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Brief paper

Energy control of a pendulum with quantized feedback[☆]

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ABSTRACT

The problem of controlling a nonlinear system to an invariant manifold using quantized state feedback is considered by the example of controlling the pendulum's energy. A feedback control law based on the speed gradient algorithm is chosen. The main result consisting in precisely characterizing allowed quantization error bounds and resulting energy deviation bounds is presented.

1. Introduction

Control theory has initially been developed under idealistic assumptions regarding information transmission in a feedback loop. More recently, however, researchers have been increasingly interested in the question of how much information is really needed to perform a desired control task, or conversely, what control objectives can be achieved with a given amount of information. Such considerations arise from applications where scarce communication resources, sensor limitations, or security concerns play a role, and are also motivated by theoretical interest in understanding the interplay between information and control.

Among the various phenomena responsible for a limited amount of information available in a feedback loop, quantization is one of the most basic and widely investigated. By a *quantizer* we mean a function that maps a continuous real-valued system signal into a piecewise constant one taking a finite set of values, thereby encoding this signal using a finite alphabet. Notable early studies of the effect of quantization on the behavior of control systems include (Curry, 1970; Delchamps, 1990; Kalman, 1956; Miller, Mousa, & Michel, 1988), and a brief overview of the recent literature can be found in Sharon and Liberzon (2012).

One approach to analysis of quantized control systems, taken in Liberzon (2003) and elsewhere, involves modeling quantization effects as additive errors. If the controller possesses suitable robustness with respect to such errors, then the system performance can be shown to degrade gracefully due to quantization. In the context of stabilizing an equilibrium, instead of global asymptotic stabilization one typically obtains two nested invariant regions such that all trajectories starting in the larger one converge to the smaller one, a fact usually established by Lyapunov arguments. While robustness to additive errors is automatic for linear systems and linear feedback controllers, for general nonlinear systems the robustness requirements can be quite restrictive and finding a controller meeting such requirements can be challenging (Liberzon, 2003).

Pendulum dynamics is a popular and important benchmark system in control theory. The problem of stabilizing the upright equilibrium, as well as the problem of controlling the pendulum's energy to a desired level, have been widely studied and call for innovative solutions. In particular, it is known (see, e.g., Shiriaev, Egeland, Ludvigsen, & Fradkov, 2001) that the upright equilibrium cannot be globally asymptotically stabilized by continuous feedback. See Angeli (2001), Åström and Furuta (2000), Rantzer and Ceragioli (2001), Shiriaev et al. (2001), Teel (1996) and the references therein for some interesting contributions to pendulum control. More generally, the problem of energy control for Hamiltonian systems was first considered in Fradkov (1996). In Shiriaev and Fradkov (2000, 2001) extended conditions for control of invariant sets were proposed with application to energy control of the pendulum.

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In this paper we consider the problem of controlling the pendulum's energy to a desired level using quantized state feedback. As the nominal feedback law, we choose one based on the *speed gradient method* from Fradkov (1980) (which stabilizes any energy level without quantization). As a candidate Lyapunov function, we choose the squared difference between the current and the desired energy levels (which decreases for the closed-loop system without quantization). We show that in the presence of sufficiently small state quantization errors, even though the Lyapunov function may not always decrease, the time periods on which it may increase and the amount by which it may increase are suitably bounded and decreasing behavior still dominates. Using these properties, we are able to establish that if the initial energy level is not too far from the desired one, then it will remain not too far from it and will eventually become close to it. While this result may appear intuitively not surprising, our main contribution lies in precisely characterizing allowed quantization error bounds and resulting energy deviation bounds.

The rest of the paper is structured as follows. In Section 2 the general problem of the pendulum's energy control using quantized state feedback is described. Our main result is presented in Section 3. Section 4 is devoted to a numerical example demonstrating the performance predicted by the main theorem.

2. Problem formulation

Consider the pendulum equations

$$\ddot{\varphi}(t) = -\frac{g}{l}\sin\varphi(t) + \frac{1}{ml^2}u(t), \tag{1}$$

where φ is a deviation angle ($\varphi = 0$ at the lower position), u is a controlling torque, g is a gravity acceleration, m and l are the mass and the length of the pendulum respectively.

Assume that $H(\varphi, \dot{\varphi})$ is the full energy of the pendulum, i.e.

$$H(\varphi, \dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 + mgl(1 - \cos\varphi).$$

Consider the problem of energy level stabilization of system (1). Let $z = [\varphi, \dot{\varphi}]^T$, $z \in \mathbb{R}^2$. Let h ($h < 2mgl$) be a positive number. Consider a set

$$X_h = \{z : 0 < H(z) \leqslant h\}.$$

Let H_* ($0 < H_* < h$) be desired energy level and the goal function be as follows

$$V(z) = \frac{1}{2}\left(H(z) - H_*\right)^2. \tag{2}$$

It is required to design a feedback law

$$u = U(z),$$

providing the achievement of the control goal

$$\lim_{t\to\infty} V\left(z(t, z_0)\right) = 0, \tag{3}$$

where the initial energy level $H(z_0)$ satisfies the following assumption:

$$z_0 \in X_h, \tag{4}$$

i.e. z_0 belongs to energy layer between 0 and h.

