Supervisory Control of Uncertain Linear Time-Varying Systems

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***Abstract*—We consider the problem of adaptively stabilizing linear plants with unknown time-varying parameters in the presence of noise, disturbances, and unmodeled dynamics using the supervisory control framework, which employs multiple candidate controllers and an estimator based switching logic to select the active controller at every instant of time. Time-varying uncertain linear plants can be stabilized by supervisory control, provided that the plant's parameter varies slowly enough in terms of mixed dwell-time switching and average dwell-time switching, the noise and disturbances are bounded and small enough in terms of L-infinity norms, and the unmodeled dynamics are small enough in the input-to-state stability sense. This work extends previously reported works on supervisory control of linear time-invariant systems with constant unknown parameters to the case of linear time-varying uncertain systems. A numerical example is included, and limitations of the approach are discussed.**

***Index Terms*—Adaptive control, input-to-state-stability, interconnected switched systems, linear time-varying systems, supervisory control.**

I. Introduction

ADAPTIVE control of uncertain time-varying plants is a challenging control problem and has attracted considerable research attention over the last several decades. Various robust adaptive control schemes for linear time-varying systems have been proposed, including direct model reference adaptive control [1], indirect adaptive pole placement control [2]–[5], and back-stepping adaptive control [6], [7] (see also, *e.g.*, [8]–[11]). These works and a majority of the literature on adaptive control of time-varying systems, more or less, employ continuously parameterized control laws in combination with continuously estimated parameters. A notably different approach is [12], where the strategy is to approximate the control input directly using sampled output data.

We present in this paper a new approach to adaptively stabilizing uncertain linear time-varying plants, using the supervisory control framework [13], [14] (see [15, Chapter 6] and the references therein for further background and related works on supervisory control; we also cover the supervisory control framework in Section III). Supervisory control differs from other adaptive control schemes (such as those mentioned in the first paragraph) in that instead of continuous parameter estimation, it discretizes the parameter space into a finite set of nominal values and employs a family of candidate controllers, one for each nominal value of the parameter. At every instant of time, an active controller is selected by an estimator-based supervisory unit using a logical decision rule. Advantages of supervisory control include i) simplicity and modularity in design: controller design amounts to controller design of known linear time-invariant systems for which various computationally efficient tools are available; and ii) the ability to handle large uncertainty (see [16] for more discussions on the advantages and drawbacks of supervisory control).

The supervisory control framework has been successfully applied to linear time-invariant systems with constant unknown parameters in the presence of unmodeled dynamics and noise [14], [17], [18]. Nonetheless, supervisory control of time-varying systems has not been studied, and it is the objective of this paper to explore this topic (see also a related problem of identification and control of time-varying systems using multiple models [19]). When parameter variation is small such that the time-varying plant can be approximated by a system with a constant parameter and small (time-varying) unmodeled dynamics, the robustness result in [18] can be applied. However, when parameter variation is large such that the previous approximation is not justified, the result in [18] is no longer applicable. The main contribution of this paper is to show that supervisory control is capable of stabilizing plants with large variation in the parameter space over time, provided that the parameter varies slowly enough in the mixed average dwell-time and dwell-time senses. Further, stabilization can be achieved in the presence of unmodeled dynamics, bounded disturbances, and bounded measurement noise, provided that the unmodeled dynamics are small in the input-to-state sense and the noises and disturbances are small in the $\mathcal{L}_\infty$ norm. The contribution can be viewed in two ways: at the qualitative level, it says that the supervisory control design provides a margin of robustness against noise, disturbances, and unmodeled dynamics; at the quantitative level, it provides a description of this robustness margin.

Another contribution, relevant to switched system research, is the use of a new class of slowly switching signals, which is quantified by both dwell-time [17] and average dwell-time [20], in stability analysis of switched systems. We use this class of switching signals to obtain an input-to-state-stability-like (ISS-like) result for interconnected switched systems (which is essential in the stability proof of supervisory control of time-varying plants). The tool used to establish stability of interconnected

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switched systems can also be used to study stability of switched nonlinear systems in which a constant switching gain among a family of ISS-Lyapunov functions of the subsystems is not available (see Remark 2).

