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STABILITY ANALYSIS OF DETERMINISTIC AND STOCHASTIC SWITCHED SYSTEMS VIA A COMPARISON PRINCIPLE AND MULTIPLE LYAPUNOV FUNCTIONS*

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Abstract. This paper presents a general framework for analyzing stability of nonlinear switched systems, by combining the method of multiple Lyapunov functions with a suitably adapted comparison principle in the context of stability in terms of two measures. For deterministic switched systems, this leads to a unification of representative existing results and an improvement upon the current scope of the method of multiple Lyapunov functions. For switched systems perturbed by white noise, we develop new results which may be viewed as natural stochastic counterparts of the deterministic ones. In particular, we study stability of deterministic and stochastic switched systems under average dwell-time switching.

Key words. switched systems, stability analysis, stochastic stability, multiple Lyapunov functions, comparison principle, two measures

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1. Introduction.

A family of continuous-time systems, together with a *switching signal* that chooses an *active subsystem* from the family at every instant of time, constitute a *switched system* [30]. Compared to *hybrid systems* [44], which currently are the focus of a large and growing interdisciplinary area of research, switched systems enable a more abstract modeling of continuous time systems with isolated switching events, which is suitable from a control-theoretic viewpoint. The abstraction is the result of modeling the switching signal as a purely time-dependent function, regardless of the mechanism of its generation. However, results obtained in this framework are then applicable to more specific hybrid systems; see, e.g., [18, 30] for a discussion. This paper is concerned with stability analysis of switched systems whose continuous dynamics are described by ordinary or stochastic differential equations.

Stability analysis by Lyapunov's direct method, in the simplest case of a single system, involves seeking a positive definite function of the states—called a Lyapunov function—that decreases along solution trajectories; see, e.g., [13, 22] for details. In case of switched systems, there are essentially two approaches to analyzing stability using Lyapunov's direct method; one involves investigating the existence of a *common Lyapunov function*, and the other utilizes *multiple Lyapunov functions*; see, e.g., [30, Chapters 2, 3] for an extensive account. The former approach is usually more challenging, although once a common Lyapunov function is found, the subsequent analysis is simple. The latter approach is usually more amenable to applications; typically, to check stability, one needs to allocate one Lyapunov function to each subsystem, trace through the sequence of values of these functions at switching instants, and verify certain monotonicity requirements on this sequence. We elaborate further on this method below.

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The *comparison principle* [22, 28] helps in stability analysis by acting upon Lyapunov's direct method. It characterizes the time evolution of a Lyapunov function along system trajectories in terms of the solution of a scalar differential equation—the *comparison system*. The stability characteristics of the original higher dimensional system can then be inferred from those of the comparison system.

Our purpose is to establish a framework for stability analysis of (a) deterministic and (b) stochastic switched systems, by combining the method of multiple Lyapunov functions with the comparison principle. Specific motivations and contributions are elaborated below.

Deterministic switched systems. Stability analysis of deterministic switched systems using multiple Lyapunov functions first appeared in [37] and has evolved over a series of articles, for instance, [6, 19, 38]. The basic idea behind this method is to utilize stability properties of individual subsystems to infer stability properties of a switched system, thereby characterizing switching signals that ensure stability. A typical result, e.g., [30, Theorem 3.1], involves verifying two conditions to check global asymptotic stability of a switched system: first, each Lyapunov function is to monotonically decay when the corresponding subsystem is active, and second, the sequence formed by the values of each Lyapunov function at the instants when the corresponding subsystem becomes active is to be monotonically decreasing. (Henceforth we shall refer to this as the *fixed-index monotonicity condition.*) The verification of the second condition apparently requires quantitative knowledge of system trajectories. However, in situations where the switching is triggered by the state crossing some switching surfaces, on which the values of relevant Lyapunov functions match, the second condition follows if the first holds. Also, slow switching with a suitable *dwell-time* (see, e.g., [30]) allows each Lyapunov function to decay sufficiently before a switching occurs, thereby satisfying the second condition. A generalization of dwell-time switching is provided by the scheme of *average dwell-time switching* [16], which has proved to be a fruitful analysis tool in supervisory control; see [30, Chapter 6] and the references therein. This scheme requires that the number of switches over an arbitrary time interval should increase at most linearly with the length of the interval but places no specific restrictions on monotonicity of Lyapunov function values at switching instants (thereby allowing violation of the second condition above). We know that under suitable hypotheses, average dwell-time switching guarantees global asymptotic stability of a switched system [16], but the original proof of this result (provided in [30, Chapter 3]) does not utilize Lyapunov functions alone. We propose an alternative approach to stability analysis, based on the observation that a time trace of Lyapunov functions corresponding to active subsystems, in a typical trajectory of a switched system, shows impulsive behavior. Trajectories of suitable scalar impulsive differential equations may be used to generate such traces and are thus natural choices for comparison systems (cf. [29, 42], where impulsive comparison systems were utilized in stability analysis of impulsive differential equations). By employing various types of impulsive differential equations as comparison systems, in our results we relax the fixed-index sequence monotonicity condition, allowing oscillations and overshoots in the sequence, and also letting unstable subsystems participate in the dynamics. Additionally, our proofs retain conventional characteristics of Lyapunov's direct method in the context of switched systems, without resorting to independent arguments as in the proof of the average dwell-time result [16]. We propose our results on deterministic switched systems in section 2.

Stochastic switched systems. There is an enormous body of literature considering effects of noise and disturbances in systems, particularly from control and communication viewpoint; see, e.g., [7, 24, 34, 45]. The use of Lyapunov functions in stability analysis of stochastic systems is a classical idea, discussed extensively in [1, 11, 14, 25, 46]. The literature on stability of stochastic *switched* systems—constituted by subsystems perturbed by a standard Wiener process—is much less extensive compared to that on deterministic switched systems. In recent times, modeling and analysis of stochastic hybrid systems have appeared in, e.g., [17, 20]. Ergodic control of switched diffusion processes appears in [12]; stabilization methods that involve arguments similar to multiple Lyapunov functions appear in [2]. Some straightforward results involving common Lyapunov functions for stochastic switched systems may be found in [8]. We propose a framework for stochastic stability analysis by utilizing statistical estimates of Lyapunov functions at switching instants and during active periods of each subsystem. We consider some general stability definitions, for instance, global asymptotic stability in the mean and global asymptotic stability in probability, and employ the method of multiple Lyapunov functions adapted to the stochastic context. Much like the deterministic case, in a typical trajectory of a stochastic switched system, a time trace of expected values of Lyapunov functions corresponding to active subsystems shows impulsive behavior. We utilize impulsive differential equations as comparison systems to build a general framework for stability analysis of stochastic switched systems. This allows for very general behavior of the expected values of Lyapunov functions corresponding to active subsystems between switching intervals. In particular, replacing the fixed-index sequence monotonicity condition in the deterministic case with a stochastic analogue involving statistical estimates of Lyapunov functions at switching instants, we obtain a natural stochastic counterpart of [30, Theorem 3.1]. In addition, our results provide sufficient conditions for global asymptotic stability in the mean under average dwell-time switching and some more specific hypotheses. We propose our results on stochastic switched systems in section 3.

The concept of *stability analysis in terms of two measures* generalizes analysis of the norm of the state vector to analysis of the behavior of more general functions of the states; see, e.g., [29, 35]. We incorporate stability analysis in terms of two measures in the framework that we build for switched systems and gain greater flexibility for our results.

We study representative notions of deterministic and stochastic stability for a reasonably large class of switched systems. Naturally, not every type of stability can be described here. However, the framework of stability analysis we propose is applicable, as it stands, to more general stability notions, as will be indicated subsequently at appropriate places.

Some Notations. For notational convenience and brevity, we adopt the following conventions. For M_1, M_2, M_3 subsets of Euclidean space, let $\mathrm{C}[M_1, M_2]$ denote the set of all continuous functions $f : M_1 \longrightarrow M_2$, and $\mathrm{C}^1[M_1, M_2]$ denote the set of all continuously differentiable functions $f : M_1 \longrightarrow M_2$. We also use $\mathrm{C}^{1,2}[M_1 \times M_2, M_3]$ to denote the set of functions $f : M_1 \times M_2 \longrightarrow M_3$ that are continuously differentiable once and twice in the first and second arguments, respectively. We denote by $|\cdot|$ the standard Euclidean norm and by $\mathbb{R}_{\geqslant 0}$ the interval $[0, \infty[$. As usual, $\circ$ between two functions denotes their composition.

We say that a function $\alpha \in \mathrm{C}[\mathbb{R}_{\geqslant 0}, \mathbb{R}_{\geqslant 0}]$ is of class $\mathcal{K}$ if α is strictly increasing with $\alpha(0) = 0$, is of class $\mathcal{K}_\infty$ if in addition $\alpha(r) \to \infty$ as $r \to \infty$; and we write $\alpha \in \mathcal{K}$

and $\alpha \in \mathcal{K}_\infty$, respectively. A function $\beta \in \mathrm{C}\big[\mathbb{R}^2_{\geqslant 0}, \mathbb{R}_{\geqslant 0}\big]$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is a function of class $\mathcal{K}$ for every fixed t and $\beta(r,t) \to 0$ as $t \to \infty$ for every fixed r; and we write $\beta \in \mathcal{KL}$.

2. Deterministic switched systems. In this section, we study stability of deterministic switched nonlinear nonautonomous systems. In section 2.1, we describe a switched system and the stability notions that we study and define the comparison systems that we employ in our analysis. We propose our comparison theorem for deterministic switched systems in section 2.2 and illustrate it in sections 2.3 and 2.4. We discuss two other stability notions in section 2.5 that are different from Lyapunov stability and demonstrate that their analysis can be similarly carried out in the proposed framework.

2.1. Preliminaries.

System description and stability definitions. We consider a family of nonlinear nonautonomous systems,

$$\dot{x} = f_p(t, x), \qquad p \in \mathcal{P}, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $\mathcal{P}$ is an index set, $f_p \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}^n]$ is a vector field measurable in the first argument and locally Lipschitz in the second (see, e.g., [41] for further details), $f_p(\cdot, 0) \equiv 0$, for every $p \in \mathcal{P}$. Let there exist a piecewise constant function (continuous from the right by convention) $\sigma : \mathbb{R}_{\geqslant 0} \longrightarrow \mathcal{P}$, which specifies at every time t the index $\sigma(t) = p \in \mathcal{P}$ of the active subsystem. A *switched system* generated by the family (2.1) and such a *switching signal* σ is

$$\dot{x} = f_\sigma(t, x), \qquad x(t_0) = x_0, \quad t \geqslant t_0, \tag{2.2}$$

where $t_0 \in \mathbb{R}_{\geqslant 0}$. It is assumed that there is no jump in the state x at the switching instants and that there is a finite number of switches on every bounded interval of time. We denote the switching instants by τ_i, $i = 1, 2, \ldots$, $\tau_0 := t_0$, and the sequence $\{\tau_i\}_{i \geqslant 0}$ is strictly increasing. The *solution of* (2.2) *as a function of time t, interpreted in the sense of Carathéodory, initialized at a given pair* (t_0, x_0)*, and under a given switching signal* σ, is denoted by $x(t)$.

To perform analysis in terms of two measures, we utilize functions belonging to the class defined by

$$\Gamma := \left\{ h \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}] \;\middle|\; \inf_{(t,x)} h(t,x) = 0 \right\}. \tag{2.3}$$

We focus on the following notion of stability in terms of two measures; see, e.g., [29] for further details on other related concepts of stability. This stability definition coincides in spirit with the class $\mathcal{KL}$ stability defined for differential inclusions in the paper [43], where the authors consider autonomous systems on an extended state space.

DEFINITION 2.1. *Let* $h^\circ, h \in \Gamma$. *The switched system* (2.2) *is said to be* (h°, h)-globally uniformly asymptotically stable ((h°, h)-GUAS) *if there exists a class* $\mathcal{KL}$ *function* β *such that for every* $(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$*, the inequality*

$$h(t, x(t)) \leqslant \beta(h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \geqslant t_0 \tag{2.4}$$

holds.

In this paper we consider stability notions that are uniform with respect to the initial time t_0, e.g., global uniform asymptotic stability. The occurrences of "uniform" in the sequel convey this particular sense. In contrast, in much of the existing literature "uniform" global asymptotic stability is used to signify uniformity over a class of switching signals; see, e.g., [30]. In situations where there is uniformity in this sense, it will be explicitly indicated at appropriate places.

Remark 2.2. Under the special case of $h^\circ(t,x) = h(t,x) = |x|$, we recover usual *global uniform asymptotic stability* (GUAS), and for autonomous systems the corresponding specializations lead to *global asymptotic stability* (GAS); see, e.g., [22] for further details on GUAS and GAS. Other examples include stability with respect to arbitrary "tubes" in $\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ with the measures h°, h being the Hausdorff point-to-set distances, stability of prescribed motion $x_{\mathrm{r}}(\cdot)$ measured by $h^\circ(t,x) = h(t,x) := |x(t) - x_{\mathrm{r}}(t)|$, and partial stability. More examples may be found in [29].

Remark 2.3. The (h°, h)-GUAS property can be rephrased in traditional $\varepsilon - \delta$ form as follows. The (h°, h)-GUAS property expressed by (2.4) holds if and only if the following properties hold simultaneously:

(S1) ((h°, h)-uniform Lyapunov stability) there exists a class $\mathcal{K}_\infty$ function δ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}_{\geqslant 0}$, we have $h^\circ(t_0, x_0) < \delta(\varepsilon) \implies h(t, x(t)) < \varepsilon \quad \forall\, t \geqslant t_0$;

(S2) ((h°, h)-uniform global asymptotic convergence) for every $r, \varepsilon > 0$, there exists a number $T(r,\varepsilon) \geqslant 0$ such that for every $t_0 \in \mathbb{R}_{\geqslant 0}$, we have $h^\circ(t_0, x_0) < r \implies h(t, x(t)) < \varepsilon \quad \forall\, t \geqslant t_0 + T(r,\varepsilon)$.

A similar equivalence is established in [43, Proposition 1]; the proof of the above can be easily constructed from the proof of this proposition, and for completeness is provided in section A.

We introduce properties that will later be required from Lyapunov functions, cf. [29]. Let $V \in \mathrm{C}\big[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}\big]$. The function V is said to be *h-positive definite* if for $(t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ there exists a function $\alpha_1 \in \mathcal{K}_\infty$ such that $\alpha_1 \circ h(t,x) \leqslant V(t,x)$, and *$h^\circ$-decrescent* if for $(t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ there exists a function $\alpha_2 \in \mathcal{K}_\infty$ such that $V(t,x) \leqslant \alpha_2 \circ h^\circ(t,x)$. In results that follow, we shall require a family of functions $\{V_p \mid p \in \mathcal{P}\}$ to be *$\mathcal{P}$-uniformly h-positive definite and h° decrescent*; i.e., there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that we have

$$\alpha_1 \circ h(t,x) \leqslant V_p(t,x) \leqslant \alpha_2 \circ h^\circ(t,x) \qquad \forall\, (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \quad \forall\, p \in \mathcal{P}. \tag{2.5}$$

With h° and h specialized to Euclidean norms, usual positive definiteness and decrescence of V_p are recovered.

Remark 2.4. We point out that if $\mathcal{P}$ is finite, or if $\mathcal{P}$ is compact and suitable continuity assumptions hold true, then (2.5) is no loss of generality.