The algorithm design is based on the *speed gradient method* (Fradkov, 1980, 2007; Fradkov & Andrievsky, 2011). According to the speed gradient method it is required to calculate the function $\omega(z, u) = \dot{V}(z)$, i.e. $\omega(z, u)$ is the speed of variation of the quantity V along the trajectories of system (1)

$$\omega(z, u) = \left(H(z) - H_*\right)B^T z\, u,$$

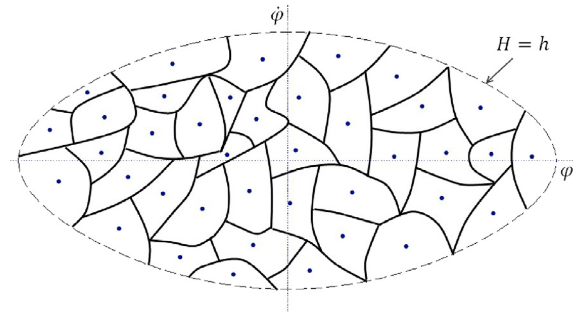

Fig. 1. Quantizer regions.

where $B = [0,\ 1]^T$. Let us find u-derivative of $\omega(z, u)$ and write down the control algorithm in the finite form

$$u = U(z) = -\gamma\frac{\partial\omega}{\partial u} = -\gamma\left(H(z) - H_*\right)B^T z, \tag{5}$$

where $\gamma > 0$.

The idea of algorithm (5) can be explained as follows (Fradkov, 2005). To achieve control goal (3), it is advisable to vary u such that V decreases. But because V does not depend on u, it is difficult to find the direction of such decrease. Instead, one can decrease $\dot{V}$ by ensuring that $\dot{V} < 0$, which is the condition that V decreases. The function $\dot{V}(z) = \omega(z, u)$ explicitly depends on u, which makes it possible to design algorithm (5).

The following theorem, characterizing the performance of control algorithm (5), can be directly concluded from Theorem 3.1 and Remark 3.1 in Fradkov (2007).

Theorem 1. *If the initial energy layer between the levels $H(z_0)$ and H_* does not contain an equilibrium of the unforced system, then the goal level H_* will be achieved in the controlled system (1), (5) for any $\gamma > 0$ from all initial conditions.*

The fulfillment of the condition in Theorem 1 follows from (4).

Let the set $\mathcal{Z} = \{z_i : z_i \in X_h, i \in \mathbb{N}\} \bigcup z_{sat}$ be a finite subset of $X_h \bigcup z_{sat}$, where $z_{sat} \in \mathbb{R}^2$. Consider quantizer $q(z) : \mathbb{R}^2 \to \mathcal{Z}$ proposed in Liberzon (2003). Assume that $Z_i = \left\{z \in \mathbb{R}^2 : q(z) = z_i\right\}$ are quantizer regions (Fig. 1), such that $\bigcup Z_i = X_h$. Hence, $q(z) = z_i$ for all $z \in Z_i$, $i \in \mathbb{N}$. When z does not belong to the union of quantization regions, the quantizer saturates, i.e. $q(z) = z_{sat}$ if $z \notin X_h$.

Suppose that only quantized measurements $q(z)$ of the state z are available. Then the state feedback law (5) is non-implementable. Hence, instead of continuous control (5) consider quantized feedback control law (5):

$$u = U(q(z)) = -\gamma\left(H(q(z)) - H_*\right)B^T q(z), \tag{6}$$

and control goal

$$\limsup_{t\to\infty} |H(z(t)) - H_*| < \varkappa_1, \tag{7}$$

where $\varkappa_1$ is some positive number.

Therefore, the problem is to find conditions of achievement of the goal (7) with quantized state feedback control (6). Note that assumption (4) is essential in the case of control algorithm (5) but can be omitted with using modifications of (5). In Shiriaev et al. (2001) it is shown that the global attractivity of the upright equilibrium can be achieved by a modification of the speed gradient energy method based on the idea of variable structure systems (VSS). However, an application of such a modified algorithm to the case of quantized measurements does not seem straightforward and is not pursued here.

3. Main result

Let $e(z) = q(z) - z = [e_1(z), e_2(z)]^T$ be a quantizer error vector. Assume that quantizer is chosen such that

$$|e_1(z)| \leqslant \Delta_1, \qquad |e_2(z)| \leqslant \Delta_2 \quad \text{for all } z \in X_h. \tag{8}$$

Hence, $|e(z)| \leqslant \sqrt{\Delta_1^2 + \Delta_2^2} = \Delta$ for all $z \in X_h$.

Law (6) can be rewritten as follows:

$$U(q(z)) = U(z) + e_u(z),$$

where

$$\begin{aligned} e_u(z) = -\gamma \Bigg(\bigg(\Big(\frac{1}{2} ml^2 (e_2(z) + \dot\varphi)^2 \\ + mgl\,(1 - \cos(\varphi + e_1(z))) - H_* \bigg) e_2(z) \\ + \left(ml^2 \left(\dot\varphi + \frac{1}{2} e_2(z) \right) e_2(z) \right. \\ \left. + mgl(\cos\varphi - \cos(\varphi + e_1(z))) \right) \dot\varphi \Bigg). \end{aligned}$$

Note that for all $z \in X_h$

$$mgl\,(1 - \cos(\varphi + e_1(z))) = H(\varphi + e_1(z), 0) \leqslant h, \tag{9}$$

and

$$\begin{aligned} &|\cos\varphi - \cos(\varphi + e_1(z))| \\ &\quad = 2\left| \sin\frac{2\varphi + e_1(z)}{2} \sin\frac{e_1(z)}{2} \right| \\ &\quad \leqslant 2\left| \sin\frac{e_1(z)}{2} \right| \leqslant 2\sin\frac{\delta}{2}, \end{aligned} \tag{10}$$

where $\delta = \min\{\pi, \Delta_1\}$.