The paper's organization is as follows. In Section II, we clarify the notation used in the paper, and in Section III, we formulate the control problem and describe the supervisory control framework. Our main result on closed-loop stability of supervisory control of uncertain time-varying linear plants is presented in Section IV. The subsequent sections are devoted to the proof of the main result: the structure of the closed-loop system is described in Section V and then formalized as an interconnected switched system in Section VI, for which we provide a stability analysis, following by the proof of the main theorem in Section VII. We provide a numerical example and a performance discussion in Section VIII and conclude the paper in Section IX with a summary of the results and a discussion of future work.

Notation

Denote by $(\cdot)_{[t_0,t)}$, $t \geqslant t_0 \geqslant 0$, the *segmentation operator* such that for a function f, $(f)_{[t_0,t)}(\tau) := f(\tau)$ if $\tau \in [t_0,t)$, and $(f)_{[t_0,t)}(\tau) := 0$ otherwise. For a vector v, denote by $|\cdot|$ the 2-norm: $|v| := (v^T v)^{1/2}$, and by $|\cdot|_\infty$ the ∞-norm: $|v|_\infty = \max|v_i|$. Denote by $\|A\|$ the induced 2-norm of a matrix A. For a function f, denote $\|(f)_{[t_0,t)}\| := \sup_{s\in[t_0,t)} |f(s)|$ and $\|f\|_\infty := \|(f)_{[0,\infty)}\|$. For $\lambda > 0$, define the $e^{-\lambda t}$*-weighted* $\mathcal{L}_2$ *norm* of a function f as

$$\|(f)_{[t_0,t)}\|_{2,\lambda} := \left(\int_{t_0}^{t} e^{-\lambda(t-\tau)}|f(\tau)|^2 d\tau\right)^{1/2}, \quad t \geqslant t_0.$$

Denote by $\|(f)_{[t_0,*)}\|_{2,\lambda}$ the function obtained when we let t be a variable in the preceding $\|\cdot\|_{2,\lambda}$ definition. For more details on the norm $\|\cdot\|_{2,\lambda}$, see, *e.g.*, [21, Chapter 3]. A *switching signal* $s : [0,\infty) \to \mathcal{P}$, where $\mathcal{P}$ is an index set, is a piecewise constant and continuous from the right function, and the discontinuities of s are called *switches* or *switching times*. We assume that there are finitely many switches in every finite interval (*i.e.,* no Zeno behavior). For a switching signal s and a time t, denote by t_s the latest switching time of s before the time t. By convention, $t_s = 0$ if t is less than or equal to the first switching time of s.

A switching signal has a *dwell-time* τ_d if every two consecutive switches are separated by at least τ_d. Denote by $N_s(T,t_0)$ the number of switches in the interval $[t_0,T)$. A switching signal has an *average dwell-time* τ_a [20] if $\exists N_0 \geqslant 0$ such that $N_s(T,t_0) \leqslant N_0 + (T-t_0)/\tau_a \ \forall T \geqslant t_0$. The number N_0 is called a *chatter bound*. When $N_0 = 1$, we recover dwell-time switching with the dwell-time being τ_a. Denote by $\mathcal{S}_{dwell}[\tau_d]$ the class of switching signals with dwell-time τ_d and by $\mathcal{S}_{ave}[\tau_a, N_0]$ the class of switching signals with average dwell-time τ_a and chatter bound N_0.

Recall that (see, *e.g.*, [22]) a continuous function $\alpha : [0,\infty) \to [0,\infty)$ is of class $\mathcal{K}$ if α is strictly increasing, and $\alpha(0) = 0$, and further, $\alpha \in \mathcal{K}_\infty$ if $\alpha(r)\to\infty$ as $r\to\infty$. A function $\beta : [0,\infty)\times[0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$ if $\beta(\cdot,t) \in \mathcal{K}$ for every fixed t, and $\beta(r,t)$ decreases to 0 as $t\to\infty$ for every fixed r. Denote by $\mathcal{J}$ the class of continuous non-decreasing functions $f : [0,\infty) \to [1,\infty)$.

II. Problem Formulation and the Supervisory Control Architecture

A. Problem Formulation

Consider uncertain time-varying plants of the following form:

$$\mathbf{P} : \begin{cases} \dot{x} = A(p^*(t))x + B(p^*(t))u + \big(\Delta_x(z,x,u,t) + w_x\big), \\ y = C(p^*(t))x + \big(\Delta_y(z,x,u,t) + w_y\big), \\ \dot{z} = f_\Delta(z,x,u,t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^\ell$ is the input, $y \in \mathbb{R}^r$ is the output, w_x and w_y are the disturbance and measurement noise, respectively, $z \in \mathbb{R}^k$ is the state of the unmodeled dynamics, and $p^* : [0,\infty) \to \mathbb{R}^{n_p}$ is the unknown time-varying parameter. We assume that A, B, and C are continuous in p^*, and f_Δ, Δ_x, and Δ_y are locally Lipschitz in x, u, z and continuous in t. We assume that p^* is nice enough so the existence and uniqueness of a solution of (1) for every initial condition and piecewise-continuous input is guaranteed.