In this section, we present our results in the absence of classical differentiability assumptions, and the directional upper right Dini derivative is utilized. For example, along the vector field of a member with index p of the family of systems (2.1) the derivative of a function $V_p \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ is defined as

$$\mathrm{D}^+_{f_p} V_p(t,x) := \limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon}\big(V_p(t+\epsilon, x + \epsilon f_p(t,x)) - V_p(t,x)\big);$$

for further details, see, e.g., [10]. We require that the functions V_p, $p \in \mathcal{P}$ are locally Lipschitz in the second argument.

Remark 2.5. For continuously differentiable functions, the upper right Dini derivative derivative reduces to the ordinary derivative. In particular, we note that for $V_p \in \mathrm{C}^1[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$, the expression simplifies to

$$\mathrm{D}_{f_p} V_p(t,x) = \frac{\mathrm{d}}{\mathrm{d}t} V_p(t,x) = \left(\frac{\partial V_p}{\partial t} + \frac{\partial V_p}{\partial x} f_p \right)(t,x),$$

which is the total derivative of V_p along solutions of the system with index p in the family (2.1).

Comparison systems and deterministic comparison principle. Let the switched system (2.2) be given, and for a given switching signal σ, let $\{\tau_i\}_{i\geqslant 1}$ be the sequence of switching instants. Let $\mathbb{R}^n \times \mathcal{P} \ni (x,p) \longmapsto y(x,p) \in \mathbb{R}_{\geqslant 0}$ be a function continuous in x. We consider scalar nonlinear nonautonomous impulsive differential equations of the type

$$\text{(2.6)} \quad \begin{cases} \dot{\xi} = \phi(t,\xi), \quad t \neq \tau_i, \\ \xi(\tau_i) = \psi_i\left(\xi\left(\tau_i^-\right), y\left(x(\tau_i), \sigma(\tau_i)\right)\right), \end{cases} \qquad \xi(t_0) = \xi_0 \geqslant 0, \quad i \geqslant 1, \quad t \geqslant t_0,$$

where $\xi \in \mathbb{R}$, the field $\phi \in \mathrm{C}\big[\mathbb{R}^2_{\geqslant 0}, \mathbb{R}\big]$ and the *reset map* $\psi_i \in \mathrm{C}\big[\mathbb{R}^2_{\geqslant 0}, \mathbb{R}_{\geqslant 0}\big]$ are such that $\phi(\cdot,0) \equiv 0$ and $\psi_i(0,\cdot) \equiv 0$, $t_0 \in \mathbb{R}_{\geqslant 0}$, τ_i, $i = 1,2,\ldots$, are the instants of the impulses, $\tau_0 := t_0$, *the sequence* $\{\tau_i\}_{i\geqslant 1}$ *is identified with the sequence of switching instants generated by* σ. Systems of the type (2.6) are utilized as *comparison systems*, as well as various special cases—with or without impulses, time-variation, and y. In this paper we shall not be utilizing time-varying ϕ; an example of this may be found in [9].

We denote by $\xi(t)$ *the solution* (2.6) *as a function of time, with the sequence* $\{\tau_i\}_{i\geqslant 0}$ *specified, and initialized at* (t_0, ξ_0). In all cases, we shall tacitly assume the existence of a unique solution to differential equations of the type (2.6). If there are multiple solutions, however, the results hold true with the largest solution in place of the (unique) solution of the comparison systems.

The GUAS property of equations of the type (2.6) is defined similarly to Definition 2.1, with h°, h specialized to the scalar norm. Formally, a system of the type (2.6), with a given sequence $\{\tau_i\}_{i\geqslant 0}$, is said to be *globally uniformly asymptotically stable* (GUAS) if there exists a function $\beta_\xi \in \mathcal{KL}$ such that for every $(t_0, \xi_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$ the inequality

$$\text{(2.7)} \qquad |\xi(t)| \leqslant \beta_\xi(|\xi_0|, t - t_0) \qquad \forall\, t \geqslant t_0$$

holds. The properties required from ϕ and ψ_i in (2.6) ensure that $\xi(\cdot) \geqslant 0$; we shall therefore omit the absolute values on ξ in the sequel.

Remark 2.6. Remark 2.3 still applies if a system of the type (2.6) replaces (2.2), with h°, h specialized to the scalar norm.

The need to compare different Lyapunov functions at switching instants, which is inherent in the multiple Lyapunov functions method, prompts us to have a function y of the system states in the reset equation of a comparison system of the type (2.6). This function will be utilized in the construction of a suitable comparison system that will render [30, Theorem 3.1] a special case of our Theorem 2.8. For yet other applications, y will not be required.

The following well-known comparison lemma for nonswitched deterministic systems is needed for subsequent developments in this section; see, e.g., [28] for a proof.

LEMMA 2.7. *Consider the system with index p in the family* (2.1). *Suppose that there exist a function* $V_p \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ *and a comparison system*

$$\Sigma: \quad \dot{\xi} = \phi(t,\xi), \qquad \xi(t_0) = \xi_0 \geqslant 0, \quad t \geqslant t_0,$$

such that $\forall (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$*, the differential inequality*

$$\mathrm{D}^+_{f_p} V_p(t,x) \leqslant \phi(t, V_p(t,x)) \tag{2.8}$$

holds. Then $V_p(\tau, x(\tau)) \leqslant \xi(\tau)$ *implies* $V_p(t, x(t)) \leqslant \xi(t) \quad \forall\, t \geqslant \tau$*, where* $x(t)$ *and* $\xi(t)$ *are the solutions of the system with index* p *in* (2.1) *and* Σ*, respectively.*

2.2. Comparison theorem for deterministic switched systems. The following result establishes a general framework for testing stability of deterministic switched systems using multiple Lyapunov functions and a comparison system.

THEOREM 2.8. *Consider the switched system* (2.2) *with a fixed switching signal* σ *generating a sequence of switching instants* $\{\tau_i\}_{i\geqslant 1}$*, and two functions* $h^\circ, h \in \Gamma$*. Suppose that there exist functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*,* $V_p \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$*,* $p \in \mathcal{P}$*, locally Lipschitz in the second argument, and a system* Σ *of the type* (2.6)*, such that*

- (i) *the family* $\{V_p \mid p \in \mathcal{P}\}$ *is* $\mathcal{P}$*-uniformly* h*-positive definite and* h°*-decrescent in the sense of* (2.5)*;*
- (ii) $\forall\,(t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ *and* $\forall\, p \in \mathcal{P}$*, we have* $\mathrm{D}^+_{f_p} V_p(t,x) \leqslant \phi(t, V_p(t,x))$*;*
- (iii) $\forall\,(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$*, there exists* $\xi_0 \in \mathbb{R}_{\geqslant 0}$ *such that* $V_{\sigma(\tau_i)}(\tau_i, x(\tau_i)) \leqslant \xi(\tau_i)$ $\forall\, i \geqslant 0$*, where* $x(t)$ *and* $\xi(t)$ *are the corresponding solutions of* (2.2) *and* Σ*, respectively;*
- (iv) Σ *is* GUAS *in the sense of* (2.7)*.*

Then (2.2) *is* (h°, h)-GUAS *in the sense of* Definition 2.1*.*

Proof. Consider the interval $[\tau_\ell, \tau_{\ell+1}[$, with ℓ an arbitrary nonnegative integer. From hypotheses (iii) and (ii), and Lemma 2.7 applied with $\tau = \tau_\ell$, we have

$$V_{\sigma(\tau_\ell)}(t, x(t)) \leqslant \xi(t) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[.$$

The above estimate in conjunction with hypothesis (iv) leads to

$$V_{\sigma(\tau_\ell)}(t, x(t)) \leqslant \xi(t) \leqslant \beta_\xi(\xi_0, t - t_0) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[.$$

In view of hypothesis (i), we have

$$\alpha_1 \circ h(t, x(t)) \leqslant \beta_\xi(\alpha_2 \circ h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[. \tag{2.9}$$

With $\beta(r,s) := \alpha_1^{-1} \circ \beta_\xi(\alpha_2(r), s)$, the estimate in (2.9) is equivalent to

$$h(t, x(t)) \leqslant \beta(h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[.$$

Clearly $\beta \in \mathcal{KL}$. The arbitrariness of ℓ implies that

$$h(t, x(t)) \leqslant \beta(h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \geqslant t_0.$$

The (h°, h)-GUAS property of (2.2) follows. ∎

Theorem 2.8 does not provide a direct method for analyzing stability of a given switched system; we need to look for a suitable comparison system and check its stability properties first. We will now demonstrate how to proceed with such a scheme of analysis and how our framework stands in relation to existing results involving multiple Lyapunov functions. In section 2.3, we generalize [30, Theorem 3.1] in terms of two measures, which involves verification of a fixed-index sequence monotonicity condition to establish the (h°, h)-GUAS property of a switched system. The comparison system utilized in the proof makes use of quantitative information of system trajectories. In section 2.4 we construct comparison systems to rederive the sufficient conditions for GUAS of switched system under average dwell-time switching given in [30, Theorem 3.2] or [16]. Although the fixed-index sequence monotonicity condition is violated, the comparison framework of Theorem 2.8 works; also we do not require explicit information of system trajectories.

2.3. Stability under fixed-index sequence monotonicity condition. For this subsection, we let $\mathcal{P}$ be a finite set of N elements.

COROLLARY 2.9. *Consider the switched system* (2.2) *and* $h^\circ, h \in \Gamma$. *Suppose that there exist functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho, U \in \mathcal{K}$, α *positive definite,* $V_p \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ *for each* $p \in \mathcal{P}$ *locally Lipschitz in the second argument, such that*

(i) *the family* $\{V_p \mid p \in \mathcal{P}\}$ *is* $\mathcal{P}$*-uniformly* h*-positive definite and* h°*-decrescent in the sense of* (2.5)*;*

(ii) $\forall (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ *and* $\forall p \in \mathcal{P}$, *we have* $\mathrm{D}^+_{f_p} V_p(t,x) \leqslant -\alpha \circ h^\circ(t,x)$*;*

(iii) *for every pair of switching time* (τ_i, τ_j), $i < j$ *such that* $\sigma(\tau_i) = \sigma(\tau_j) = p \in \mathcal{P}$ *and* $\sigma(\tau_k) \neq p$ *for* $\tau_i < \tau_k < \tau_j$, *the inequality*

$$V_p(\tau_j, x(\tau_j)) - V_p(\tau_i, x(\tau_i)) \leqslant -U \circ h^\circ(\tau_i, x(\tau_i)) \tag{2.10}$$

holds, where $x(t)$ *is the solution of* (2.2) *initialized at* (t_0, x_0)*;*

(iv) $\forall (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$, *we have* $\alpha_2 \circ h^\circ(t,x) \leqslant \rho \circ \alpha_1 \circ h(t,x)$.

Then (2.2) *is* (h°, h)-GUAS.

Proof. We define a candidate impulsive differential comparison system of the type (2.6):

$$\Sigma : \quad \begin{cases} \dot{\xi} = -\alpha \circ \alpha_2^{-1}(\xi), & t \neq \tau_i, \\ \xi(\tau_i) = V_{\sigma(\tau_i)}(\tau_i, x(\tau_i)), \end{cases} \qquad i \geqslant 0, \quad t \geqslant t_0. \tag{2.11}$$

By its very definition, Σ satisfies hypotheses (ii)–(iii) of Theorem 2.8. Hypothesis (i) of Theorem 2.8 is satisfied by our hypothesis (i). To verify hypothesis (iv) of Theorem 2.8, we shall first prove uniform Lyapunov stability of Σ and then prove its global uniform asymptotic convergence, in view of Remark 2.6.

Consider the interval $[\tau_0, \tau_1[$. From hypothesis (ii) we have

$$V_{\sigma(\tau_0)}(\tau_1, x(\tau_1)) \leqslant V_{\sigma(\tau_0)}(\tau_0, x(\tau_0)).$$

Combining with hypothesis (i), we reach

$$\alpha_1 \circ h(\tau_1, x(\tau_1)) \leqslant \alpha_2 \circ h^\circ(t_0, x_0). \tag{2.12}$$

For every $p \in \mathcal{P}$, we have from hypothesis (i)

$$V_p(\tau_1, x(\tau_1)) \leqslant \alpha_2 \circ h^\circ(\tau_1, x(\tau_1)),$$

and therefore by hypothesis (iv),

$$V_p(\tau_1, x(\tau_1)) \leqslant \rho \circ \alpha_1 \circ h(\tau_1, x(\tau_1)).$$

In view of (2.12), we get

$$V_p(\tau_1, x(\tau_1)) \leqslant \rho \circ \alpha_2 \circ h^\circ(t_0, x_0). \tag{2.13}$$

Consider now the interval $[\tau_1, \tau_2[$. From hypothesis (ii) we have

$$V_{\sigma(\tau_1)}(\tau_2, x(\tau_2)) \leqslant V_{\sigma(\tau_1)}(\tau_1, x(\tau_1)).$$

Combining with hypothesis (i) and applying (2.13) with $p = \sigma(\tau_2)$, we get

$$\alpha_1 \circ h(\tau_2, x(\tau_2)) \leqslant \rho \circ \alpha_2 \circ h^\circ(t_0, x_0). \tag{2.14}$$

Now, for all $p \in \mathcal{P}$, we have

$$V_p(\tau_2, x(\tau_2)) \leqslant \alpha_2 \circ h^\circ(\tau_2, x(\tau_2)),$$

so by hypothesis (iv),

$$V_p(\tau_2, x(\tau_2)) \leqslant \rho \circ \alpha_1 \circ h(\tau_2, x(\tau_2)).$$

Now (2.14) gives

$$V_p(\tau_2, x(\tau_2)) \leqslant \rho \circ \rho \circ \alpha_2 \circ h^\circ(t_0, x_0).$$

It is not difficult to see that the worst-case situation for maximum possible overshoot of the function V_σ occurs when the switching signal σ visits every element of the set $\mathcal{P}$ without repetition until $\mathcal{P}$ is exhausted. Let $\tau_{j^\star}$ be the first switching instant after all the subsystems that participate in the dynamics have become active at least once since initialization at $t = t_0$. Define the function

$$\rho^j := \underbrace{\rho \circ \ldots \circ \rho}_{j \text{ times}}.$$

From the above computations, it is easy to see that

$$\xi(\tau_{j^\star}) = V_{\sigma(\tau_{j^\star})}(\tau_{j^\star}, x(\tau_{j^\star})) \leqslant \rho^{\mathrm{N}-1} \circ \alpha_2 \circ h^\circ(t_0, x_0).$$

Clearly $\rho^{\mathrm{N}-1} \in \mathcal{K}$. Define the function

$$\gamma(\cdot) := \max\left\{\alpha_2(\cdot), \rho \circ \alpha_2(\cdot), \ldots, \rho^{\mathrm{N}-1} \circ \alpha_2(\cdot)\right\}.$$

From (2.10) it follows that $\xi(t) \leqslant \gamma \circ h^\circ(t_0, x_0) \, \forall \, t \geqslant t_0$. Therefore, by hypothesis (i) and the definition of ξ_0 in (2.11),

$$\xi(t) \leqslant \gamma \circ \alpha_1^{-1}(\xi_0) \qquad \forall \, t \geqslant t_0. \tag{2.15}$$

It remains to prove uniform global asymptotic convergence of Σ.

We distinguish two cases.

Case 1. *Switching stops in finite time.* Since σ eventually attains a constant value, say, from the κth switching instant, it follows that there are no impulses after $t = \tau_\kappa$. That is to say, the system (2.11) becomes an autonomous scalar ordinary differential equation after $t = \tau_\kappa$, with negative right-hand side for nonzero $\xi(\tau_\kappa)$. Therefore, $\xi(t)$ monotonically decreases to $0 \, \forall \, t \geqslant \tau_\kappa$. In conjunction with (2.15) which shows uniform Lyapunov stability of Σ, we conclude that (2.11) is GUAS. Theorem 2.8 now guarantees that (2.2) is (h°, h)-GUAS.