From (8) to (10) and $\dot\varphi \leqslant \sqrt{\frac{2h}{ml^2}}$, one obtains

$$|e_u(z)| \leqslant \gamma\,\Delta_e \quad \text{for all } z \in X_h, \tag{11}$$

where

$$\Delta_e = \frac{1}{2} ml^2 \Delta_2^3 + \frac{3l\sqrt{2mh}}{2} \Delta_2^2 + (4h - H_*)\Delta_2 + 2g\sqrt{2mh}\sin\frac{\delta}{2}.$$

Denote for any $b > a \geqslant 0$ the set

$$V^{-1}_{[a,b]} = \{z \in X_h : a \leqslant V(z) \leqslant b\},$$

and for any $d > c \geqslant 0$ the set

$$H^{-1}_{[c,d]} = \{z \in X_h : c \leqslant H(z) \leqslant d\}.$$

It is easy to see that

$$V^{-1}_{[a,b]} = H^{-1}_{\left[H_* + \sqrt{2a},\, H_* + \sqrt{2b}\right]} \cup H^{-1}_{\left[H_* - \sqrt{2b},\, H_* - \sqrt{2a}\right]}.$$

Let h_* be a positive constant, satisfying the condition $h_* < \min\{H_*, h - H_*\}$. Consider the following functions of scalar variable y:

$$f_1(y) = H_* - h_* - \frac{3ml^2\,\Delta_e^2}{y^2},$$

$$f_2(y) = \frac{g}{l}\sqrt{1 - \left(1 - \frac{f_0(y)}{mgl}\right)^2} - \frac{\gamma\,\Delta_e}{ml^2}\left(\frac{h_*\sqrt{6}}{y} + 1\right),$$

where $f_0(y) = \min\{2mgl - H_* - h_*,\, f_1(y)\}$,

$$f_3(y) = 4h_*^2 - y^2 - \frac{16\sqrt{6}\,\gamma\,\Delta_e^3}{f_2(y)\,y},$$

$$f_4(y) = \frac{1}{12}\, y^2 \arccos\left(1 - \frac{f_1(y)}{mgl}\right) f_2(y) - \Delta_e^2.$$

The main result is the following theorem.

Theorem 2. *If the following system of inequalities*

$$\begin{aligned} &0 < y < 2h_*, \\ &f_1(y) > 0, \quad f_2(y) > 0, \quad f_3(y) > 0, \quad f_4(y) > 0 \end{aligned} \tag{12}$$

is feasible with respect to y, then for any solution $y = h_1$ of (12) and for any given initial condition $z(0) \in H^{-1}_{[H_*-\varkappa_2, H_*+\varkappa_2]}$ trajectories of closed-loop system (1), (6) satisfy $z(t) \in H^{-1}_{[H_*-h_*, H_*+h_*]}$ for all $t \geqslant 0$ and there exists $T > 0$ such that $z(t) \in H^{-1}_{[H_*-\varkappa_1, H_*+\varkappa_1]}$ for all $t \geqslant T$, where

$$\varkappa_1 = \sqrt{\frac{1}{4}h_1^2 + \frac{2\sqrt{6}\,\gamma\,\Delta_e^3}{h_1 f_2(h_1)}}, \qquad \varkappa_2 = \sqrt{h_*^2 - \frac{2\sqrt{6}\,\gamma\,\Delta_e^3}{h_1 f_2(h_1)}},$$

i.e. the goal (7) is achieved for $\varkappa_1$ as above.

Corollary 1. *For any $\tilde\varkappa_1, \tilde\varkappa_2$ satisfying $\tilde\varkappa_1 < \tilde\varkappa_2 < h_*$ there exist sufficiently small Δ_1, Δ_2 such that for any given initial condition $z(0) \in H^{-1}_{[H_*-\tilde\varkappa_2, H_*+\tilde\varkappa_2]}$ trajectories of closed-loop system (1), (6) satisfy $z(t) \in H^{-1}_{[H_*-h_*, H_*+h_*]}$ for all $t \geqslant 0$ and there exists $T > 0$ such that $z(t) \in H^{-1}_{[H_*-\tilde\varkappa_1, H_*+\tilde\varkappa_1]}$ for all $t \geqslant T$.*

Proof of Corollary 1. Let $h_1 = 2\tilde\varkappa_1$. If $\Delta_1 \to 0$ and $\Delta_2 \to 0$, then $\Delta_e \to 0$. Hence, $\varkappa_2 \to h_*$.

Since the negative terms in $f_1(h_1)$, $f_2(h_1)$ and $f_4(h_1)$ vanish and $f_3(h_1) \to 4(h_*^2 - \tilde\varkappa_1^2) > 0$ if $\Delta_e \to 0$ and $\varkappa_2 \to h_*$, it is easy to see that $y = 2\tilde\varkappa_1$ is a solution of system (12).