Our objective is to use the supervisory control framework [13], [14] to stabilize the uncertain plant (1) in the presence of noise, disturbances, and unmodeled dynamics.

B. Switched System Approximation of Time-Varying Plants

Assumption 1: A compact set $\Omega \subset \mathbb{R}^{n_p}$ *is known such that* $p^*(t) \in \Omega \ \forall t$.

We proceed by approximating the time-varying system (1) by a switched system plus unmodeled dynamics in the following way. We divide Ω into a finite number of subsets Ω_i such that $\bigcup_{i\in\mathcal{P}} \Omega_i = \Omega$, and $\Omega_i \cap \Omega_j \in \partial\Omega_i \cap \partial\Omega_j$, where $\mathcal{P} = \{1,\ldots,m\}$, m is the number of subsets, and $\partial\Omega_i$ is the boundary of the set Ω_i. How to divide and what the number of subsets is are interesting research questions of their own and are not pursued here (see [23]), but intuitively, we want the sets Ω_i small in some sense. Define the signal $s : [0,\infty) \to \mathcal{P}$

$$s(t) := i : p^*(t) \in \Omega_i \tag{2}$$

such that s is continuous from the right. Because p^* are not known, the signal s is not known *a priori*. We assume that the sets Ω_i, A, B, C, and p^* "behave well" in the sense that the signal s in (2) is a well-defined switching signal without chattering. Right continuity of σ can always be ensured by setting the value of σ to be the limit from the right at the time the signal p^* crosses the boundary shared by two or more subsets; if p^* travels along the shared boundary of some sets, right continuity can still be ensured by carefully defined convention. Chattering of s could possibly occur for a general p^* and general regions Ω_i but there exist works that address the issue of how to design regions to avoid chattering (see, e.g., [24]). Generally, fast varying parameters (such as p^* with large derivatives) and a large number of subsets (the size of $\mathcal{P}$ is large) imply fast switching signal s.

Assumption 2: The signal s *defined in (2) is a well-defined switching signal.*

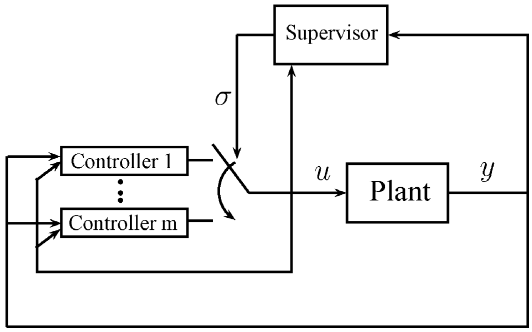


Fig. 1. Supervisory control framework.

For every subset $\Omega_i, i \in \mathcal{P}$, pick a nominal value $q_i \in \Omega_i$. Let $A_i := A(q_i), B_i := B(q_i)$, and $C_i := C(q_i)$. We can rewrite the plant (1) as

$$\mathbf{P}: \begin{cases} \dot{x} = A_{s(t)}x + B_{s(t)}u + \left[\delta_A(t)x + \delta_B(t)u\right] \\ \quad\quad + \left(\Delta_x(z,x,u,t) + w_x\right), \\ y = C_{s(t)}x + \left[\delta_C(t)x\right] + \left(\Delta_y(z,x,u,t) + w_y\right), \\ \dot{z} = f_\Delta(z,x,u,t) \end{cases} \quad (3)$$

where $\delta_A(t) := A(p^*(t)) - A_{s(t)}$, $\delta_B(t) := B(p^*(t)) - B_{s(t)}$, and $\delta_C(t) := C(p^*(t)) - C_{s(t)}$. The terms inside the square brackets are those due to the process of approximating the time-varying plant by a switched system, and the terms inside the parentheses are the unmodeled dynamics.

Assumption 3: (A_i, B_i) are stabilizable, and (A_i, C_i) are detectable $\forall i \in \mathcal{P}$.