Case 2. *Switching continues indefinitely.* Consider the restatement of the inequality (2.10) with $\xi(\tau_i)$ as defined in (2.11)

$$\begin{aligned}\xi(\tau_j) - \xi(\tau_i) &\leqslant -U \circ h^\circ(\tau_i, x(\tau_i)) \\ &\leqslant -U \circ \alpha_2^{-1} \circ V_p(\tau_i, x(\tau_i)) \\ &= -U \circ \alpha_2^{-1} \circ \xi(\tau_i).\end{aligned}$$

The pair (τ_i, τ_j) satisfies the condition in hypothesis (iii). Clearly, $\{\xi(\tau_i)\}_{\{i \geqslant 0 | \sigma(\tau_i) = p\}}$ is a positive (for nonzero x_0) monotonically decreasing sequence and must attain a limit, say, $c \geqslant 0$. If $c \neq 0$, then

$$\xi(\tau_j) - \xi(\tau_i) \leqslant -U \circ \alpha_2^{-1}(c)$$

for all (τ_i, τ_j) satisfying hypothesis (iii), which means that for some large enough j, say, j', $\xi(\tau_{j'}) < 0$. In view of (2.11) this means $V_{\sigma(\tau_{j'})}(\tau_{j'}, x(\tau_{j'})) < 0$, contradicting the hypothesis. Therefore, the subsequence $\{\xi(\tau_i)\}_{\{i \geqslant 0 | \sigma(\tau_i)=p\}}$ attains the limit 0 as $i \uparrow \infty$. For all time t between any two switching instants (τ_i, τ_j) satisfying hypothesis (iii), there is a uniform bound on $\xi(t)$ given by (2.15). For each $p \in \mathcal{P}$, the subsequence $\{\xi(\tau_i)\}_{\{i \geqslant 0 | \sigma(\tau_i)=p\}}$ attains the limiting value of 0, implying global asymptotic convergence. Combining with Lyapunov stability proved in (2.15) above, we conclude that Σ is GUAS.

By Theorem 2.8 now we conclude that (2.2) is (h°, h)-GUAS. ◻

Remark 2.10. We note that in the above proof, Σ makes explicit use of state information of (2.2)—in the notation of (2.6), we use

$$\psi_i\left(\xi\left(\tau_i^-\right), y(x(\tau_i), \sigma(\tau_i))\right) = V_{\sigma(\tau_i)}(\tau_i, x(\tau_i))$$

in the reset equation of Σ. On the other hand, $\xi(\tau_i^-)$ is not utilized.

Remark 2.11. Hypothesis (iv) in Corollary 2.9 essentially is a technical requirement to ensure that the measure h is nontrivial. This guarantees that the different Lyapunov functions at switching instants can be estimated from the initial condition. For $h^{\circ}(t, x) = h(t, x) = |x|$, this property is automatic (just let $\rho := \alpha_2 \circ \alpha_1^{-1}$), and then Corollary 2.9 becomes identical to [30, Theorem 3.1].

2.4. Stability under average dwell-time switching. In this subsection we rederive an existing result on global asymptotic stability under average dwell-time switching via our Theorem 2.8. We no longer retain the assumption that $\mathcal{P}$ is finite. Although we specialize to Euclidean norms and autonomous switched systems—to be able to use the aforesaid result in situ—the analysis can be readily generalized to two measures and nonautonomous switched systems.

Let us consider the autonomous switched system

$$\dot{x} = f_{\sigma}(x), \qquad x(t_0) = x_0, \quad t \geqslant t_0, \tag{2.16}$$

where $x \in \mathbb{R}^n$, $f_p \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}^n]$ is locally Lipschitz for every $p \in \mathcal{P}$, $f_p(0) = 0$. The switching signal σ is said to have *average dwell-time* $\tau_a > 0$ [16] if there exists a positive number $N_{\circ}$ such that the number of switches $N_{\sigma}(T, t)$ on the interval $[t, T[$ satisfies

$$N_{\sigma}(T, t) \leqslant N_{\circ} + \frac{T - t}{\tau_a} \qquad \forall\, T \geqslant t \geqslant t_0. \tag{2.17}$$

We investigate the conditions on the average dwell-time of the switching signal σ such that (2.16) is GAS. The available result is as follows. (For a detailed discussion and proof, see, e.g., [30, Theorem 3.2].)

THEOREM 2.12 (see [16]). *Consider the switched system* (2.16). *Let there exist functions* $V_p \in \mathrm{C}^1[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ *for every* $p \in \mathcal{P}$, $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ *and a positive number* $\lambda_{\circ}$ *such that*

$$\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|) \qquad \forall\, p \in \mathcal{P} \tag{2.18}$$

and

$$\frac{\partial V_p}{\partial x}(x) f_p(x) \leqslant -\lambda_{\circ} V_p(x) \qquad \forall\, x \in \mathbb{R}^n. \tag{2.19}$$

Suppose also that there exists a positive constant μ *such that*

$$V_p(x) \leqslant \mu V_q(x) \qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, p, q \in \mathcal{P}. \tag{2.20}$$

Then (2.16) *is* GAS *for every switching signal* σ *with average dwell-time* $\tau_a > \frac{\ln \mu}{\lambda_\circ}$.

Remark 2.13. The condition (2.20) imposes a restriction on permissible Lyapunov functions. Since this is a global result, it does not hold if, for example, some of the Lyapunov functions are quadratic and some others are quartic. Also, in view of interchangeability of p and q in (2.20), it follows that $\mu > 1$, excluding the trivial case of $\mu = 1$, which implies that there is a *common Lyapunov function* for the switched system [30].

For a fixed index $p \in \mathcal{P}$, the values of the Lyapunov function V_p at every switching instant τ_j with $\sigma(\tau_j) = p$ form a sequence $\{V_p(x(\tau_j))\}_{\{j\geqslant 0 | \sigma(\tau_j)=p\}}$. As discussed in section 1, results like [30, Theorem 3.1] provide sufficient conditions for stability of the switched system under the assumption that the sequences $\{V_p(x(\tau_j))\}_{\{j\geqslant 0|\sigma(\tau_j)=p\}}$ are monotonically decreasing for every $p \in \mathcal{P}$. But σ with an average dwell-time permits overshoots and oscillations in each of these sequences; thus [30, Theorem 3.1] is inapplicable. We mentioned in section 1 that currently the problem is tackled by independent arguments utilizing auxiliary functions, as in [30]; the proof does not utilize the Lyapunov functions alone. In the framework of Theorem 2.8, we can dispense with such auxiliary functions and easily rederive Theorem 2.12, as we now demonstrate.

To the end of this subsection, we assume that the hypotheses of Theorem 2.12 hold.

An impulsive differential equation as a comparison system. Consider an impulsive differential system of the type (2.6) with

$$\psi_i\left(\xi\left(\tau_i^-\right), y(x(\tau_i), \sigma(\tau_i))\right) := \mu\xi(\tau_i^-), \quad i \geqslant 1, \quad \mu > 0,$$

as the reset equation and

$$\phi(t, \xi) := -\lambda_\circ \xi, \quad \lambda_\circ > 0.$$

The complete system stands as

$$\Sigma' : \quad \begin{cases} \dot{\xi} = -\lambda_\circ \xi, \quad t \neq \tau_i, \\ \xi(\tau_i) = \mu\xi(\tau_i^-), \quad \mu > 0, \end{cases} \qquad \xi(t_0) = V_{\sigma(t_0)}(x_0), \quad i \geqslant 1, \quad t \geqslant t_0. \tag{2.21}$$

From (2.18) it follows that hypothesis (i) of Theorem 2.8 is satisfied with $h^\circ(t,x) = h(t,x) = |x|$. Further, from (2.19) and (2.20) together with the initial condition in (2.21), it follows that hypotheses (ii) and (iii) of Theorem 2.8 are satisfied, respectively. Now we investigate stability of Σ'.

Let $T > 0$ be arbitrary. Consider the evolution of the system (2.21) from $t = t_0$ through $t = T$. Let there be $N_\sigma(T, t_0)$ switches on this interval, and let $\nu := N_\sigma(T, t)$, where $N_\sigma(T, t)$ is as defined in (2.17)

$$\xi(\tau_{i+1}^-) = \xi(\tau_i)\mathrm{e}^{-\lambda_\circ(\tau_{i+1}-\tau_i)}, \quad 0 \leqslant i \leqslant \nu,$$

and

$$\xi(T) = \xi(\tau_\nu)\mathrm{e}^{-2\lambda_\circ(T-\tau_\nu)}.$$

Combining with the reset equation of Σ' and iterating over i, it follows that

$$\xi(T) = \xi(t_0)\mu^{\nu}\mathrm{e}^{-\lambda_\circ(T-t_0)}. \tag{2.22}$$

Using the definition of ν, (2.22) leads to

$$\xi(T) = \xi(t_0)\mu^{N_\circ}\mathrm{e}^{\lambda_\circ t_0}\mathrm{e}^{-(\lambda_\circ - \ln\mu/\tau_a)T}. \tag{2.23}$$

To ensure $\xi(T) \to 0$ as $T \uparrow \infty$, it is sufficient to have $\tau_a > \frac{\ln\mu}{\lambda_\circ}$. This guarantees the convergence of the impulsive differential system (2.21) to zero as time increases to infinity. Stability of Σ' follows directly from (2.23)—the estimate $\xi(t) \leqslant \xi(t_0)\mu^{N_\circ}\mathrm{e}^{\lambda_\circ t_0}$ holds if $\tau_a > \frac{\ln\mu}{\tau_a}$. We conclude that Σ' is GAS, considering Remark 2.6. Therefore, hypothesis (iv) of Theorem 2.8 is also satisfied.

Intuitively,

- the minimum rate of decay of the Lyapunov function corresponding to each active subsystem is captured by the continuous dynamics of Σ'; and
- the maximum jump in the values of two Lyapunov functions corresponding to two consecutively active subsystems is captured by the reset equation of Σ'.

We conclude that by Theorem 2.8 the switched system (2.16) is GAS for switching signals with $\tau_a > \frac{\ln\mu}{\lambda_\circ}$.

An ordinary differential equation as a comparison system. Consider the following scalar autonomous differential equation, a special case of (2.6), for a candidate comparison system:

$$\Sigma'' : \quad \dot{\xi} = \left(\frac{\ln\mu}{\tau_a} - \lambda_\circ\right)\xi, \qquad \xi(t_0) = \mu^{N_\circ}\mathrm{e}^{\lambda_\circ t_0}V_{\sigma(t_0)}(x_0), \quad t \geqslant t_0. \tag{2.24}$$

The solution of Σ'' is

$$\xi(t) = \mu^{N_\circ}V_{\sigma(t_0)}(x_0)\mathrm{e}^{-(\lambda_\circ - \ln\mu/\tau_a)(t-t_0)} \qquad \forall\, t \geqslant t_0. \tag{2.25}$$

Let average dwell-time of σ be τ_a, and $\nu := N_\sigma(T, t_0)$ be the number of switches on $[t_0, T[$. Considering the least rate of decay for Lyapunov functions corresponding to active subsystems, we have for an arbitrary $T \geqslant 0$,

$$V_{\sigma(\tau_i)}(x(\tau_i^-)) \leqslant V_{\sigma(\tau_i)}(x(\tau_i))\mathrm{e}^{-\lambda_\circ(\tau_{i+1}-\tau_i)}, \quad 0 \leqslant i \leqslant \nu,$$

and

$$V_{\sigma(\tau_\nu)}(x(T)) \leqslant V_{\sigma(\tau_\nu)}(x(T))\mathrm{e}^{-\lambda_\circ(T-\tau_\nu)}.$$

Combining with (2.20) at switching instants and iterating over i, we reach the estimate

$$V_{\sigma(\tau_\nu)}(x(T)) \leqslant \mu^{\nu}V_{\sigma(t_0)}(x_0)\mathrm{e}^{-\lambda_\circ T} = \mu^{N_\circ}\mathrm{e}^{\lambda_\circ t_0}V_{\sigma(t_0)}(x_0)\mathrm{e}^{-(\lambda_\circ - \ln\mu/\tau_a)T}. \tag{2.26}$$

Clearly, $\lambda_\circ > \frac{\ln\mu}{\tau_a}$ ensures global asymptotic stability of Σ''. By Theorem 2.8 with $h^\circ(t,x) = h(t,x) = |x|$, it follows that (2.16) is GAS for switching signals with $\tau_a > \frac{\ln\mu}{\lambda_\circ}$.

From (2.18) it follows that hypothesis (i) of Theorem 2.8 is satisfied with $h^\circ(t,x) = h(t,x) = |x|$. Further, from the discussion above, it follows that hypotheses (ii)–(iv) of Theorem 2.8 are also satisfied

Intuitively,

- the initial condition of Σ'' captures the maximum possible overshoot in V_σ—this corresponds to the situation when all $N_\circ$ switches occur very close to $t = t_0$;
- $\xi(\cdot)$ forms an envelope of the sequence $\{V_{\sigma(\tau_i)}(x(\tau_i))\}_{i\geqslant 0}$ over the interval $[t_0, T]$.

We conclude that by Theorem 2.8 the switched system (2.16) is GAS for switching signals with $\tau_a > \frac{\ln\mu}{\lambda_\circ}$. This agrees with the assertion of Theorem 2.12.

Remark 2.14. We note that in contrast to (2.11), the comparison systems (2.21) and (2.24) do not utilize state information in the function y directly. However (2.21) utilizes $\xi(\tau_i^-)$ in the reset equation; cf. Remark 2.10.

Remark 2.15. It is easy to see from (2.22) and (2.25) that for switching signals with average dwell-time bounded away from $\frac{\ln\mu}{\lambda_\circ}$, we have GAS of (2.16). For instance, for $\lambda \in\ ]0, \lambda_\circ[$ if $\tau_a \geqslant \frac{\ln\mu}{\lambda_\circ-\lambda}$, then the GAS property of (2.16) follows. Under this situation, (2.16) is GAS *uniformly* over all switching signals with $\tau_a \geqslant \frac{\ln\mu}{\lambda_\circ-\lambda}$. In prevailing literature, (2.16) is said to be *globally "uniformly" asymptotically stable* over all such switching signals.

Our comparison-based approach enables us to work with simple scalar differential equations which provide upper bounds of Lyapunov functions, rather than analyze the complicated evolution of the Lyapunov functions themselves. This provides new insights into average dwell-time switching, as is illustrated above, and moreover we can derive new switching rules that extend average dwell-time, as the following two remarks illustrate.

Remark 2.16. Let $N_\sigma(T, t)$ denote the number of switches on the interval $[t, T[$. As another illustration of Theorem 2.8, consider the switched system (2.2), and suppose that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $V_p \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$, $p \in \mathcal{P}$, and real numbers $m > 1, \lambda_\circ > 0, \mu > 1$, such that

(i) the family $\{V_p \mid p \in \mathcal{P}\}$ is $\mathcal{P}$-uniformly h-positive definite and h°-decrescent in the sense of (2.5);
(ii) $\forall\,(t, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ and $\forall\, p \in \mathcal{P}$, we have $\mathrm{D}^+_{f_p} V_p(t, x) \leqslant -\lambda_\circ V_p^m(t, x)$;
(iii) $\forall\,(t, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ and $\forall\, p, q \in \mathcal{P}$, we have $V_p(t, x) \leqslant \mu V_q(t, x)$.