Finally, from $\varkappa_2 \to h_*$ one obtains $H^{-1}_{[H_*-\tilde\varkappa_2, H_*+\tilde\varkappa_2]} \subset H^{-1}_{[H_*-\varkappa_2, H_*+\varkappa_2]}$. Therefore, Corollary 1 directly follows from Theorem 2. □

Remark 1. Since chattering on the boundaries between the quantization regions is possible, the right-hand side of the differential equation (1), (6) is discontinuous, and its solutions are to be interpreted in the sense of Filippov. At the points of discontinuity, the Filippov solutions are directed along the convex hull of initial directions that are on both sides of the discontinuity. If the vector fields (on both sides of the boundary of the quantization regions) are directed toward the boundary, then sliding modes can occur. (On the other hand, if the trajectory crosses the boundary, then the solution satisfies the differential equation almost everywhere and it is a standard Carathéodory solution.) However, this issue will not play a significant role in the subsequent stability analysis. The reason is that, according to Filippov (see Filippov, 1988, Section 15, page 155), we have that (a) after adding convex combinations at the points of discontinuity of a vector field, the upper bound on $\dot V$ that we will establish remains valid (does not increase), and (b) the resulting solutions are absolutely continuous and differentiable almost everywhere.

The proof of Theorem 2 is based on the following auxiliary statement that can be proved along the lines of Lemma 1 in Polushin, Fradkov, Putov, and Rogov (1999).

Lemma 1. *Let there exist a constant $\alpha \geqslant 0$, such that $\dot V(z) \leqslant \alpha$ for all $z \in X_h$ for almost all $t \geqslant 0$, where $\dot V(z)$ is the time derivative of $V(z)$ along the trajectories of closed-loop system (1), (6). Let there exist positive constants A, v_0, v_3 $(0 < v_0 < v_3 < \infty)$ and continuous functions $\beta, \epsilon : V^{-1}_{[v_0,v_3]} \to \mathbb{R}^+ \setminus \{0\}$, such that for arbitrary trajectory $z(\cdot)$ of system (1), (6) if*

$$z(t) \in V^{-1}_{[v_0,v_3]} \setminus \Pi \quad \forall t \in [t_1, t_2] \text{ for some } 0 < t_1 < t_2,$$

where

$$\Pi = \left\{ z \in X_h : \dot V(z) < -\beta(z) \text{ for almost all } t \geqslant 0 \right\}, \text{ then}$$

(a) $t_2 - t_1 \leqslant A$, and

(b) *there exists $t_3 > t_2$, such that $V(z(t_3)) - V(z(t_1)) \leqslant -\epsilon(z(t_1))$, and for all $t \in (t_2, t_3)$ from $z(t) \in V^{-1}_{[v_0,v_3]}$ it follows $z(t) \in \Pi$.*

Suppose $\alpha A < \frac{1}{2}(v_3 - v_0)$. Denote $v_1 = v_0 + \alpha A$, $v_2 = v_3 - \alpha A$. If $z(0) \in V^{-1}_{[0,v_2]}$, then $z(t) \in V^{-1}_{[0,v_3]}$ for all $t \geqslant 0$ and there exists $T > 0$ such that $z(t) \in V^{-1}_{[0,v_1]}$ for all $t \geqslant T$.

Proof of Theorem 2. Let $v_0 = \frac{1}{8}h_1^2$, $v_3 = \frac{1}{2}h_*^2$. Hence, from $0 < h_1 < 2h_*$ it follows that $0 < v_0 < v_3$. By direct calculations

$$\dot V(z) = (H(z) - H_*)\,\dot\varphi\,(e_u(z) - \gamma\,(H(z) - H_*)\,\dot\varphi).$$

- Estimate $\dot V$.

Since $\left(\sqrt{\frac{\gamma}{2}}(H(z) - H_*)\,\dot\varphi - \sqrt{\frac{1}{2\gamma}}e_u(z)\right)^2 = \frac{\gamma}{2}(H(z) - H_*)^2\,\dot\varphi^2 - (H(z) - H_*)\,\dot\varphi\,e_u(z) + \frac{1}{2\gamma}e_u^2(z) \geqslant 0$,

$$\begin{aligned}\dot V(z) &\leqslant -\gamma(H(z) - H_*)^2\,\dot\varphi^2 + \frac{\gamma}{2}(H(z) - H_*)^2\,\dot\varphi^2 \\ &\quad + \frac{1}{2\gamma}e_u^2(z) \leqslant -\frac{\gamma}{2}(H(z) - H_*)^2\,\dot\varphi^2 + \frac{1}{2\gamma}e_u^2(z) \\ &\leqslant \frac{\gamma\,\Delta_e^2}{2}.\end{aligned} \tag{13}$$

Let $\alpha = \frac{\gamma\Delta_e^2}{2}$. Therefore, $\dot V \leqslant \alpha$ for all $z \in X_h$ for almost all $t \geqslant 0$.

- Let the function $\beta(z) = const = \frac{\alpha}{2}$. Prove that for arbitrary trajectory $z(t)$ if $z(t) \in V^{-1}_{[v_0,v_3]}$ and $\dot V \geqslant -\frac{\alpha}{2}$ for all $t \in [t_1, t_2]$ for some $t_1 < t_2$, then there exists $A \geqslant 0$ such that $t_2 - t_1 \leqslant A$.