C. The Supervisory Control Framework

The supervisory control framework [13], [14], [16], [17] consists of a family of candidate controllers and a supervisor that orchestrates the switching among the controllers (see the architecture of supervisory control in Fig. 1). The supervisory control scheme described below is essentially the same as those in [14] with a particular type of multi-estimator; the reader is referred to [14] for further in-depth discussion.

1) Multi-Controller: A family of candidate controllers, parameterized by p, are designed such that the controller with index $p \in \mathcal{P}$ stabilizes the linear time-invariant plant $\dot{x} = A_p x + B_p u,\ y = C_p x$. Denote a state-space realization of the controller with an index p as

$$\mathbf{C}_p: \begin{cases} \dot{x}_\mathbb{C} = F_p(x_\mathbb{C}, u, y) \\ u_p = H_p x_\mathbb{C} \end{cases} \quad (4)$$

where $x_\mathbb{C} \in \mathbb{R}^{n_\mathbb{C}}$, F_p is a linear function, and $H_p \in \mathbb{R}^{n_\mathbb{C} \times \ell}$.

2) Supervisor: The supervisor comprises a multi-estimator, monitoring signals, and a switching logic.

Multi-Estimator: A multi-estimator is a bank of estimators, each of which takes in the input u and the output y and produce the estimated output $y_p, p \in \mathcal{P}$. A multi-estimator must have the property that at least one of the output estimation errors $\hat{y}_p - y$ is small for all u. We use the following particular observer-based multi-estimator whose state is $x_\mathbb{E} = (\hat{x}_1, \ldots, \hat{x}_m) \in \mathbb{R}^{mn}$ and whose dynamics are

$$\begin{cases} \dot{\hat{x}}_p = A_p\hat{x}_p + B_p u + L_p(\hat{y}_p - y) \\ \hat{y}_p = C_p\hat{x}_p \end{cases} \quad p \in \mathcal{P} \quad (5)$$

where L_p are such that $A_p + L_pC_p$ are Hurwitz for all $p \in \mathcal{P}$. We set the initial state $\hat{x}_p(0) = 0$ for all $p \in \mathcal{P}$. Let $\tilde{y}_p := \hat{y}_p - y$ be the output estimation errors.

Monitoring Signals: Monitoring signals are functions of certain norm of the output estimation errors, and they are used in the switching logic to produce the switching signal (see the switching logic (7) below). We use the following particular type of monitoring signals μ_p, $p \in \mathcal{P}$, which is the $e^{-\lambda t}$-weighted $\mathcal{L}_2$-norm of the output estimation errors [20]:

$$\mu_p(t) = \varepsilon + \gamma\|(\tilde{y}_p)_{[0,t)}\|^2_{2,\lambda} \quad (6)$$

for some design constants $\{\varepsilon, \lambda, \gamma\} > 0$. The signal μ_p can be implemented as ε plus the output of the linear filter $\dot{\xi} = -\lambda\xi + |\tilde{y}_p|^2$ with $\xi(0) = 0$. The constant ε is to ensure that the switching signal generated by the particular switching logic below is a slow switching signal—a property necessary for stability proof (see the (28b) and (29b)).

Switching Logic: A switching logic produces a switching signal that indicates at every instant of time the active controller. We use the *scale-independent hysteresis switching logic* [20]

$$\sigma(t) := \begin{cases} \underset{q\in\mathcal{P}}{\operatorname{argmin}}\, \mu_q(t) & \text{if } \exists\, q \in \mathcal{P} \text{ such that} \\ & (1+h)\mu_q(t) \leqslant \mu_{\sigma(t^-)}(t), \\ \sigma(t^-) & \text{else} \end{cases} \quad (7)$$

where $h > 0$ is a *hysteresis constant*.

Altogether, the supervisory control law is given by

$$\begin{cases} \dot{x_\mathbb{C}} = F_\sigma(x_\mathbb{C}, u, y), \\ u = H_\sigma x_\mathbb{C} \end{cases} \quad (8)$$

in view of (4), where σ is as in (7).

D. Design Parameters

The design parameters h, λ, and γ must satisfy certain conditions to ensure closed-loop stability. The relationship among these parameters involves the so-called *injected systems* [25], which are defined below.