Let us find conditions on $N_\sigma(T, t_0)$ such that (2.2) is (h°, h)-GUAS.

Consider the impulsive differential system of the type (2.6):

$$\overline{\Sigma}: \quad \begin{cases} \dot{\xi} = -\lambda_\circ \xi^m, \quad t \neq \tau_i, \\ \xi(\tau_i) = \mu\xi(\tau_i^-), \quad \mu > 0, \end{cases} \qquad \xi(t_0) = V_{\sigma(t_0)}(x(t_0)), \quad i \geqslant 1, \quad t \geqslant t_0.$$

A straightforward analysis leads to

$$\xi(T) = \frac{\mu^{N_\sigma(T,t_0)}\xi(t_0)}{\left(1 + \lambda_\circ(m-1)(\xi(t_0))^{m-1}\left(\sum_{i=1}^{N_\sigma(T,t_0)+1} \mu^{(m-1)(i-1)}(\tau_i - \tau_{i-1})\right)\right)^{1/(m-1)}},$$

where we let $\tau_{N_\sigma(T,t_0)+1} := T$. Since $\mu > 1$, we obtain

$$\xi(T) \leqslant \frac{\mu^{N_\sigma(T,t_0)}\xi(t_0)}{\left(1 + \lambda_\circ(m-1)(\xi(t_0))^{m-1}(T - t_0)\right)^{1/(m-1)}}.$$

A little calculation shows that if there exists $\epsilon > 0$ and $K > 0$ such that
(2.27)

$$N_\sigma(T, t_0) \leqslant \frac{1}{\ln \mu} \ln \left(\frac{K \Big(1 + \lambda_\circ (m-1) \left(\alpha_2 \circ h^\circ(t_0, x_0)\right)^{m-1} (T - t_0)\Big)^{\left(\frac{1}{m-1} - \epsilon\right)}}{\alpha_2 \circ h^\circ(t_0, x_0)} \right),$$

then $\overline{\Sigma}$ is GUAS. Just as in the case of Σ' and Σ'' above, it is easy to verify that the hypotheses of Theorem 2.8 hold with $\Sigma = \overline{\Sigma}$. It then follows by Theorem 2.8 that (2.2) is (h°, h)-GUAS. Note that unlike the case of average dwell-time, the switching law in (2.27) that guarantees (h°, h)-GUAS of (2.2) depends on the initial conditions (t_0, x_0).

Remark 2.17. In the context of Theorem 2.8, it is not necessary to assume that each subsystem in the family (2.1) is (h°, h)-GUAS for the switched system (2.2) to be (h°, h)-GUAS. This is evident from hypothesis (ii), which does not require ϕ to be negative definite in the second argument. It is possible that none of the individual subsystems is (h°, h)-GUAS but the switched system is, provided each subsystem is active for a small enough time.

For instance, consider the switched system (2.16) and suppose that switching is such that the number of switches $N_\sigma(T, t_0)$ on the interval $[t_0, T[$ is given by

$$N_\sigma(T, t_0) \geqslant \frac{T - t_0}{\delta_a} - N_\circ, \tag{2.28}$$

where $N_\circ, \delta_a > 0$. Such switching signals were called "reverse average dwell-time switching signals" in [15]. Let there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $V_p \in \mathrm{C}^1[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$, $p \in \mathcal{P}$, and real numbers $\mu' \in]0, 1[$ and $\lambda_\circ > 0$, such that
(i) the estimate (2.18) holds;
(ii) $\frac{\partial V_p}{\partial x} f_p(x) \leqslant \lambda_\circ V_p(x) \quad \forall\, x \in \mathbb{R}^n, \quad \forall\, p \in \mathcal{P}$;
(iii) $V_{\sigma(\tau_i)}(x(\tau_i)) \leqslant \mu' V_{\sigma(\tau_i^-)}(x(\tau_i^-)) \quad \forall\, i \geqslant 1$;
(iv) $\delta_a < -\ln \mu' / \lambda_\circ$.
Then (2.16) is GAS. Indeed, consider the comparison system

$$\widetilde{\Sigma}: \quad \begin{cases} \dot{\xi} = \lambda_\circ \xi, & t \neq \tau_i, \\ \xi(\tau_i) = \mu' \xi(\tau_i^-), \end{cases} \qquad \xi(t_0) = V_{\sigma(t_0)}(t_0, x_0), \quad i \geqslant 1, \quad t \geqslant t_0. \tag{2.29}$$

It is easy to see that hypothesis (i) with $h^\circ(t, x) = h(t, x) = |x|$, and hypotheses (ii) and (iii) of Theorem 2.8 are satisfied. Following the constructions for Σ' in (2.21) above, it is not difficult to show that the solution of $\widetilde{\Sigma}$ is given by

$$\xi(T) = (\mu')^{N_\circ} V_{\sigma(t_0)}(x(t_0)) \mathrm{e}^{(\lambda_\circ + \ln \mu' / \delta_a)(T - t_0)}, \tag{2.30}$$

where $T > t_0$. Clearly, from (2.30) it follows that $\widetilde{\Sigma}$ is GAS if $\lambda_\circ + \ln \mu' / \delta_a < 0$. In other words, if the condition (iv) above holds, hypothesis (iv) of Theorem 2.8 is also satisfied; Theorem 2.8 now guarantees that (2.16) is GAS. However, this situation is admittedly restrictive; the condition (iii) above holds only under special situations.

Remark 2.18. Consider a hybrid system described by a partition of its continuous state space into regions via fixed switching surfaces (*guards*) and fixed continuous dynamics in each region [30, 44]. Every trajectory of such a hybrid system corresponding to a fixed initial condition can be realized as a trajectory of the switched system (2.2), for a suitable time-dependent switching signal σ. For hybrid system trajectories corresponding to different initial conditions, the resulting switching signals are in general different, and stability properties of the corresponding switched system realizations are different. Therefore, we cannot conclude stability of the hybrid system from stability of the associated switched system. However, it is possible to conclude stability of the hybrid system from the switched system provided we have uniform stability with respect to a suitable class of switching signals—namely, the class of switching signals obtained from the hybrid system by varying its initial condition.

We have stated our Theorem 2.8 for a fixed switching signal σ; no uniformity with respect to σ is claimed. However, we have noted in Remark 2.15 that under specific hypotheses, we do have uniform stability over the class of switching signals with sufficiently large average dwell-time. It turns out that in supervisory control algorithms based on state-dependent hysteresis switching (utilizing guards with memory), the switching signal is effectively constrained to precisely such a class, thereby ensuring uniform stability; see, e.g., [30] and the references therein. Identification of other useful classes of switching signals for which we can conclude stability for hybrid systems in this way remains to be studied.

2.5. Remarks on other stability notions. Although not covered by the results presented so far, the comparison framework of Theorem 2.8 is general enough to describe various other stability behavior. The classical comparison principle has been successfully applied to describe *strict stability*, *total stability*, *practical stability*, and *finite time stability*, among others. The idea has also been applied in the context of partial, impulsive, stochastic, and functional differential equations and integral equations; see [29] for further details. In this subsection we study two other notions of stability of switched systems in our framework of Theorem 2.8, namely, uniform practical stability and finite time stability.

Practical stability. We briefly study one representative notion of practical stability, defined below; see, e.g., [27] for other definitions and details.

DEFINITION 2.19. *Let $h^\circ, h \in \Gamma$ and the pair (λ, A), $\lambda \in]0, A[$ be given. The system* (2.2) *is said to be* (h°, h)-uniformly practically stable with respect to (λ, A) *if for every $t_0 \in \mathbb{R}_{\geqslant 0}$, the property*

$$h^\circ(t_0, x_0) < \lambda \implies h(t, x(t)) < A \qquad \forall\, t \geqslant t_0 \tag{2.31}$$

holds for all solutions of (2.2).

With h°, h specialized to Euclidean norms, we recover usual uniform practical stability.

Remark 2.20. Uniform practical stability of impulsive differential equations of the type (2.6), for a given sequence $\{\tau_i\}_{i\geqslant 0}$, is identical to Definition 2.19, with h°, h specialized to absolute values.

The following result, which we state without proof, provides sufficient conditions for (h°, h) uniform practical stability of the switched system (2.2) with respect to a given pair (λ, A); see [9] for a proof. In section 3.5 we state and prove a stochastic version of this result. We define the "open tube" (or cylinder) of radius $r > 0$ in terms of a measure $h \in \Gamma$ to be the set

$$\mathcal{B}(h, r) = \{(t, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \mid h(t, x) < r\}.$$

PROPOSITION 2.21. *Consider the switched system* (2.2) *with a given σ, $h^\circ, h \in \Gamma$, and let the pair (λ, A), $\lambda \in]0, A[$ be given. Suppose that there exist functions $\alpha_1, \alpha_2 \in$*

$\mathcal{K}_\infty$, $V_p \in \mathrm{C}\big[\mathcal{B}(h,A) \cap \big((\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n)\backslash\mathcal{B}(h^\circ,\lambda)\big),\ \mathbb{R}_{\geqslant 0}\big]$ *for each* $p \in \mathcal{P}$ *locally Lipschitz in the second argument, and a system* Σ *of the type* (2.6)*, such that*

(i) *the family* $\{V_p \mid p \in \mathcal{P}\}$ *is* $\mathcal{P}$*-uniformly* h*-positive definite and* h°*-decrescent in the sense of* (2.5)*;*

(ii) $\forall\,(t,x) \in \mathcal{B}(h,A) \cap \big((\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n)\backslash\mathcal{B}(h^\circ,\lambda)\big)$ *and* $\forall\,p \in \mathcal{P}$*, we have* $\mathrm{D}^+_{f_p} V_p(t,x) \leqslant \phi(t, V_p(t,x))$*;*

(iii) $V_{\sigma(\tau_i)}(\tau_i, x(\tau_i)) \leqslant \xi(\tau_i) \quad \forall\,i \geqslant 0$*, where* $x(t)$ *and* $\xi(t)$ *are solutions of* (2.2) *and* Σ*, respectively;*

(iv) Σ *is uniformly practically stable with respect to* $(\alpha_2(\lambda), \alpha_1(A))$*.*

Then (2.2) *is* (h°, h)*-uniformly practically stable with respect to* (λ, A)*.*

Finite time stability. We provide sufficient conditions for *finite time stability* in case of autonomous switched systems only. The definition of finite-time stability in terms of two measures is proposed in Definition 2.22; for further details on finite-time stability with Euclidean norms, see [3].

For autonomous switched systems, we specialize the class of functions Γ in (2.3) to the corresponding autonomous version $\Gamma_a := \{h \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}] \mid \inf_x h(x) = 0\}$. For simplicity, we only consider special cases of h° and h below.

Definition 2.22. *Let* $h^\circ, h \in \Gamma_a$ *such that* $\ker h^\circ = \ker h = \{0\}$ *and* h°, h *are positive definite, radially unbounded. The switched system* (2.16) *is said to be* (h°, h)-finite time stable *if for every* $\varepsilon > 0$ *the following two properties hold simultaneously:*

- *The finite time convergence property holds; i.e., there exists a function* $T : \mathbb{R}^n\setminus\{0\} \longrightarrow \mathbb{R}_{>0}$ *called the settling time function such that for every* $x_0 \in \mathbb{R}^n\setminus\{0\}$*,* $x(t)$ *is defined on the interval* $[0, T(x_0)[$ *with* $x(t) \in \mathbb{R}^n\setminus\{0\}$ *and* $\lim_{t\uparrow T(x_0)} x(t) = 0$*, where* $x(t)$ *is the solution of* (2.16) *with initial condition* x_0*.*
- *Lyapunov stability holds; i.e., there exists a function* $\delta \in \mathcal{K}_\infty$ *such that* $h^\circ(x_0) < \delta(\varepsilon) \implies h \circ x(t) < \varepsilon \quad \forall\,t \in [0, T(x_0)[$*.*

With h°, h specialized to Euclidean norms, we recover finite time stability in the sense of [3].

Remark 2.23. Finite time stability of impulsive differential equations of the type (2.6) is identical to Definition 2.22, with h°, h specialized to absolute values.

Proposition 2.24. *Consider the switched system* (2.2) *with a given* σ*, and* $h^\circ, h \in \Gamma_a$*. Suppose that there exist functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*, locally Lipschitz* $V_p \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ *for each* $p \in \mathcal{P}$ *and a system* Σ *of the type* (2.6)*, such that*

(i) $\forall\,x \in \mathbb{R}^n$ *and* $\forall\,p \in \mathcal{P}$*, we have* $\alpha_1 \circ h(x) \leqslant V_p(x) \leqslant \alpha_2 \circ h^\circ(x)$*;*

(ii) $\forall\,x \in \mathbb{R}^n$ *and* $\forall\,p \in \mathcal{P}$*, we have* $\mathrm{D}^+_{f_p} V_p(x) \leqslant \phi(t, V_p(x))$*;*

(iii) $\forall\,x_0 \in \mathbb{R}^n$*, there exists* $\xi_0 \in \mathbb{R}_{\geqslant 0}$ *such that* $V_{\sigma(\tau_i)}(\tau_i, x(\tau_i)) \leqslant \xi(\tau_i) \quad \forall\,i \geqslant 0$*, where* $x(t)$ *and* $\xi(t)$ *are the corresponding solutions of* (2.16) *and* Σ*, respectively;*

(iv) Σ *is finite time stable.*

Then (2.16) *is finite time stable.*

The proof of this Proposition is not difficult and is omitted.

3. Stochastic switched systems. In this section, we study stability of stochastic switched nonlinear nonautonomous systems of the Itô type. In section 3.1, we describe a stochastic switched system and define the notions of stability that we study. We propose our comparison theorem for stochastic switched systems in section 3.2 and illustrate it in sections 3.3 and 3.4. We discuss two other stability notions in section 3.5 and demonstrate that their analysis can be similarly carried out in the framework of the proposed results.

3.1. Preliminaries.

System description and stability definitions. Let $\mathbf{\Omega} := (\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space, where Ω is the sample space, $\mathcal{F}$ is the Borel σ-algebra on Ω, and P is a probability measure on the measurable space $(\Omega, \mathcal{F})$.

We consider a family of nonlinear nonautonomous Itô systems,

$$\mathrm{d}x = f_p(t,x)\mathrm{d}t + G_p(t,x)\mathrm{d}w, \qquad p \in \mathcal{P}, \tag{3.1}$$

where $x \in \mathbb{R}^n$, $\mathcal{P}$ is an index set, $f_p \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}^n]$ is a vector field, $G_p \in \mathrm{C}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}^{n\times m}]$ is a diffusion rate matrix function, $f_p(\cdot, 0) \equiv 0$ and $G_p(\cdot, 0) \equiv 0$ for every $p \in \mathcal{P}$, w is an m-dimensional normalized Wiener process defined on the probability space $\mathbf{\Omega}$, and $\mathrm{d}x$ is a stochastic differential of x. We assume that f_p and G_p are smooth enough to ensure existence and uniqueness of the corresponding solution process; for precise conditions see, e.g., [36]. A *switched system* generated by the family (3.1) and a *switching signal* σ, defined similarly to section 2.1, is

$$\mathrm{d}x = f_\sigma(t,x)\mathrm{d}t + G_\sigma(t,x)\mathrm{d}w, \qquad x(t_0) = x_0, \quad t \geqslant t_0, \tag{3.2}$$

where $t_0 \in \mathbb{R}_{\geqslant 0}$. It is assumed, just as in section 2.1, that there is no jump in the state x at the switching instants, and there is a finite number of switches on every bounded interval of time. We denote the switching instants by $\tau_i,\ i = 1, 2, \ldots,$ with $\tau_0 := t_0$, and the sequence $\{\tau_i\}_{i\geqslant 0}$ is strictly increasing. The *solution process of* (3.2) *as a function of time* t*, initialized at a given pair* (t_0, x_0)*, and under a given switching signal* σ, is denoted by $x(t)$. In what follows, expected values at the (deterministic) initial condition are to be identified with their actual values; see also Remark 3.3.