From $V(z) \in [v_0,\ v_3]$ it follows that $|H(z) - H_*| \in \left[\frac{1}{2}h_1,\ h_*\right]$.

Since $-\frac{\gamma}{2}\,(H(z) - H_*)^2\,\dot\varphi^2 + \frac{\gamma}{2}\Delta_e^2 \geqslant \dot V(z) \geqslant -\frac{\gamma\Delta_e^2}{4}$,

$$(H(z) - H_*)^2\,\dot\varphi^2 \leqslant \frac{3\Delta_e^2}{2}.$$

Hence,

$$|\dot\varphi| \leqslant \frac{\Delta_e\sqrt{6}}{h_1}. \tag{14}$$

Estimate $|\ddot\varphi|$.

$$\begin{aligned}|\ddot\varphi| &= \left|\frac{g}{l}\sin\varphi - \frac{1}{ml^2}(U(z) + e_u(z))\right| \\ &\geqslant \frac{g}{l}\,|\sin\varphi| - \frac{1}{ml^2}|U(z) + e_u(z)|.\end{aligned} \tag{15}$$

Since

$$\begin{aligned}|U(z) + e_u(z)| &\leqslant |U(z)| + |e_u(z)| \\ &\leqslant \gamma\,|H(z) - H_*|\cdot|\dot\varphi| + \gamma\,\Delta_e \\ &\leqslant \gamma\,h_*\frac{\Delta_e\sqrt{6}}{h_1} + \gamma\,\Delta_e = \gamma\,\Delta_e\left(\frac{h_*\sqrt{6}}{h_1} + 1\right),\end{aligned}$$

$$|\ddot\varphi| \geqslant \frac{g}{l}\,|\sin\varphi| - \frac{\gamma\,\Delta_e}{ml^2}\left(\frac{h_*\sqrt{6}}{h_1} + 1\right).$$

Now let us estimate $|\sin\varphi|$. From (14) and $|H(z) - H_*| \in \left[\frac{1}{2}h_1,\ h_*\right]$ one obtains (see Fig. 2)

$$|\sin\varphi| \geqslant |\sin\varphi_{min}|,$$

where φ_{min} is such that

$$P(\varphi_{min}) = \min\left\{2mgl - H_* - h_*,\ H_* - h_* - \frac{3ml^2\,\Delta_e^2}{h_1^2} = f_1(h_1)\right\} = f_0(h_1),$$

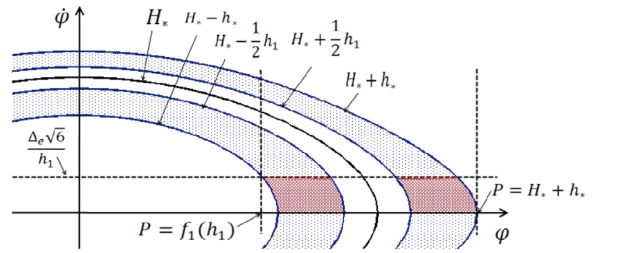

Fig. 2. Sinus estimation. From the symmetry it is sufficient to consider the first quadrant of the phase plane, i.e. $Z_+ = \{z : \varphi > 0, \dot\varphi > 0\}$. Red areas denote the set $Z_+ \bigcap \left\{z : |\dot\varphi| \leqslant \frac{\Delta_e\sqrt{6}}{h_1}\right\} \bigcap \left\{z : |H(z) - H_*| \in \left[\frac{1}{2}h_1,\ h_*\right]\right\}$.. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

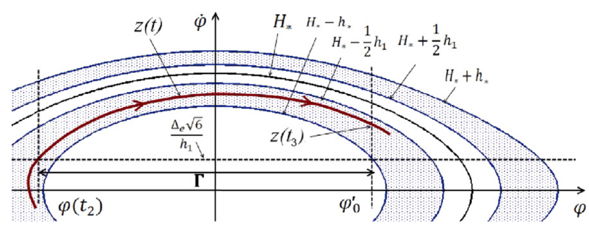

Fig. 3. Strip Γ.

and $P(\varphi) = mgl(1 - \cos\varphi)$ is the potential energy. Hence, $|\sin\varphi| \geqslant \sqrt{1 - \left(1 - \frac{f_0(h_1)}{mgl}\right)^2}$ and

$$|\ddot\varphi| \geqslant \frac{g}{l}\sqrt{1 - \left(1 - \frac{f_0(h_1)}{mgl}\right)^2} - \frac{\gamma\,\Delta_e}{ml^2}\left(\frac{h_*\sqrt{6}}{h_1} + 1\right) = f_2(h_1).$$

Therefore, $A = \frac{2\sqrt{6}\,\Delta_e}{h_1 f_2(h_1)}$, and inequality $\alpha A < \frac{1}{2}(v_3 - v_0)$ is fulfilled if and only if $f_3(h_1) > 0$.

- Let us prove that there exist $t_3 > t_2$ and $\epsilon > 0$ such that

  $V(z(t_3)) - V(z(t_1)) \leqslant -\epsilon(z(t_1))$

  and from $z(t) \in V^{-1}_{[v_0,v_3]}$ it follows $\dot V(t) < -\frac{\alpha}{2}$ for almost all $t \in (t_2,\ t_3)$.

