of agent j. If agent j is within distance ρ of i then they must have merged; suppose not. The setting is shown in Fig. 11: Agent i is initially located at A and the ray $\overrightarrow{AC}$ can be imagined as the right boundary of agent i's field of view (shaded area in figure); it cannot see anything to the right of $\overrightarrow{AC}$. Let $|AC|=\rho$, the merging radius. At that instant, agent i's field-of-view cone is rotating around A with angular velocity ω_i. To stay out of i's windshield, agent j's velocity projection on the direction perpendicular to AC must be greater than $\rho\omega_i$. If $\rho\omega_i > v_j$, then once v_i starts to rotate, it will be able to get j back into its windshield again. This condition is equivalent to (40). □

Proof of Proposition 17: First let us assume that i travels at constant speed v_i. In Fig. 12, assume that agent i is initially located at A and turns clockwise around O with radius r. As it turns around, the intersection of all discs of radius ρ with i as the moving center is again a disc (the shaded area C) of radius $\rho-r$ centered at O (this can be easily shown: any point inside C is in all of the moving discs of radius ρ and any point outside C is not in at least one of the moving discs). Since any agent falling into C will merge with i, to avoid being found by i, an agent, say j, must move outside C and as i turns two full cycles, j must turn at least one full cycle to avoid appearing in i's windshield. Hence, if it takes less time for i to turn two cycles than for j to turn one cycle

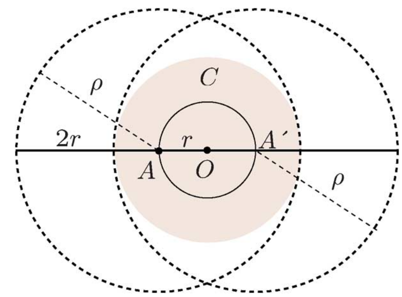

Fig. 12. Agent at A turns along a circle with center O of radius r, leaving the small solid circle as its trace. The two large dotted circles are of distance ρ to the agent at A and A'. The shaded disc denotes the intersection of all discs with radius ρ from the agent as it turns a full cycle.

$$\frac{2\times 2\pi r}{v_i} < \frac{2\pi(\rho-r)}{v_j} \tag{48}$$

agent i will then find an existing agent or merge with it. The inequality (48) yields the condition

$$\omega_i = \frac{v_i}{r} > \frac{2v_j+v_i}{\rho}. \tag{49}$$

Selecting (41) is then enough to guarantee that G regains liveness and the system will rendezvous. If v_i is not constant, since ω_i remains a constant by assumption, the intersection must have a minimum radius of $v_{\min}/\omega_i$ from O. In this case, we need

$$\frac{2\times 2\pi}{\omega_i} < \frac{2\pi(\rho-v_{\min}/\omega_i)}{v_j} \tag{50}$$

which is also satisfied by (41). Lastly, (40) from Proposition 16 is also satisfied. □

Appendix C
Direct Cyclic Pursuit

If the windshield of an agent is a single point ($\phi=0$), every agent moves directly toward its target, and we obtain a version of the classic cyclic pursuit, or n-bug problem. We call this variety the *direct cyclic pursuit* problem to distinguish it from the Dubins car cyclic pursuit. If we are not sure whether such pursuit is cyclic, we call it *direct pursuit*. Such cases can be viewed as limiting cases as the windshield goes from an open interval to a point. Here we show that our approach also applies to this special case. Allowing the pursuit formation to be an arbitrary polygon and using the general form of θ_i defined for self-intersecting polygon, (9) becomes

$$\dot{V} = \sum_i -v_i(1+\cos\theta_i) \tag{51}$$

with $0<\theta_i\le\pi$ and $\sum\theta_i \le (n-2)\pi$. Since for at least one i, $\theta_i \le ((n-2)/n)\pi$, $\dot{V}$ is always strictly negative for fixed n and any set of $v_i \ge \epsilon$ for some fixed $\epsilon>0$. The intree and cycle plus branch cases also follow. We obtain the following corollary.

Corollary 19: Direct pursuit of n agents with lower bounded speed and arbitrary connected, single-target assignment graph will rendezvous in finite time.

We can say a little more if $v_i = 1$ (or any constant) holds for all agents.

Corollary 20: Unit speed direct pursuit of n agents with connected, single-target assignment graph has the property

$$\dot{V} \leq \max\left\{-1, -n\left(1+\cos\left(\frac{n-2}{n}\pi\right)\right)\right\} =: -\delta \quad (52)$$

and will rendezvous in time no more than V_0/δ (recall that V_0 is the value of V at $t = 0$).

Proof: Writing $\dot{V} = h + f$ as that of (13), we readily see that $h = -n$ for such systems. (52) is then easily obtained by substituting in (22). □

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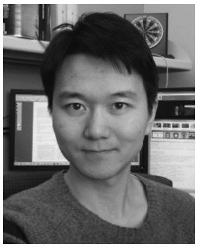

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