Let $|x|_q := (\mathsf{E}[|x|^q])^{1/q}$ denote the qth mean of a random variable x defined on $\mathbf{\Omega}$. We will have occasion to use *Jensen's inequality*: if $\varphi \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}]$ is concave and x is a random variable on $\mathbf{\Omega}$, then $\mathsf{E}[\varphi(x)] \leqslant \varphi(\mathsf{E}[x])$. Also, we need *Chebyshev's inequality*: for $\varepsilon > 0$, $\psi \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$, x a random variable on $\mathbf{\Omega}$, we have $\mathsf{P}[\psi(x) \geqslant \varepsilon] \leqslant \mathsf{E}[\psi(x)]/\varepsilon$.

Consider the system with index p in the family (3.1). Let $V \in \mathrm{C}^{1,2}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$. By Itô's formula we have the stochastic differential of V as

$$\mathrm{d}V(t,x) = \mathscr{L}_p V(t,x)\mathrm{d}t + V_x(t,x)G_p(t,x)\mathrm{d}w(t),$$

where

$$\mathscr{L}_p V(t,x) := V_t(t,x) + V_x(t,x)f_p(t,x) + \frac{1}{2}\operatorname{tr}\left(V_{xx}(t,x)G_p(t,x)G_p^{\mathrm{T}}(t,x)\right) \tag{3.3}$$

is the *infinitesimal generator* for the system with index p in (3.1) acting on the function V, and V_t, V_x, V_{xx} denote the partial differentials of $V(t,x)$ with respect to t, x, and twice with respect to x, respectively, and tr denotes the trace of a matrix; see, e.g., [14].

We focus on the following two general notions of stochastic stability in terms of two measures, which belong to the set Γ in (2.3).

DEFINITION 3.1. *Let* $h^\circ, h \in \Gamma$. *The stochastic switched system* (3.2) *is said to be* (h°, h)-globally uniformly asymptotically stable in the mean ((h°, h)-GUAS-M) *if there exists a function* $\beta \in \mathcal{KL}$ *such that for every* $(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$*, the inequality*

$$\mathsf{E}[h(t, x(t))] \leqslant \beta(h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \geqslant t_0 \tag{3.4}$$

holds.

DEFINITION 3.2. *The stochastic switched system* (3.2) *is said to be* (h°, h)-globally uniformly asymptotically stable in probability ((h°, h)-GUAS-P) *if for every* $\eta \in]0,1[$, *there exists a function* $\beta \in \mathcal{KL}$ *such that for every* $(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$, *the inequality*

$$\mathsf{P}\Big[h(t, x(t)) \geqslant \beta(h^\circ(t_0, x_0), t - t_0)\Big] < \eta \qquad \forall\, t \geqslant t_0 \tag{3.5}$$

holds.

Remark 3.3. We have stated our stability definitions in terms of deterministic initial condition. However, the results of this section hold true for the general case of stochastic initial condition as well. For instance, if we consider replacing $h^\circ(t_0, x_0)$ with $\mathsf{E}[h^\circ(t_0, x_0)]$ or $h^\circ(t_0, \mathsf{E}[x_0])$ in either (3.4) or (3.5), all the results hold true, with minor straightforward modifications in the proofs.

Remark 3.4. The (h°, h)-GUAS-M property is equivalent to the simultaneous verification of the following properties:

(SM1) there exists a class $\mathcal{K}_\infty$ function δ such that for every $\varepsilon > 0$, $t_0 \in \mathbb{R}_{\geqslant 0}$, we have $h^\circ(t_0, x_0) < \delta(\varepsilon) \implies \mathsf{E}[h(t, x(t))] < \varepsilon \quad \forall\, t \geqslant t_0$;

(SM2) for every $r, \varepsilon > 0$, there exists a number $T(r, \varepsilon) \geqslant 0$ such that for every $t_0 \in \mathbb{R}_{\geqslant 0}$, we have $h^\circ(t_0, x_0) < r \implies \mathsf{E}[h(t, x(t))] < \varepsilon \quad \forall\, t \geqslant t_0 + T(r, \varepsilon)$.

The proof follows directly from the proof of the equivalence in Remark 2.3 presented in Appendix A, with $\mathsf{E}[h(t, x(t))]$ replacing $h(t, x(t))$.

The (h°, h)-GUAS-P property is equivalent to the simultaneous verification of the following properties:

(WP1) for every $\eta' \in]0,1[$, there exists a function $\delta \in \mathcal{K}_\infty$ such that for every $\varepsilon > 0$, $t_0 \in \mathbb{R}_{\geqslant 0}$, we have $h^\circ(t_0, x_0) < \delta(\varepsilon) \implies \mathsf{P}[h(t, x(t)) \geqslant \varepsilon] < \eta' \quad \forall\, t \geqslant t_0$; and

(WP2) for every $\eta'' \in]0,1[$, $r, \varepsilon' > 0$, there exists a number $T(r, \varepsilon) \geqslant 0$ such that for every $t_0 \in \mathbb{R}_{\geqslant 0}$, we have $h^\circ(t_0, x_0) < r \implies \mathsf{P}[h(t, x(t)) \geqslant \varepsilon'] < \eta'' \quad \forall\, t \geqslant t_0 + T(r, \varepsilon')$.

Establishing this equivalence takes some more work; an outline may be found at the end of Appendix B.

Remark 3.5. We recover a notion essentially equivalent to *global uniform asymptotic stability in the qth mean* (GUAS-M$_q$) from Definition 3.1 with $h^\circ(t, x) = h(t, x) = |x|^q$, $q \geqslant 1$; see, e.g., [14]. With the same h° and h, Definition 3.2 yields *global uniform asymptotic stability in probability* (GUAS-P) in the sense of [23].

Remark 3.6. The (h°, h)-GUAS-P property of (3.2) follows from its (h°, h)-GUAS-M property. To see this, pick $\eta \in]0,1[$ and let there exist a function $\beta \in \mathcal{KL}$ such that (3.4) is satisfied. Consider a second function $\overline{\beta} \in \mathcal{KL}$ such that $\overline{\beta}(r, s) > \beta(r, s)/\eta$ for all $(r, s) \in \mathbb{R}^2_{\geqslant 0}$. Utilizing Chebyshev's inequality, we now have for every $t \geqslant t_0$

$$\begin{aligned}\mathsf{P}\big[h(t, x(t)) \geqslant \overline{\beta}(h^\circ(t_0, x_0), t - t_0)\big] &\leqslant \frac{\mathsf{E}[h(t, x(t))]}{\overline{\beta}(h^\circ(t_0, x_0), t - t_0)} \\ &\leqslant \frac{\beta(h^\circ(t_0, x_0), t - t_0)}{\overline{\beta}(h^\circ(t_0, x_0), t - t_0)} < \eta,\end{aligned}$$

which is the (h°, h)-GUAS-P property.

Comparison systems and stochastic comparison principle. We utilize the comparison system (2.6); see section 2.1 for the definitions. We need the following stochastic version of the comparison principle; see, e.g., [26] for a proof, where the authors also consider stochastic comparison systems.

LEMMA 3.7. *Consider the stochastic system with index p in the family* (3.1). *Suppose that there exist a function $V \in C^{1,2}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ and a comparison system*

$$\Sigma : \quad \dot{\xi} = \phi(t, \xi), \qquad \xi(t_0) = \xi_0 \geqslant 0, \quad t \geqslant t_0,$$

where ϕ is concave in the second argument, such that for all $(t, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$, the differential inequality

$$\mathscr{L}_p V(t, x) \leqslant \phi(t, V(t, x))$$

is valid. Then $\mathsf{E}[V(\tau, x(\tau))] \leqslant \xi(\tau)$ implies $\mathsf{E}[V(t, x(t))] \leqslant \xi(t) \quad \forall\, t \geqslant \tau$, where $x(t)$ and $\xi(t)$ are the solution process of the system with index p in (3.1) *and the solution of Σ, respectively.*

3.2. Comparison theorem for stochastic switched systems. The following result establishes a general framework for testing stability of stochastic switched systems using multiple Lyapunov functions and a comparison system.

THEOREM 3.8. *Consider the stochastic switched system* (3.2) *with a fixed switching signal σ generating a sequence of switching instants $\{\tau_i\}_{i \geqslant 1}$ and two functions $h^\circ, h \in \Gamma$. Suppose that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $V_p \in C^{1,2}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ for each $p \in \mathcal{P}$, and a system Σ of the type* (2.6), *such that*

- (i) *α_1 is convex, and ϕ is concave in the second argument;*
- (ii) *the family $\{V_p \mid p \in \mathcal{P}\}$ is $\mathcal{P}$-uniformly h-positive definite and h°-decrescent in the sense of* (2.5)*;*
- (iii) *$\forall\, (t, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ and $\forall\, p \in \mathcal{P}$, the estimate $\mathscr{L}_p V_p(t, x) \leqslant \phi(t, V_p(t, x))$ holds;*
- (iv) *$\forall\, (t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$, there exists $\xi_0 \in \mathbb{R}_{\geqslant 0}$ such that $\mathsf{E}\big[V_{\sigma(\tau_i)}(\tau_i, x(\tau_i))\big] \leqslant \xi(\tau_i)$ for all $i \geqslant 0$, where $x(t)$ and $\xi(t)$ are the solution process of* (3.2) *and the solution of Σ, respectively, for these initial conditions;*
- (v) *Σ is* GUAS *in the sense of* (2.7).

Then (3.2) *is (h°, h)-*GUAS-M *in the sense of Definition* 3.1.

Proof. Consider the interval $[\tau_\ell, \tau_{\ell+1}[$, with ℓ an arbitrary nonnegative integer. Combining hypotheses (iv), (iii), and (i), and Lemma 3.7 with $\tau = \tau_\ell$, we have

$$\mathsf{E}\big[V_{\sigma(\tau_\ell)}(t, x(t))\big] \leqslant \xi(t) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[.$$

The above estimate, in conjunction with hypothesis (v) and (2.7), yields

$$\mathsf{E}\big[V_{\sigma(\tau_\ell)}(t, x(t))\big] \leqslant \beta_\xi(\xi_0, t - t_0) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[. \tag{3.6}$$

The function α_1 being convex by hypothesis (i), taking expectations and using Jensen's inequality in (2.5) and considering (3.6), we reach the estimate

$$\alpha_1 \circ \mathsf{E}[h(t, x(t))] \leqslant \beta_\xi(\xi_0, t - t_0) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[. \tag{3.7}$$

The arbitrariness of ℓ in (3.7) leads to

$$\mathsf{E}[h(t, x(t))] \leqslant \beta(h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \geqslant t_0,$$

where $\beta(r, s) := \alpha_1^{-1} \circ \beta_\xi(\alpha_2(r), s)$. The function β being of class $\mathcal{KL}$, it follows that (3.2) is (h°, h)-GUAS-M. ∎

The following obvious Corollary, which we merely state, follows almost immediately from Theorem 3.8 in the light of Remark 3.6.

COROLLARY 3.9. *Suppose that the hypotheses of Theorem* 3.8 *hold true. Then* (3.2) *is* (h°, h)-GUAS-P *in the sense of* (3.5).

Remark 3.10. The hypothesis on convexity of α_1 in Corollary 3.9 actually is not necessary for (h°, h)-GUAS-P of (3.2). In Theorem 3.24 we will prove a stronger property without this convexity assumption, which will imply the (h°, h)-GUAS-P property of (3.2). For the moment, however, we shall work with (h°, h)-GUAS-M.

Theorem 3.8, like its deterministic counterpart Theorem 2.8, does not provide a direct method for analyzing stability of a given stochastic switched system. We will now demonstrate how to proceed with such a scheme of analysis and propose a few more specific results. In section 3.3 we provide sufficient conditions for (h°, h)-GUAS-M under a fixed-index sequence monotonicity condition. The comparison system utilized in the proof of this result utilizes quantitative information of system trajectories. In section 3.4 we provide sufficient conditions for GUAS-M$_q$ of a stochastic switched system under average dwell-time switching, where the fixed-index sequence monotonicity assumption imposed in section 3.3 is violated. We construct two comparison systems that do not require quantitative information of system trajectories.

3.3. Stochastic stability under fixed-index sequence monotonicity condition. For this subsection we let $\mathcal{P}$ be a finite set with N elements. The following result provides a stochastic version of Corollary 2.9.

COROLLARY 3.11. *Consider the stochastic switched system* (3.2) *and* $h^\circ, h \in \Gamma$. *Suppose there exist functions* $\alpha, \alpha_1, \alpha_2, \rho, U \in \mathcal{K}_\infty$, $V_p \in \mathrm{C}^{1,2}[\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ *for each* $p \in \mathcal{P}$, *such that*

(i) α_1, $\alpha \circ \alpha_2^{-1}$ *and* $U \circ \alpha_2^{-1}$ *are convex, and* ρ *is concave;*

(ii) *the family* $\{V_p \mid p \in \mathcal{P}\}$ *is* $\mathcal{P}$*-uniformly* h*-positive definite and* h°*-decrescent in the sense of* (2.5)*;*

(iii) $\forall\, (t, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ *and* $\forall\, p \in \mathcal{P}$, *we have* $\mathscr{L}_p V_p(t, x) \leqslant -\alpha \circ h^\circ(t, x)$*;*

(iv) *for every pair of switching times* (τ_i, τ_j), $i < j$ *such that* $\sigma(\tau_i) = \sigma(\tau_j) = p$ *and* $\sigma(\tau_k) \neq p$ *for* $\tau_i < \tau_k < \tau_j$, *the inequality*

$$\mathsf{E}[V_p(\tau_j, x(\tau_j))] - \mathsf{E}[V_p(\tau_i, x(\tau_i))] \leqslant -\mathsf{E}[U \circ h^\circ(\tau_i, x(\tau_i))] \tag{3.8}$$

holds, where $x(t)$ *is the solution process of* (3.2) *initialized at* (t_0, x_0)*;*

(v) $\forall\, (t, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$, *we have* $\alpha_2 \circ h^\circ(t, x) \leqslant \rho \circ \alpha_1 \circ h(t, x)$.

Then (3.2) *is* (h°, h)-GUAS-M.

Proof. We define a candidate impulsive differential comparison system of the type (2.6):

$$\Sigma : \quad \begin{cases} \dot{\xi} = -\alpha \circ \alpha_2^{-1}(\xi), & t \neq \tau_i, \\ \xi(\tau_i) = \mathsf{E}\left[V_{\sigma(\tau_i)}(\tau_i, x(\tau_i))\right], \end{cases} \qquad i \geqslant 0, \quad t \geqslant t_0. \tag{3.9}$$

By its very definition, Σ satisfies hypotheses (iii)–(iv) of Theorem 3.8. Hypotheses (i)–(ii) of Theorem 3.8 are satisfied by our hypotheses (i)–(ii). To verify hypothesis (v) of Theorem 3.8, we shall first prove Lyapunov stability of Σ and then prove its global asymptotic convergence, in view of Remark 2.6.