Consider a strip

$$\begin{aligned}\Gamma &= \left\{(\varphi, \dot\varphi) : \varphi \in (\varphi(t_2),\ \varphi'_0);\ |\varphi'_0| = \varphi_0,\ \operatorname{sign}\varphi'_0 \right. \\ &\quad \left. = -\operatorname{sign}\varphi(t_2)\right\},\end{aligned}$$

where φ_0 is such that $P(\varphi_0) = H_* - h_* - \frac{3ml^2\,\Delta_e^2}{h_1^2} = f_1(h_1)$ and $0 < \varphi_0 < \pi$, i.e.

$$\varphi_0 = \arccos\left(1 - \frac{f_1(h_1)}{mgl}\right).$$

By definition of t_2 (time instant, when the trajectory enters $V^{-1}_{[v_0,v_3]} \cap \Pi$ from $V^{-1}_{[v_0,v_3]}\backslash\Pi$) we have $\frac{d|\dot\varphi(t_2)|}{dt} > 0$, hence, $\operatorname{sign}\ddot\varphi(t_2) = \operatorname{sign}\dot\varphi(t_2)$. From $f_3(h_1) > 0$ it follows that $\left|\frac{g}{l}\sin\varphi(t_2)\right| > \left|\frac{u(t_2)+e(t_2)}{ml^2}\right|$. Then $\operatorname{sign}\ddot\varphi(t_2) = -\operatorname{sign}\varphi(t_2)$. Therefore, $\operatorname{sign}\dot\varphi(t_2) = -\operatorname{sign}\varphi(t_2)$. By definition of φ_0 it follows that $|\dot\varphi| > \frac{\Delta_e\sqrt{6}}{h_1}$ on the strip Γ (see Fig. 3). Thus,

$$\dot V(t) \leqslant -\frac{\gamma}{2}(H(z) - H_*)^2\,\dot\varphi^2 + \frac{\gamma\,\Delta_e^2}{2} \leqslant -\frac{\gamma\,\Delta_e^2}{4} = -\frac{\alpha}{2}.$$

Now let t_3 be the first time instant $t > t_2$ when the trajectory $z(t)$ reaches the line $\varphi(t) = \varphi'_0$, and let us estimate $V(t_3) - V(t_1)$.

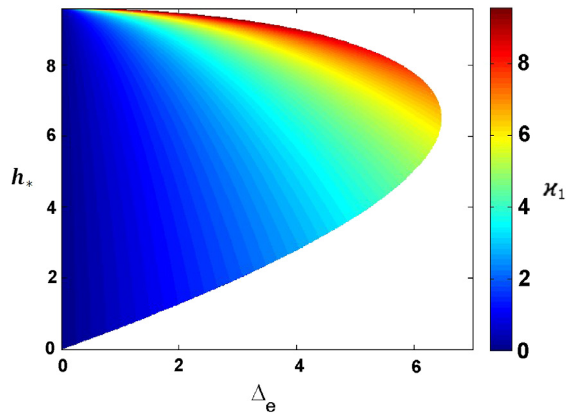

Fig. 4. The dependence of minimum possible $\varkappa_1$ on Δ_e and h_*. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

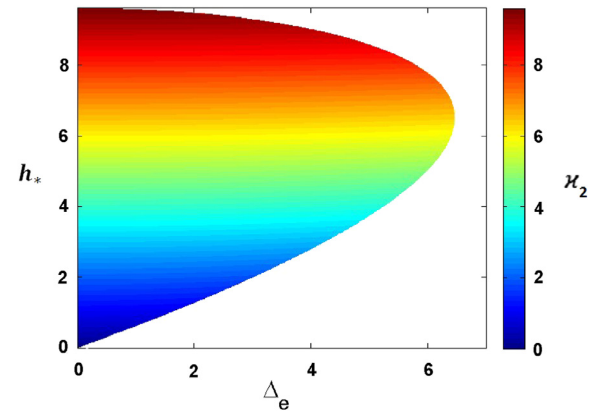

Fig. 5. The dependence of maximum possible $\varkappa_2$ on Δ_e and h_*. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

From (13)

$$V(t_3) - V(t_2) \leqslant -\frac{\gamma}{2}\int_{t_2}^{t_3} (H(z) - H_*)^2\,\dot\varphi^2 dt + \alpha\,(t_3 - t_2)$$
$$\leqslant -\frac{1}{8}\gamma h_1^2 \int_{t_2'}^{t_3}\left(\dot\varphi^2 - \frac{4\Delta_e^2}{h_1^2}\right)dt$$
$$\leqslant -\frac{1}{8}\gamma h_1^2 \int_{t_2'}^{t_3}\left(\dot\varphi^2 - \frac{2}{3}\dot\varphi^2\right)dt = -\frac{1}{24}\gamma h_1^2 \int_{t_2'}^{t_3}\dot\varphi^2 dt,$$

where t_2' is such that $\varphi(t_2') = \operatorname{sign}\varphi(t_2)\cdot\min\{|\varphi(t_2)|,\ \varphi_0\}$ and $t_2 \leqslant t_2'$. From Cauchy–Schwarz inequality one has

$$\int_{t_2'}^{t_3}\dot\varphi^2 dt \geqslant \frac{1}{t_3 - t_2'}\left(\int_{t_2'}^{t_3}\dot\varphi\,dt\right)^2.$$