An injected system with index p is obtained by combining the controller with index p with the multi-estimator and takes $y - \hat{y}_p = \tilde{y}_p$ as the input. For the multi-estimator (5), the injected system with index $p \in \mathcal{P}$ is of the form

$$\dot{x}_{\mathbb{CE}} = \mathbf{A}_p x_{\mathbb{CE}} + \mathbf{B}_p\tilde{y}_p \quad (9)$$

where $x_{\mathbb{CE}} := \begin{pmatrix} x_\mathbb{E} \\ x_\mathbb{C} \end{pmatrix} \in \mathbb{R}^{n_{\mathbb{CE}}}$, $n_{\mathbb{CE}} := nm + n_\mathbb{C}$, is the state of the injected system, and $\mathbf{A}_p$ is a Hurwitz matrix (see Appendix A for detail on how to arrive at (9)). Then there exists a family of quadratic Lyapunov functions $V_p(x_{\mathbb{CE}}) = x_{\mathbb{CE}}^T P_p x_{\mathbb{CE}}, P_p \succ 0$ (P_p is positive definite) such that $\forall p \in \mathcal{P}$

$$a_1|x_{\mathbb{CE}}|^2 \leqslant V_p(x_{\mathbb{CE}}) \leqslant a_2|x_{\mathbb{CE}}|^2 \quad (10a)$$

$$\frac{\partial V_p(x_{\mathbb{CE}})}{\partial x}(\mathbf{A}_p x_{\mathbb{CE}} + \mathbf{B}_p\tilde{y}_p) \leqslant -\lambda_0 V_p(x_{\mathbb{CE}}) + \gamma_0|\tilde{y}_p|^2 \quad (10b)$$

for some constants $\{a_1, a_2, \lambda_0, \gamma_0\} > 0$ (the existence of such common constants for the family of injected systems is guaranteed because $\mathcal{P}$ is finite). In fact, one can take $-\lambda_0$ to be twice the maximum (negative) real part of the eigenvalues of the matrices $\mathbf{A}_p$ over all $p \in \mathcal{P}$. There also $\exists \mu \geqslant 1$ such that

$$V_q(x_{\mathbb{CE}}) \leqslant \mu V_p(x_{\mathbb{CE}}) \quad \forall x_{\mathbb{CE}} \in \mathbb{R}^{n_{CE}}, \forall p, q \in \mathcal{P}. \qquad (11)$$

We can always pick $\mu = a_2/a_1$ but there may be other smaller μ satisfying (11) (for example, $\mu = 1$ if V_p are the same for all p even though $a_2/a_1 > 1$).

Let $-\hat{\lambda}$ be twice the maximum (negative) real part of eigenvalues of the matrices $A_p + L_p C$ over all $p \in \mathcal{P}$

$$\hat{\lambda} := -2 \max_{p \in \mathcal{P}} \{Re(\mathrm{eig}(A_p + L_p C_p))\}. \qquad (12)$$

For switched plants ($\delta_{\mathbf{P}} = 0$, and there are no unmodeled dynamics), the constant ε in (6) can be chosen arbitrarily. The larger ε is, the larger the ultimate bound of the closed-loop states will be. For the original plant with unmodeled dynamics, we need ε to be small enough, and the bound on ε depends on the bounds on the unmodeled dynamics. This quantification on ε will be made precise in Theorem 1's statement in Section IV. The parameters h, λ, and γ are chosen such that

$$\ln(1+h) < m \ln \mu \qquad (13)$$

$$(\kappa+1)\lambda < \lambda_0, \; \kappa := \frac{m \ln \mu}{\ln(1+h)} \qquad (14)$$

$$\lambda < \hat{\lambda} \qquad (15)$$

where h is as in (7), μ is as in (11), λ is as in (6), μ is as in (11), $\hat{\lambda}$ is as in (12), λ_0 is as in (10b), γ is as in (6), and γ_0 is as in (10b).

Remark 1: We can give the conditions (13), (14), and (15) the following interpretations: (13) means that the switching logic must be active enough (smaller h) to cope with changing parameters in the plant; (14) implies that the "learning rate" λ of the monitoring signals must be slower in some sense than the "convergence rate" λ_0 of the injected systems; and (15) can be seen as saying the "learning rate" λ must be slower than the "estimation rate" $\hat{\lambda}$ of the multi-estimator. For the case of time-invariant plants (i.e A, B, C are constant matrices), we only need the condition (14), not the extra conditions (15) and (13), to prove stability of the closed-loop system [14] (the condition (14) can be rewritten as $\ln(1+h)/(\lambda m) > \ln \mu/(\lambda_0 - \lambda)$, exactly as in [14]).