Consider the time interval $[\tau_0, \tau_1[$. From hypothesis (iii), we have

$$\mathsf{E}\left[V_{\sigma(\tau_0)}(\tau_1, x(\tau_1))\right] \leqslant \mathsf{E}\left[V_{\sigma(\tau_0)}(\tau_0, x_0)\right].$$

Combining with hypothesis (ii), using Jensen's inequality we reach

$$\mathsf{E}[\alpha_1(h(\tau_1, x(\tau_1)))] \leqslant \alpha_2 \circ h^\circ(t_0, x_0). \tag{3.10}$$

For every $p \in \mathcal{P}$, we have from hypothesis (i)

$$\mathsf{E}[V_p(\tau_1, x(\tau_1))] \leqslant \mathsf{E}[\alpha_2 \circ h^\circ(\tau_1, x(\tau_1))]\,,$$

and therefore by hypothesis (v),

$$\mathsf{E}[V_p(\tau_1, x(\tau_1))] \leqslant \mathsf{E}[\rho \circ \alpha_1 \circ h(\tau_1, x(\tau_1))]\,. \tag{3.11}$$

In view of hypothesis (ii), we apply Jensen's inequality to (3.11) and use hypothesis (ii) to get

$$\mathsf{E}[V_p(\tau_1, x(\tau_1))] \leqslant \rho(\mathsf{E}[\alpha_1 \circ h(\tau_1, x(\tau_1))]) \leqslant \rho \circ \alpha_2 \circ h^\circ(t_0, x_0). \tag{3.12}$$

Consider now the interval $[\tau_1, \tau_2[$. From hypothesis (iii) we have

$$\mathsf{E}\big[V_{\sigma(\tau_1)}(\tau_2, x(\tau_2))\big] \leqslant \mathsf{E}\big[V_{\sigma(\tau_1)}(\tau_1, x(\tau_1))\big]\,.$$

Combining with hypothesis (ii) and applying (3.12) with $p = \sigma(\tau_2)$, we get

$$\mathsf{E}[\alpha_1 \circ h(\tau_2, x(\tau_2))] \leqslant \rho \circ \alpha_2 \circ h^\circ(t_0, x_0). \tag{3.13}$$

For all $p \in \mathcal{P}$, we have

$$\mathsf{E}[V_p(\tau_2, x(\tau_2))] \leqslant \mathsf{E}[\alpha_2 \circ h^\circ(\tau_2, x(\tau_2))]\,,$$

so by hypothesis (iv) and Jensen's inequality we get

$$\mathsf{E}[V_p(\tau_2, x(\tau_2))] \leqslant \rho(\mathsf{E}[\alpha_1 \circ h(\tau_2, x(\tau_2))]).$$

Now (3.12) gives

$$\mathsf{E}[V_p(\tau_2, x(\tau_2))] \leqslant \rho \circ \rho \circ \alpha_2 \circ h^\circ(t_0, x_0).$$

It is not difficult to see that the worst-case situation for maximum possible overshoot of the function $\mathsf{E}[V_\sigma]$ occurs when the switching signal σ visits every element of the set $\mathcal{P}$ without repetition until $\mathcal{P}$ is exhausted. Let $\tau_{j^\star}$ be the first switching instant after all the subsystems that participate in the dynamics have become active at least once since initialization at $t = t_0$. From the above computations, it is easy to see that, with

$$\rho^j := \underbrace{\rho \circ \ldots \circ \rho}_{j \text{ times}},$$

the estimate

$$\xi(\tau_{j^\star}) \leqslant \rho^{\mathrm{N}-1} \circ \alpha_2 \circ h^\circ(t_0, x_0),$$

is valid. Define the function

$$\gamma(\cdot) := \max\left\{\alpha_2(\cdot), \rho \circ \alpha_2(\cdot), \ldots, \rho^{\mathrm{N}-1} \circ \alpha_2(\cdot)\right\}.$$

From the above arguments and (3.8), it follows that $\xi(t) \leqslant \gamma \circ h^\circ(t_0, x_0)$. In view of the definition of ξ_0 and hypothesis (ii), this leads to

$$\xi(t) \leqslant \gamma \circ \alpha_1^{-1}(\xi_0) \qquad \forall\, t \geqslant t_0. \tag{3.14}$$

It remains to prove uniform global asymptotic convergence of Σ.

We distinguish two cases.

Case 1. *Switching stops in finite time.* Since σ eventually becomes constant from the κth switching instant, it follows that there are no impulses after $t = \tau_\kappa$. Therefore, the system (3.9) becomes an autonomous scalar ordinary differential equation after $t = \tau_\kappa$, with negative right-hand side for nonzero $\xi(\tau_\kappa)$. It follows that $\xi(t)$ monotonically decreases to 0 for all $t \geqslant \tau_\kappa$. From (3.14) uniform Lyapunov stability of Σ follows. Thus, (3.9) is GUAS. Theorem 3.8 now guarantees that (3.2) is (h°, h)-GUAS-M.

Case 2. *Switching continues indefinitely.* Consider the restatement of the inequality (3.8) with $\xi(\tau_i)$ as in (3.9)

$$\begin{aligned}\xi(\tau_j) - \xi(\tau_i) &\leqslant -\mathsf{E}[U \circ h^\circ(\tau_i, x(\tau_i))] = -\mathsf{E}\left[U \circ \alpha_2^{-1} \circ \alpha_2 \circ h^\circ(\tau_i, x(\tau_i))\right] \\ &\leqslant -U \circ \alpha_2^{-1}\left(\mathsf{E}[\alpha_2 \circ h^\circ(\tau_i, x(\tau_i))]\right),\end{aligned}$$

where we have utilized Jensen's inequality and convexity of $U \circ \alpha_2^{-1}$ in hypothesis (i). The infinite sequence $\{\xi(\tau_i)\}_{\{i \geqslant 0 | \sigma(\tau_i) = p\}}$ is monotonically nonincreasing and therefore must attain a limit, say, $c \geqslant 0$. Taking limits as $i \uparrow \infty$ on both sides of (3.8), we have

$$c - c \leqslant - \lim_{\substack{i \uparrow \infty \\ \sigma(\tau_i) = p}} U \circ \alpha_2^{-1}\left(\mathsf{E}[\alpha_2 \circ h^\circ(\tau_i, x(\tau_i))]\right),$$

which leads to

$$\lim_{\substack{i \uparrow \infty \\ \sigma(\tau_i) = p}} \mathsf{E}[\alpha_2 \circ h^\circ(\tau_i, x(\tau_i))] = 0.$$

Considering hypothesis (ii) and the reset equation in (3.9), we have

$$\lim_{\substack{i \uparrow \infty \\ \sigma(\tau_i) = p}} \xi(\tau_i) = \lim_{\substack{i \uparrow \infty, \\ \sigma(\tau_i) = p}} \mathsf{E}\left[V_{\sigma(\tau_i)}(\tau_i, x(\tau_i))\right] = 0. \tag{3.15}$$

Combining (3.14) and (3.15), we conclude that (3.9) is GUAS.

By Theorem 3.8, there exists a function $\beta \in \mathcal{KL}$ such that

$$\mathsf{E}[h(t, x(t))] \leqslant \beta(h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \geqslant t_0,$$

and we conclude that (3.2) is (h°, h)-GUAS-M. $\square$

Remark 3.12. We note that in the above proof, Σ makes explicit use of state information of (3.2)—in the notation of (2.6), we use

$$\psi_i\left(\xi\left(\tau_i^-\right), y(x(\tau_i), \sigma(\tau_i))\right) = \mathsf{E}\left[V_{\sigma(\tau_i)}(\tau_i, x(\tau_i))\right]$$

in the reset equation. However, $\xi(\tau_i^-)$ is not utilized; cf. Remark 2.10.

Remark 3.13. For Euclidean norms, the function $\rho \in \mathcal{K}_\infty$ always exists if the function $\alpha_2 \circ \alpha_1^{-1}$ is concave (cf. Remark 2.11). Also, if α_1, α_2, ρ, and U are quadratic, as is typically the case for linear systems, hypothesis (i) is always satisfied.

Remark 3.14. It readily follows that for autonomous switched stochastic systems and h°, h specialized to Euclidean norms, Corollary 3.11 gives global asymptotic stability in the mean. Corollary 3.9 then implies global asymptotic stability in probability, which is derived in [8] without the aid of the comparison framework.

3.4. Stochastic stability under average dwell-time switching. In this subsection we investigate conditions on the average dwell-time τ_a of a switching signal such that (3.2) has the GUAS-M$_q$ property. We no longer retain the assumption that $\mathcal{P}$ is finite. We specialize to Euclidean norms for simplicity and propose the following result, which may be regarded as a stochastic counterpart of Theorem 2.12. A generalization of Theorem 3.15 to two measures and nonautonomous stochastic switched systems is readily done.

Let us consider the autonomous stochastic switched system

$$\mathrm{d}x = f_\sigma(x)\mathrm{d}t + G_\sigma(x)\mathrm{d}w, \qquad x(t_0) = x_0, \quad t \geqslant t_0, \tag{3.16}$$

where $x \in \mathbb{R}^n$, $f_p \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}^n]$, $G_p \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}^{n\times m}]$, $f_p(0) = 0$, $G_p(0) = 0$ for every $p \in \mathcal{P}$, w is an m-dimensional Wiener process on the probability space $\mathbf{\Omega}$. We assume that f_p and G_p are smooth enough to ensure existence and uniqueness of the corresponding solution processes for every $p \in \mathcal{P}$; see, e.g., [36] for precise conditions.

We need the definition of average dwell-time from (2.17) for the following result.

THEOREM 3.15. *Consider the switched system* (3.2). *Suppose that there exist functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $V_p \in \mathrm{C}^1[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ *for each* $p \in \mathcal{P}$, *and a positive number* $\lambda_\circ$ *such that*

$$\alpha_1(|x|^q) \leqslant V_p(x) \leqslant \alpha_2(|x|^q) \qquad \forall\, x \in \mathbb{R}^n \tag{3.17}$$

with α_1 *convex,* $q \geqslant 1$, *and*

$$\mathscr{L}_p V_p(x) \leqslant -\lambda_\circ V_p(x) \qquad \forall\, x \in \mathbb{R}^n. \tag{3.18}$$

Suppose also that there exists a positive constant μ *such that for each* $t \in \mathbb{R}_{\geqslant 0}$,

$$V_{p_1}(x) \leqslant \mu V_{p_2}(x) \qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, p_1, p_2 \in \mathcal{P}. \tag{3.19}$$

Then (3.2) *is* GUAS-M$_q$ *for every switching signal* σ *with average dwell-time* $\tau_a > \frac{\ln \mu}{\lambda_\circ}$.

For a single system, (3.18) is a condition that implies global exponential stability in the qth mean [14], under an additional condition of $\alpha_i(r) = k_i r$, $i = 1, 2$. Theorem 3.15 is particularly simple for autonomous linear stochastic switched systems as we now show.

Consider a stochastic switched autonomous linear system

$$\mathrm{d}x = A_\sigma x \mathrm{d}t + B_\sigma x \mathrm{d}w_t, \qquad x(0) = x_0, \quad t \geqslant 0, \tag{3.20}$$

where $\sigma(t) \in \mathcal{P}$, $x \in \mathbb{R}^n$, $A_p, B_p \in \mathbb{R}^{n\times n}$, w is a scalar normalized Wiener process. For quadratic Lyapunov functions $V_p(x) = x^{\mathrm{T}} P_p x$, where P_p, $p \in \mathcal{P}$ is a positive definite symmetric matrix, with the aid of (3.3) the condition (3.18) simplifies to

$$A_p^{\mathrm{T}} P_p + P_p A_p + B_p P_p B_p^{\mathrm{T}} + \lambda_\circ P_p \leqslant 0, \tag{3.21}$$

which is a linear matrix inequality. To get the quadratic Lyapunov functions V_p, it is necessary to solve (3.21) for each $p \in \mathcal{P}$; see [5] for related methods of solution of such linear matrix inequalities, and see [14] for a discussion on quadratic Lyapunov functions in stability analysis of stochastic linear systems. Note that (3.19) is automatically satisfied if we can find such $V_p = x^{\mathrm{T}} P_p x$, with P_p satisfying (3.21).

We provide two different proofs of Theorem 3.15 to illustrate the versatility of our comparison framework.

An impulsive differential equation as a comparison system.

Proof. Consider an impulsive differential system of the type (2.6) with

$$\psi_i\left(\xi\left(\tau_i^-\right), y(x(\tau_i), \sigma(\tau_i))\right) = \mu\xi(\tau_i^-), \quad i \geqslant 1, \quad \mu > 0,$$

as the reset equation, and

$$\phi(t, \xi) = -\lambda_\circ \xi, \quad \lambda_\circ > 0.$$

Note that concavity of ϕ with respect to ξ is trivially satisfied. The complete system stands as

$$\text{(3.22)} \quad \Sigma' : \quad \begin{cases} \dot{\xi} = -\lambda_\circ \xi, & t \neq \tau_i, \\ \xi(\tau_i) = \mu\xi(\tau_i^-), & \mu > 0, \end{cases} \qquad \xi(t_0) = V_{\sigma(t_0)}(x_0), \quad i \geqslant 1, \quad t \geqslant t_0.$$

This system Σ' is the same as (2.21). From (3.17) it follows that hypothesis (ii) of Theorem 3.8 is satisfied with $h^\circ(t, x) = h(t, x) = |x|^q$. Further, from (3.18) and (3.19) together with the initial condition in (3.22), it follows that hypotheses (iii) and (iv) of Theorem 3.8 are satisfied, respectively.

Intuitively,

- the minimum rate of decay of the expected values of Lyapunov functions corresponding to each active subsystem is captured by the vector field of Σ'; and
- the maximum jump in the values of two Lyapunov functions corresponding to two consecutively active subsystems is captured by the reset equation of Σ' (since from (3.19) it follows that $\mathsf{E}[V_{p_1}(x(\tau_i))] \leqslant \mu\mathsf{E}\big[V_{p_2}(x(\tau_i^-))\big]$ for every $i \geqslant 1$, as in section 2.4, $\mu \geqslant 1$).

As in section 2.4, with $\tau_a > \frac{\ln\mu}{\lambda_\circ}$ the GUAS property of Σ' follows, which verifies hypothesis (v) of Theorem 3.8. In view of Remark 3.5, by Theorem 3.8 we conclude that (3.2) is GUAS-M$_q$. ∎

An ordinary differential equation as a comparison system.

Proof. Consider the following scalar autonomous system as a candidate comparison system:

$$\text{(3.23)} \qquad \Sigma'' : \quad \dot{\xi} = \left(\frac{\ln\mu}{\tau_a} - \lambda_\circ\right)\xi, \qquad \xi(t_0) = \mu^{N_\circ} \mathrm{e}^{\lambda_\circ t_0} V_{\sigma(t_0)}(x_0), \quad t \geqslant t_0,$$

Note that concavity of the vector field of Σ'' with respect to ξ is trivially satisfied. Let the average dwell-time of σ be τ_a, and let $\nu := N_\sigma(T, t_0)$. Considering the least rate of decay of the expected values of Lyapunov functions corresponding to each active subsystem given by (3.18), we have for an arbitrary $T \geqslant 0$,

$$\mathsf{E}\big[V_{\sigma(\tau_i)}(x(\tau_{i+1}^-))\big] \leqslant \mathsf{E}\big[V_{\sigma(\tau_i)}(x(\tau_i))\big]\, \mathrm{e}^{-\lambda_\circ(\tau_{i+1} - \tau_i)}, \quad 0 \leqslant i \leqslant \nu,$$

and

$$\mathsf{E}\big[V_{\sigma(\tau_\nu)}(x(T))\big] \leqslant \mathsf{E}\big[V_{\sigma(\tau_\nu)}(x(\tau_\nu))\big]\, \mathrm{e}^{-\lambda_\circ(T - \tau_\nu)}.$$

Combining with the reset equation, we have

$$\mathsf{E}\big[V_{\sigma(\tau_\nu)}(x(T))\big] \leqslant \mu^\nu V_{\sigma(t_0)}(x_0) \mathrm{e}^{-\lambda_\circ(T - t_0)}.$$

The solution ξ of (3.23) is identical to (2.25). From (3.17) it follows that hypothesis (ii) of Theorem 3.8 is satisfied with $h^\circ(t,x) = h(t,x) = |x|^q$. Further, from (3.18) and (3.19) together with the initial condition in (3.23), it follows that hypotheses (iii) and (iv) of Theorem 3.8 are satisfied, respectively.