Hence,

$$V(t_3) - V(t_2) \leqslant -\frac{1}{24}\gamma h_1^2\,\frac{1}{t_3 - t_2'}\left(\int_{t_2'}^{t_3}\dot\varphi\,dt\right)^2$$
$$= -\frac{1}{24}\gamma h_1^2\,\frac{1}{t_3 - t_2'}\left(\varphi(t_3) - \varphi(t_2')\right)^2 \leqslant -\frac{1}{6}\gamma h_1^2\,\frac{1}{t_3 - t_2'}\varphi_0^2.$$

Since

$$t_3 - t_2' \leqslant \frac{\left|\varphi(t_3) - \varphi(t_2')\right|}{|\dot\varphi|} \leqslant \frac{2\,\varphi_0\,h_1}{\Delta_e\sqrt{6}},$$

$V(t_3) - V(t_2) \leqslant -\frac{1}{2\sqrt6}\gamma h_1 \varphi_0 \Delta_e.$

Denote $\epsilon(z) = const = \frac{1}{2\sqrt6}\gamma h_1\varphi_0\Delta_e - \alpha A$. Therefore, if $f_4(h_1) > 0$, then

$V(t_3) - V(t_1) \leqslant -\epsilon < 0.$

Thus, all conditions of Lemma 1 are fulfilled. The statement of Theorem 1 follows. □

4. Numerical example

Consider system (1), (6) with the following parameters: $m = 1, l = 1, g = 9.8, h = 2mgl - 0.01 = 19.59, H_* = 9.6$.

In Theorem 2 the initial parameter h_* is such that $z(t) \in H^{-1}_{[H_*-h_*, H_*+h_*]}$ for all $t \geqslant 0$. Moreover, for small Δ_e the value of $\varkappa_2$ is close to h_*, i.e. one can say that h_* mainly plays the role of an initial condition parameter. Let $\gamma = 0.1$. The colored areas on Figs. 4 and 5 show those h_* and Δ_e for which system of inequalities (12) is feasible. Furthermore, the color scales on Figs. 4 and 5 provide the values of minimum possible $\varkappa_1$ and maximum possible $\varkappa_2$ respectively.

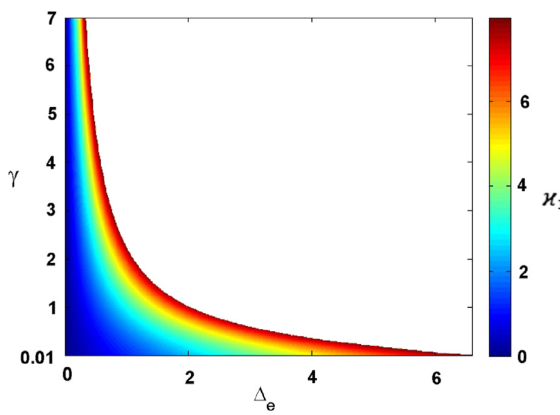

Fig. 6. The dependence of minimum possible $\varkappa_1$ on Δ_e and γ.

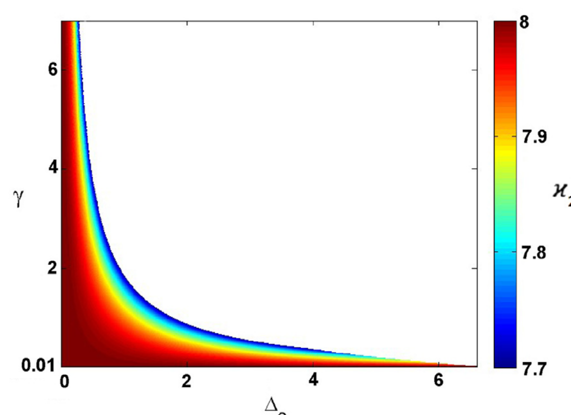

Fig. 7. The dependence of maximum possible $\varkappa_2$ on Δ_e and γ.

From Fig. 4 one can see that with increasing h_* the attraction domain $H^{-1}_{[H_*-\varkappa_1, H_*+\varkappa_1]}$ also increases. Therefore, it is important to prioritize between the accuracy of convergence and the width of initial area.

Let $h_* = 8$. We show the influence of the control gain γ on the initial and attraction domains. The colored areas on Figs. 6 and 7 show those γ and Δ_e for which system of inequalities (12) is feasible.

Note that from Theorem 2 one obtains that $\Delta_e < 6.5$, however, for any large γ there exists a small enough Δ_e such that system (12) is feasible.

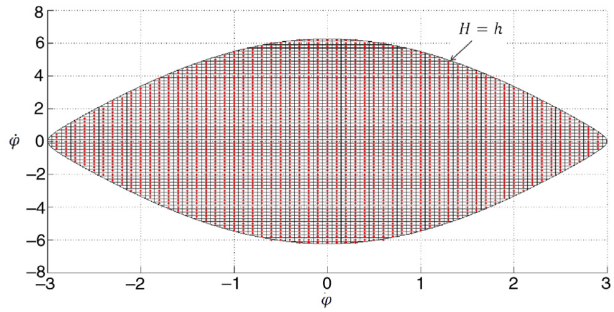

Fig. 8. Quantizer regions.