Intuitively,

- the initial condition of Σ'' captures the maximum possible overshoot in $\mathsf{E}[V_\sigma]$—this corresponds to the situation when all $N_\circ$ switches occur very close to $t = t_0$;
- $\xi(\cdot)$ forms an envelope of the sequence $\big\{\mathsf{E}\big[V_{\sigma(\tau_i)}(x(\tau_i))\big]\big\}_{i\geqslant 0}$ over the interval $[t_0, T]$.

As in section 2.4, $\lambda_\circ > \frac{\ln\mu}{\tau_a}$ ensures global uniform asymptotic stability of Σ''; this verifies Theorem 3.8 hypothesis (v). It follows that by Theorem 3.8, (3.2) is GUAS-M$_q$ for switching signals with $\tau_a > \frac{\ln\mu}{\lambda_\circ}$ in view of Remark 3.5. ∎

Remark 3.16. We note that in contrast to (3.9), the comparison systems (3.22) and (3.23) do not utilize state information in the form of the function y directly; cf. Remark 3.12.

Remark 3.17. From the solutions of (3.22) and (3.23) it is clear that for $\tau_a > \frac{\ln\mu}{\lambda_\circ}$, GUAS of Σ' and Σ'' are ensured, which in turn imply that (2.2) is GUAS-M$_q$. In other words, the GUAS-M$_q$ property of (3.2) is "uniform" over all switching signals with $\tau_a > \frac{\ln\mu}{\lambda_\circ}$. We therefore say that (3.2) is *globally "uniformly" asymptotically stable in the qth mean* over all such switching signals; see also Remark 2.15.

Remark 3.18. What we stated in Remark 2.17 for deterministic switched systems carries over to the stochastic case quite easily; it is not difficult to show that in the context of Theorem 3.8, under suitable hypotheses each subsystem may be exponentially unstable while the switched system (3.2) remains (h°, h)-GUAS-M$_q$.

Namely, consider the stochastic switched system (3.2), and suppose that σ is such that $N_\sigma(t,t_0)$ obeys (2.28) for some $N_\circ, \delta_a > 0$. Let there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $V_p \in \mathrm{C}^1[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$, $p \in \mathcal{P}$, and real numbers $\mu' \in]0,1[$ and $\lambda_\circ > 0$, such that

(i) the estimate (3.17) holds;
(ii) $\forall\, x \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ and $\forall\, p \in \mathcal{P}$, we have $\mathscr{L}_p V_p(x) \leqslant \lambda_\circ V_p(x)$;
(iii) $\mathsf{E}\big[V_{\sigma(\tau_i)}(x(\tau_i))\big] \leqslant \mu' \mathsf{E}\Big[V_{\sigma(\tau_i^-)}(x(\tau_i^-))\Big] \quad \forall\, i \geqslant 1$;
(iv) $\delta_a < -\ln\mu'/\lambda_\circ$.

Then (3.2) is GUAS-M$_q$.

It may be verified that the comparison system $\widetilde{\Sigma}$ in (2.29) is a suitable comparison system under the above hypotheses, and the assertion follows from Theorem 3.8.

Remark 3.19. We mentioned in Remark 2.18 that for a system with state-dependent switching, in general we cannot conclude stability for more than one initial condition from the associated switched system (2.2). The situation is more complicated in the case of a hybrid system with continuous dynamics perturbed by a Wiener process; now the switching signals σ corresponding to different trajectories are different even for a fixed initial condition, making direct analysis of such a system difficult. The switched system (3.2) provides for simpler stability analysis for a fixed initial condition, but once again we cannot conclude stability of the hybrid system under variations in initial conditions from stability of the switched system. Theorem 3.8 is stated without any claim of uniformity with respect to initial conditions. However, in Theorem 3.15 we obtained uniform stability of a switched system perturbed by a Wiener process over the class of σ defined by a suitable average dwell-time, without taking into account an underlying hybrid system. This class of signals is potentially useful for hybrid systems where variations in the initial condition preserve average dwell-time switching; see [39] for some results in this direction.

3.5. Remarks on other stability notions. In this subsection we study two notions of stochastic stability of switched systems, different from the ones considered so far, and utilize the framework of Theorem 3.8.

Stochastic practical stability. Stochastic practical stability is concerned with practical stability of systems perturbed by a Wiener process with respect to prespecified domains in the state space. We briefly study a representative notion of stochastic practical stability below; see [40] for a version of the definition for nonswitched systems and with $h^\circ(t,x) = h(t,x) = |x|^q$.

DEFINITION 3.20. *Let* $h^\circ, h \in \Gamma$ *and the pair* (λ, A), $\lambda \in \,]0, A[$ *be given. The stochastic switched system* (3.2) *is said to be* (h°, h)-uniformly practically stable in the mean with respect to (λ, A) *if for every* $t_0 \in \mathbb{R}_{\geqslant 0}$*, the property*

$$h^\circ(t_0, x_0) < \lambda \implies \mathsf{E}[h(t, x(t))] < A \qquad \forall\, t \geqslant t_0 \tag{3.24}$$

holds for all solution processes.

The following result provides sufficient conditions for (h°, h)-uniform practical stability of (3.2) in the mean with respect to a given pair (λ, A).

PROPOSITION 3.21. *Consider the stochastic switched system* (3.2) *with a given* σ, $h^\circ, h \in \Gamma$, *and let the pair* (λ, A), $\lambda \in \,]0, A[$ *be given. Suppose that there exist functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $V_p \in \mathrm{C}^{1,2}\big[\mathcal{B}(h, A) \cap \big((\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n) \setminus \mathcal{B}(h^\circ, \lambda)\big), \mathbb{R}_{\geqslant 0}\big]$ *for each* $p \in \mathcal{P}$, *and a system* Σ *of the type* (2.6), *such that*

(i) α_1 *is convex, and* ϕ *is concave in the second argument;*
(ii) *the family* $\{V_p \mid p \in \mathcal{P}\}$ *is* $\mathcal{P}$*-uniformly* h*-positive definite and* h°*-decrescent in the sense of* (2.5)*;*
(iii) $\forall\, (t,x) \in \mathcal{B}(h, A) \cap \big((\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n) \setminus \mathcal{B}(h^\circ, \lambda)\big)$ *and* $\forall\, p \in \mathcal{P}$, *we have* $\mathscr{L}_p V_p(t,x) \leqslant \phi(t, V_p(t,x))$*;*
(iv) $\forall\, (t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$, *there exists* $\xi_0 \in \mathbb{R}_{\geqslant 0}$ *such that* $\mathsf{E}\big[V_{\sigma(\tau_i)}(\tau_i, x(\tau_i))\big] \leqslant \xi(\tau_i)$ *for all* $i \geqslant 0$, *where* $x(t)$ *and* $\xi(t)$ *are the solution process of* (3.2) *and the solution of* Σ, *respectively, for these initial conditions;*
(v) Σ *is uniformly practically stable with respect to* $(\alpha_2(\lambda), \alpha_1(A))$.

Then (3.2) *is* (h°, h)*-uniformly practically stable in the mean with respect to* (λ, A).

Proof. In view of Remark 2.20, the uniform practical stability of Σ with respect to $(\alpha_2(\lambda), \alpha_1(A))$ implies that for every $t_0 \in \mathbb{R}_{\geqslant 0}$,

$$\xi_0 < \alpha_2(\lambda) \implies \xi(t) < \alpha_1(A) \qquad \forall\, t \geqslant t_0. \tag{3.25}$$

Select and arbitrary x_0 such that $h^\circ(t_0, x_0) < \lambda$, and pick $\xi_0 = \alpha_2 \circ h^\circ(t_0, x_0)$ and an arbitrary nonnegative integer ℓ. From hypothesis (iv) and Lemma 3.7 at time $\tau = \tau_\ell$, we have

$$\mathsf{E}\big[V_{\sigma(\tau_\ell)}(\tau_\ell, x(\tau_\ell))\big] \leqslant \xi(t) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[. \tag{3.26}$$

Combining (3.26) with (3.25) over the time interval $[\tau_\ell, \tau_{\ell+1}[$, and using hypothesis (iii), we have

$$\mathsf{E}[\alpha_1 \circ h(t, x(t))] < \alpha_1(A) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[. \tag{3.27}$$

Since α_1 is convex by hypothesis (i), by Jensen's inequality in (3.27), we reach

$$\alpha_1(\mathsf{E}[h(t, x(t))]) < \alpha_1(A) \qquad \forall\, t \in [\tau_\ell, \tau_{\ell+1}[. \tag{3.28}$$

The inequality (3.28), together with the arbitrariness of ℓ, indicate that (3.24) holds. The (h°, h)-uniform practical stability in the mean of (3.2) with respect to (λ, A) follows. ∎

Strong global uniform asymptotic stability in probability. We present a stronger version of global asymptotic stability in probability below; see, e.g., [33] for a version of the definition for nonswitched systems and Euclidean norms.

DEFINITION 3.22. *Let $h^\circ, h \in \Gamma$. The stochastic switched system* (3.2) *is said to be* (h°, h)-strongly globally uniformly asymptotically stable in probability ((h°, h)-SGUAS-P) *if for every $\eta \in]0,1[$, there exists a function $\beta \in \mathcal{KL}$ such that for every $(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$, the inequality*

$$\mathsf{P}\left[\sup_{t \geqslant t_0} h(t, x(t)) \geqslant \beta(h^\circ(t_0, x_0), t - t_0)\right] < \eta \tag{3.29}$$

holds.

LEMMA 3.23. *The (h°, h)-SGUAS-P property is equivalent to the simultaneous verification of the following two properties:*

(SP1) *for every $\eta' \in]0,1[$, there exists a function $\delta \in \mathcal{K}_\infty$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}_{\geqslant 0}$, we have*

$$h^\circ(t_0, x_0) < \delta(\varepsilon) \implies \mathsf{P}\left[\sup_{t \geqslant t_0} h(t, x(t)) \geqslant \varepsilon\right] < \eta'; \tag{3.30}$$

(SP2) *for every $\eta'' \in]0,1[$, $r, \varepsilon' > 0$, there exists a number $\widetilde{T}(r, \varepsilon') > 0$ such that for every $t_0 \in \mathbb{R}_{\geqslant 0}$, we have*

$$h^\circ(t_0, x_0) < r \implies \mathsf{P}\left[\sup_{t \geqslant t_0 + \widetilde{T}(r,\varepsilon')} h(t, x(t)) \geqslant \varepsilon'\right] < \eta''. \tag{3.31}$$

A proof of this result is provided in Appendix B.

It is clear that the (h°, h)-SGUAS-P property is stronger than the (h°, h)-GUAS-P property. However, the same hypotheses as those of Corollary 3.9 ensure this stronger property, as we prove below. In fact, the hypotheses can be slightly weaker—we can do away with the convexity assumption of α_1 in Theorem 3.8.

THEOREM 3.24. *Suppose the hypotheses of Theorem* 3.8 *hold true, with α_1 not necessarily convex. Then* (3.2) *is (h°, h)-SGUAS-P.*

Proof. In view of Lemma 3.23 it suffices to prove (SP1)–(SP2). We first prove (SP1). Let $\eta' \in]0,1[$ and $\varepsilon > 0$ be given. Since Σ is GUAS by hypothesis (v), in view of Remark 2.6 there exists a function $\delta_\xi \in \mathcal{K}_\infty$ such that for every $t_0 \in \mathbb{R}_{\geqslant 0}$, we have

$$\xi_0 < \delta_\xi(\eta' \alpha_1(\varepsilon)) \implies \xi(t) < \eta' \alpha_1(\varepsilon) \qquad \forall\, t \geqslant t_0.$$

Let $\delta(\cdot) := \alpha_2^{-1} \circ \delta_\xi(\eta' \alpha_1(\cdot))$, where we have suppressed the dependence of δ on η', which is implied. Choose an arbitrary x_0 such that $h^\circ(t_0, x_0) < \delta(\varepsilon)$ and let $\xi_0 := \alpha_2 \circ h^\circ(t_0, x_0)$. Then from hypotheses (iii), (iv) and Lemma 3.7, it follows that

$$\mathsf{E}\big[V_{\sigma(t)}(t, x(t))\big] \leqslant \xi(t) < \eta' \alpha_1(\varepsilon) \qquad \forall\, t \geqslant t_0. \tag{3.32}$$

Claim 1. We have $\mathsf{P}[\sup_{t \geqslant t_0} h(t, x(t)) \geqslant \varepsilon] < \eta'$.

Indeed, let τ_ε be the first exit time of $x(t)$ from $\mathcal{B}(h,\varepsilon)$, i.e., $\tau_\varepsilon := \inf\{t \geqslant t_0 \mid h(t,x(t)) \geqslant \varepsilon\} \leqslant \infty$. Therefore, from (3.32), we have

$$\mathsf{E}\big[V_{\sigma(\tau_\varepsilon\wedge t)}(\tau_\varepsilon\wedge t, x(\tau_\varepsilon\wedge t))\big] < \eta'\alpha_1(\varepsilon) \qquad \forall\, t \geqslant t_0. \tag{3.33}$$

Fix an arbitrary $t' > t_0$. We have from (3.33),

$$\eta'\alpha_1(\varepsilon) > \mathsf{E}\big[V_{\sigma(\tau_\varepsilon\wedge t')}(\tau_\varepsilon\wedge t', x(\tau_\varepsilon\wedge t'))\big] \geqslant \mathsf{E}\big[\mathbf{1}_{\{\tau_\varepsilon\leqslant t'\}}V_{\sigma(\tau_\varepsilon)}(\tau_\varepsilon, x(\tau_\varepsilon))\big]\,, \tag{3.34}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. From hypothesis (ii) and the definition of τ_ε, it follows that

$$\mathsf{E}\big[\mathbf{1}_{\{\tau_\varepsilon\leqslant t'\}}V_{\sigma(\tau_\varepsilon)}(\tau_\varepsilon, x(\tau_\varepsilon))\big] \geqslant \mathsf{E}\big[\mathbf{1}_{\{\tau_\varepsilon\leqslant t'\}}\alpha_1\circ h(\tau_\varepsilon, x(\tau_\varepsilon))\big] = \mathsf{P}[\tau_\varepsilon\leqslant t']\,\alpha_1(\varepsilon). \tag{3.35}$$

Combining (3.34) and (3.35) we get $\eta'\alpha_1(\varepsilon) > \alpha_1(\varepsilon)\mathsf{P}[\tau_\varepsilon \leqslant t']$, and considering the definition of τ_ε, this leads to

$$\mathsf{P}\left[\sup_{t\in[t_0,t']} h(t,x(t)) \geqslant \varepsilon\right] < \eta'.$$

Since $t' > t_0$ is arbitrary, we have

$$\mathsf{P}\left[\sup_{t\geqslant t_0} h(t,x(t)) \geqslant \varepsilon\right] < \eta',$$

whence Claim 1 is verified. The (SP1) property (3.30) of (3.2) follows.