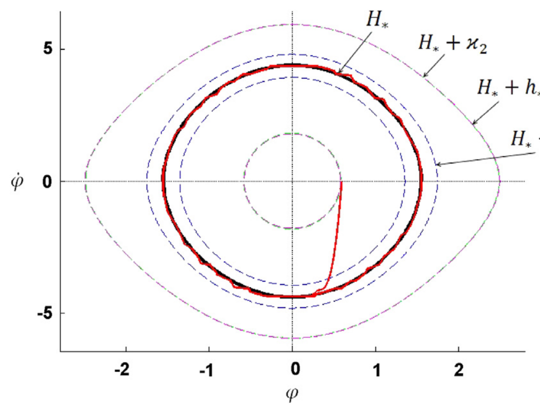

Fig. 9. Phase portrait. $\Delta_1 = 0.001$, $\Delta_2 = 0.002$ ($\Delta_e = 0.199$), $\gamma = 4$.

From Fig. 6 one can see that for given Δ_e the attraction domain $H^{-1}_{[H_* - \varkappa_1, H_* + \varkappa_1]}$ is smaller for smaller γ.

Now consider a quantizer $q(z)$ such that

$$Z_i = \left\{\varphi_k < \varphi < \varphi_k + \tau_\varphi,\ \dot{\varphi}_s < \dot{\varphi} < \dot{\varphi}_s + \tau_{\dot{\varphi}}\right\} \bigcap X_h$$

and $z_i = \left(\varphi_k + \frac{\tau_\varphi}{2},\ \dot{\varphi}_s + \frac{\tau_{\dot{\varphi}}}{2}\right)$ (see Fig. 8), where $\tau_\varphi = 0.002$, $\tau_{\dot{\varphi}} = 0.004$.

Then $\Delta_1 = 0.001$, $\Delta_2 = 0.002$ (hence, $\Delta_e = 0.199$). For $\Delta_e = 0.199$ system (12) is feasible for $0 < \gamma \leqslant 11.5$. Fig. 9 illustrates the phase portrait for $\gamma = 4$, where using Theorem 2 one obtains that $\varkappa_1 = 1.9$, $\varkappa_2 = 7.96$, i.e. for any given initial condition $z(0) \in H^{-1}_{[1.64,\ 17.56]}$ trajectories of closed-loop system (1), (6) satisfy $z(t) \in H^{-1}_{[1.6,\ 17.6]}$ for all $t \geqslant 0$ and there exists $T > 0$ such that $z(t) \in H^{-1}_{[7.7,\ 11.5]}$ for all $t \geqslant T$.

Better results can be obtained for smaller γ (although the convergence time T will increase). For example $\varkappa_1 = 0.259$, $\varkappa_2 = 7.9999$ for $\gamma = 0.1$, i.e. for any given initial condition $z(0) \in H^{-1}_{[1.6001,\ 17.5999]}$ trajectories of closed-loop system (1), (6) satisfy $z(t) \in H^{-1}_{[1.6,\ 17.6]}$ for all $t \geqslant 0$ and there exists $T > 0$ such that $z(t) \in H^{-1}_{[9.341,\ 9.859]}$ for all $t \geqslant T$ (see Fig. 10).

5. Epilogue

The problem of partial nonlinear control using quantized state feedback was considered by the example of controlling the pendulum's energy, which contains all the difficulties that are typical for nonlinear partial stable systems. As the nominal feedback law, a control based on the *speed gradient method* was chosen. The main contribution lies in precisely characterizing allowed quantization error bounds and resulting energy deviation bounds.

Numerical example shows that one can choose a smaller gain γ to decrease the attraction domain, although convergence time in this case has increased.

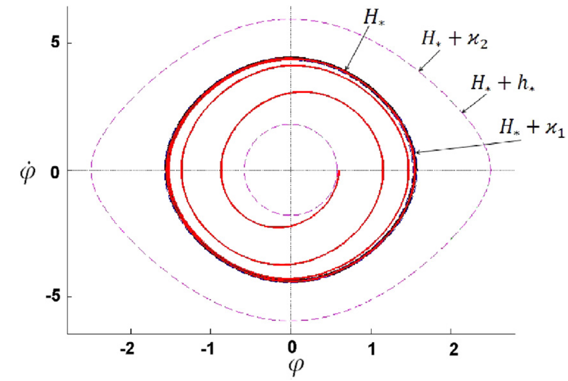

Fig. 10. Phase portrait. $\Delta_e = 0.199$, $\gamma = 0.1$.

Next steps in this research are the extension of the results to the case of systems with more than one degree of freedom (possibly with friction or noise) and to the presence of time delays in addition to quantization. Note that we have already considered time delays in cart–pendulum systems (see Ananyevskiy, Seifullaev, Nikitin, & Fradkov, 2014) and obtained some experimental results for a cart–pendulum system. Therefore, another task is to demonstrate our results by further experiments. Estimation of the convergence time is also interesting.

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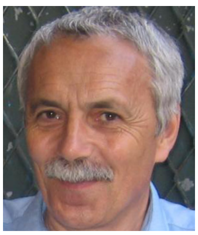

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