We now sketch the proof of (SP2), which is very similar to the proof of (SP1). Let $\eta'' \in\,]0,1[$ and $r, \varepsilon' > 0$ be given. Since Σ is GUAS by hypothesis (v), in view of Remark 2.6 there exists a number $T(\alpha_2(r), \eta''\alpha_1(\varepsilon')) \geqslant 0$ such that for every $t_0 \in \mathbb{R}_{\geqslant 0}$ we have

$$\xi_0 < \alpha_2(r) \implies \xi(t) < \eta''\alpha_1(\varepsilon') \qquad \forall\, t \geqslant t_0 + T(\alpha_2(r), \eta''\alpha_1(\varepsilon')).$$

Choose x_0 such that $h^\circ(t_0, x_0) < r$ and let $\xi_0 := \alpha_2\circ h^\circ(t_0,x_0)$. Then from hypotheses (iii), (iv) and Lemma 3.7, it follows that

$$\mathsf{E}\big[V_{\sigma(t)}(t,x(t))\big] \leqslant \xi(t) < \eta''\alpha_1(\varepsilon') \qquad \forall\, t \geqslant t_0 + T(\alpha_2(r), \eta''\alpha_1(\varepsilon')). \tag{3.36}$$

Claim 2. We have $\mathsf{P}\left[\sup_{t\geqslant t_0+T(\alpha_2(r),\eta''\alpha_1(\varepsilon'))} h(t,x(t)) \geqslant \varepsilon'\right] < \eta''$.

Indeed, defining $\tau_{\varepsilon'} := \inf\{t \geqslant t_0 + T(\alpha_2(r), \eta''\alpha_1(\varepsilon')) \mid h(t,x(t)) \geqslant \varepsilon'\} \leqslant \infty$, and following the steps of the above proof of (SP1) with τ'_ε replacing τ_ε, for an arbitrary $t'' > t_0 + T(\alpha_2(r), \eta''\alpha_1(\varepsilon'))$, we obtain

$$\eta''\alpha_1(\varepsilon') > \mathsf{E}\big[V_{\sigma(\tau_{\varepsilon'}\wedge t'')}(\tau_{\varepsilon'}\wedge t'', x(\tau_{\varepsilon'}\wedge t''))\big] \geqslant \mathsf{E}\big[\mathbf{1}_{\{\tau_{\varepsilon'}\leqslant t''\}}V_{\sigma(\tau_{\varepsilon'})}(\tau_{\varepsilon'}, x(\tau_{\varepsilon'}))\big]\,.$$

This leads to $\eta''\alpha_1(\varepsilon') > \mathsf{P}[\tau_{\varepsilon'} \leqslant t'']\,\alpha_1(\varepsilon')$. Since t'' is arbitrary, by the definition of $\tau_{\varepsilon'}$ we get

$$\mathsf{P}\left[\sup_{t\geqslant t_0+\widetilde{T}(r,\varepsilon')} h(t,x(t)) \geqslant \varepsilon'\right] < \eta'',$$

where $\widetilde{T}(r,\varepsilon') := T(\alpha_2(r), \eta''\alpha_1(\varepsilon'))$, suppressing the dependence on η'', which is implied. This verifies Claim 2, and hence the (SP2) property (3.31) of (3.2).

We conclude that (3.2) is (h°, h)-SGUAS-P. ∎

Since (h°, h)-SGUAS-P implies (h°, h)-GUAS-P, we obtain the (h°, h)-GUAS-P property of (3.2) without the necessity of α_1 being convex; cf. Remark 3.10. See also [23] for a proof of the GUAS-P property of a single stochastic system without the convexity assumption on α_1.

4. Conclusion. We have established a general framework for stability analysis of deterministic and stochastic switched systems. In section 2 we have unified representative existing results on deterministic switched systems and provided illustrations of how we can improve upon the scope of applicability of Lyapunov's second method to switched systems. In section 3 we have established new results on stability of stochastic switched systems. We have carried out analysis in terms of two measures and demonstrated how our framework applies to various stability notions. To conclude, we make the following comments with an eye toward possible future work.

In this paper we have considered disturbances in the form of a Wiener process affecting the states of a switched system, and we have not paid attention to the mechanism of switching signal generation. A common way to generate a switching signal, which has received a lot of attention in recent literature, is via a Markov chain with state space $\mathcal{P}$; see, e.g., [2, 4, 21]. In situations where both Markovian switching and a Wiener process are present, we have to keep in mind two different probability spaces in general: one that generates the disturbances, and the other which governs the Markovian switching. Some results on stability analysis of piecewise deterministic systems with Markovian switching are under development by the authors and will be presented separately.

Although we have utilized only scalar Lyapunov functions in this paper, there can be a parallel development utilizing *vector Lyapunov functions*. For details on this method, see, e.g., [32] and also [29] for an extensive development and discussion on vector Lyapunov functions used in conjunction with analysis using two measures.

A suitably modified version of the comparison framework can be used for systems with inputs also; details for the case of input-to-state stability of switched systems will be reported elsewhere.

Appendix A. Equivalence of definitions of (h°, h)-GUAS.

Proof of the claim in Remark 2.3. *Sufficiency.* Assume that the (S1) property holds. With a fixed $\varepsilon > 0$ we have

$$h^\circ(t_0, x_0) \leqslant \delta(\varepsilon) \implies h(t, x(t)) \leqslant \varepsilon \qquad \forall\, t \geqslant t_0,$$

and the above implication holds uniformly over t_0. Defining the function $\varphi \in \mathcal{K}_\infty$ such that $\varphi(\cdot) := \delta^{-1}(\cdot)$, it follows that

$$h(t, x(t)) \leqslant \varphi \circ h^\circ(t_0, x_0) \qquad \forall\, t \geqslant t_0, \tag{A.1}$$

since $\varepsilon > 0$ is arbitrary.

Now we assume that the (S2) property holds.

A straightforward generalization of [31, Lemma 3.1] leads to the existence of a family of mappings $\{T_r\}_{r>0}$ with the properties

- for every fixed $r > 0$, $T_r \in \mathrm{C}[\mathbb{R}_{>0}, \mathbb{R}_{>0}]$ is surjective and strictly decreasing, and
- for every fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is strictly increasing with r and $\lim_{r\uparrow\infty} T_r(\varepsilon) = \infty$,

such that $h^\circ(t_0, x_0) < r$ implies $h(t, x(t)) < \varepsilon \quad \forall\, t \geqslant t_0 + T_r(\varepsilon)$.

Let $\psi_r(\cdot) := T_r^{-1}(\cdot)$, $r \in]0,\infty[$. Then $\psi_r \in \mathrm{C}[\mathbb{R}_{>0}, \mathbb{R}_{>0}]$ is surjective and strictly decreasing. We write $\psi_r(0) = \infty$, considering $\lim_{t\downarrow 0} \psi_r(t) = \infty$.

Claim. For $h^\circ(t_0, x_0) < r$ we have $h(t, x(t)) \leqslant \psi_r(t)$ for $t \geqslant t_0$.

Proof. It follows from the definition of T_r that for an arbitrary $r, \varepsilon > 0$,

$$h^\circ(t_0, x_0) < r \implies h(t, x(t)) < \varepsilon \qquad \forall\, t \geqslant t_0 + T_r(\varepsilon).$$

Since $t - t_0 = T_r \circ \psi_r(t - t_0)$, for $t > t_0$, we have $h(t, x(t)) < \psi_r(t - t_0)$. Combining this and $\psi_r(0) = \infty$, the validity of the claim follows.

For arbitrary $s, t \in \mathbb{R}_{\geqslant 0}$, let

$$\overline{\psi}(s,t) := \min\left\{\varphi(s), \inf_{r\in\,]s,\infty[} \psi_r(t)\right\}.$$

By definition of φ and the above claim, for every $(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$,

$$h(t, x(t)) \leqslant \overline{\psi}(h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \geqslant t_0.$$

The function $\overline{\psi}$ need not be of class $\mathcal{KL}$, so we majorize it as follows. By definition, $\overline{\psi}(\cdot, t)$ is a nondecreasing function and for some fixed s, $\overline{\psi}(s,t) \to 0$ as $t \uparrow \infty$. Define

$$\widetilde{\psi}(s,t) := \int_s^{s+1} \overline{\psi}(v,t)\mathrm{d}v + \frac{s}{(1+s)(1+t-t_0)}.$$

It is straightforward to prove that

- $\widetilde{\psi}(s,t)$ is increasing with s for every $t \in \mathbb{R}_{\geqslant 0}$,
- $\widetilde{\psi}(s,t)$ decreases to 0 as $t \uparrow \infty$ for every fixed $s \in \mathbb{R}_{\geqslant 0}$, and
- $\widetilde{\psi}(s,t) \geqslant \overline{\psi}(s,t)$.

Then defining $\beta(s,t) := \sqrt{\varphi(s)\widetilde{\psi}(s,t)}$, it follows that for an arbitrary $(t_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$,

$$h(t, x(t)) \leqslant \sqrt{\big(\varphi \circ h^\circ(t_0, x_0)\big)\big(\overline{\psi}(h^\circ(t_0, x_0), t - t_0)\big)} \leqslant \beta(h^\circ(t_0, x_0), t - t_0) \qquad \forall\, t \geqslant t_0.$$

The proof of sufficiency is complete.

Necessity.

- To see (S1), consider the inequality (2.4) at the initial condition. We recover a class $\mathcal{K}_\infty$ function $\beta(\cdot, 0) =: \gamma(\cdot)$ relating h° and h as $h(t,x) \leqslant \gamma \circ h^\circ(t,x)$ for all $(t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$. Now for $\varepsilon > 0$, consider $\delta(\cdot) = \gamma^{-1}(\cdot)$.
- To see (S2), consider $r, \varepsilon > 0$ given and the existence of T follows from the property of the class $\mathcal{KL}$ function β.

The proof of necessity is complete. ◻

Appendix B. Equivalence of definitions of (h°, h)-SGUAS-P.

Proof of Lemma 3.23. First we show that (h°, h)-SGUAS-P implies (SP1)–(SP2).

We note that by the (h°, h)-SGUAS-P property, there exists a function $\beta \in \mathcal{KL}$ such that (3.29) holds. Let $\eta', \eta'' \in]0,1[$, $r, \varepsilon, \varepsilon' > 0$ and $t_0 \in \mathbb{R}_{\geqslant 0}$ be given. We choose $\eta = \eta' \wedge \eta''$. From the deterministic equivalence claimed in Remark 2.3, we have two numbers $\delta(\varepsilon) > 0$ and $T(r, \varepsilon') \geqslant 0$ such that from the condition $\sup_{t\geqslant t_0} h(t, x(t)) < \beta(h^\circ(t_0, x_0), t - t_0)$ we get

(i) $h^\circ(t_0, x_0) < \delta(\varepsilon) \implies \sup_{t\geqslant t_0} h(t, x(t)) < \varepsilon$, and

(ii) $h^\circ(t_0, x_0) < r \implies \sup_{t\geqslant t_0 + T(r,\varepsilon')} h(t, x(t)) < \varepsilon'$.

Choose $x_0 \in \mathbb{R}^n$ such that $h^\circ(t_0, x_0) < \delta(\varepsilon) \wedge r$. Define the sets

$$\begin{aligned} \Omega'_t &:= \{h(t, x(t)) < \varepsilon\}, \\ \Omega''_t &:= \{h(t, x(t)) < \varepsilon'\} \cup \{t < t_0 + T(r, \varepsilon')\}. \end{aligned} \tag{B.1}$$

Note that the set $\{t < t_0 + T(r, \varepsilon')\}$ merely ensures that the probability measure of the set $\{h(t, x(t)) < \varepsilon'\}$ is meaningful only after $t \geqslant t_0 + T(r, \varepsilon')$, which is what we need. In the light of (i)–(ii) and the definitions in (B.1), we know that (3.29) implies that $\mathsf{P}[\cap_{t\geqslant t_0} (\Omega'_t \cap \Omega''_t)] \geqslant 1 - \eta$. But by our choice of η, this means that $\mathsf{P}[\cap_{t\geqslant t_0} (\Omega'_t \cap \Omega''_t)] \geqslant (1-\eta') \vee (1-\eta'')$. Therefore, $\mathsf{P}[\cap_{t\geqslant t_0} \Omega'_t] \geqslant 1-\eta'$ and $\mathsf{P}[\cap_{t\geqslant t_0} \Omega''_t] \geqslant 1 - \eta''$, which are the properties (SP1) and (SP2) in (3.30) and (3.31), respectively.

Now we show that (SP1)–(SP2) implies (h°, h)-SGUAS-P.

Let $\eta \in]0, 1[$, $r, \varepsilon, \varepsilon' > 0$ and $t_0 \in \mathbb{R}_{\geqslant 0}$ be given. We choose $\eta' = \eta'' = \eta/2$. From (SP1)–(SP2), we get two numbers $\delta(\varepsilon) > 0$ and $T(r, \varepsilon') \geqslant 0$. Choose x_0 such that $h^\circ(t_0, x_0) < \delta(\varepsilon) \wedge r$. Using the definitions of Ω'_t and Ω''_t in (B.1), from (3.30) and (3.31) we have $\mathsf{P}[\cap_{t\geqslant t_0} \Omega'_t] \geqslant 1-\eta/2$, and $\mathsf{P}[\cap_{t\geqslant t_0} \Omega''_t] \geqslant 1-\eta/2$. We know that for $A, B \subset \Omega$ such that $\mathsf{P}[A] \geqslant 1-\eta/2$ and $\mathsf{P}[B] \geqslant 1-\eta/2$, we have $\mathsf{P}[A \cap B] \geqslant 1-\eta$. (This follows from the simple observation that $1 \geqslant \mathsf{P}[A \cup B] = \mathsf{P}[A] + \mathsf{P}[B] - \mathsf{P}[A \cap B]$.) It follows that $\mathsf{P}[\cap_{t\geqslant t_0} (\Omega'_t \cap \Omega''_t)] \geqslant 1 - \eta$, which is just

$$\mathsf{P}\left[\left\{\sup_{t\geqslant t_0} h(t, x(t)) < \varepsilon\right\} \bigcap \left\{\sup_{t\geqslant t_0 + T(r,\varepsilon')} h(t, x(t)) < \varepsilon'\right\}\right] \geqslant 1 - \eta.$$

In view of Remark 2.3 and (i)–(ii) above, since $r, \varepsilon, \varepsilon' > 0$ were arbitrary, it follows that there exists a function $\beta \in \mathcal{KL}$ such that $\mathsf{P}\big[\sup_{t\geqslant t_0} h(t, x(t)) < \beta(h^\circ(t_0, x_0), t - t_0)\big] \geqslant 1 - \eta$, which is the (h°, h)-SGUAS-P property (3.29). □

We remark that with little alteration of the above proof we obtain the equivalence of the definitions of (h°, h)-GUAS-P claimed in Remark 3.4. Indeed, for arbitrary fixed $t > t_0$, we need to eliminate the intersections of the sets Ω'_t and Ω''_t over $t \geqslant t_0$ in the equations above to establish the equivalence for time t. Since t is an arbitrary choice, we can conclude the validity of the result for every $t \geqslant t_0$.

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