III. Main Result

Assumption 4: For the plant (1), the unmodeled dynamics of z is input-to-state stable (ISS) [26] with respect to x and u:

$$|z(t)| \leqslant \beta_z(|z(0)|, t) + \gamma_z^1(\|x\|_{[0,t)}) + \gamma_z^2(\|u\|_{[0,t)}) \; \forall t \geqslant 0 \quad (16)$$

for some $\beta_z \in \mathcal{KL}$, $\gamma_z^1, \gamma_z^2 \in \mathcal{K}_\infty$. The unmodeled dynamics Δ_x and Δ_y satisfy

$$|\Delta_x(z,x,u,t)| \leqslant \delta_\Delta \big(\gamma_x^0(|z(0)|)e^{-\lambda_\Delta t} + \gamma_x^1(\|x\|_{[0,t)}) + \gamma_x^2(\|u\|_{[0,t)})\big) \quad (17a)$$

$$|\Delta_y(z,x,u,t)| \leqslant \delta_\Delta \big(\gamma_y^0(|z(0)|)e^{-\lambda_\Delta t} + \gamma_y^1(\|x\|_{[0,t)}) + \gamma_y^2(\|u\|_{[0,t)})\big) \quad (17b)$$

for all z, x, u, and t and for some $\{\lambda_\Delta, \delta_\Delta\} \geqslant 0$ with respect to some given $\{\gamma_x^i, \gamma_y^i\} \in \mathcal{K}_\infty, i = 0, 1, 2$.

The following constant $\delta_{\mathbf{P}}$ quantifies how well the time-varying plant (1) without the unmodeled dynamics, noise, and disturbances can be approximated by the nominal switched system $\dot{x} = A_s x + B_s u, \; y = C_s x$:

$$\delta_{\mathbf{P}} := \max\{\|\delta_A\|_\infty, \|\delta_B\|_\infty, \|\delta_C\|_\infty\}. \qquad (18)$$

When $\delta_{\mathbf{P}} = 0$, p^* is a switching signal, and hence, the plant is a switched plant (if further p^* is a constant signal, then the original plant, without unmodeled dynamics, is a linear-time invariant system).

Slow switching signals are often characterized by dwell-time or average dwell-time switching; see the paper [27] for an in-depth discussion on various types of dwell-time switching. For stability results in this paper, we define the class of *hybrid dwell-time* signals $\mathcal{S}_{hybrid}[\tau_d, \tau_a, N_0]$, which is characterized by three numbers—a dwell-time, an average dwell-time, and a chatter bound—as follows:

$$\mathcal{S}_{hybrid}[\tau_d, \tau_a, N_0] := \mathcal{S}_{dwell}[\tau_d] \cap \mathcal{S}_{ave}[\tau_a, N_0]. \qquad (19)$$

When $\tau_d = 0$ (which means the dwell-time can be infinitesimally small), we have $\mathcal{S}_{hybrid}[0, \tau_a, N_0] = \mathcal{S}_{ave}[\tau_a, N_0]$. Let

$$X(t) := \max_{p \in \mathcal{P}} \{|x(t)|, |z(t)|, |x_{\mathbb{C}}(t)|, |x_{\mathbb{E}}(t)|, \mu_p(t)\} \quad (20)$$

$$\gamma_c := \max_{p,q \in \mathcal{P}} \big\|[C_p \quad -C_q]\big\|. \qquad (21)$$

Theorem 1: Consider the uncertain plant (1). Suppose that Assumptions 1, 2, 3, and 4 hold. Consider the supervisory control scheme with the multi-controller (4) with the state $x_{\mathbb{C}}$, the multi-estimator (5) with the state $x_{\mathbb{E}}$, the monitoring signals (6) with the states μ_p, and the switching logic (7). Suppose that the design parameters satisfy (13), (14), and (15). For every $\overline{X}_0 > 0$, there exist a function $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ and numbers $\{\bar{w}_x, \bar{w}_y, \bar{\delta}, \varepsilon\} > 0$ such that if $\{\delta_{\mathbf{P}}, \gamma_c, \delta_\Delta\} \leqslant \bar{\delta}$, $\|w_x\|_\infty \leqslant \bar{w}_x$, and $\|w_y\|_\infty \leqslant \bar{w}_y$, for all $|X(0)| < \overline{X}_0$ and for every $s \in \mathcal{S}_{hybrid}[\delta_d, \bar{\tau}, \overline{N}]$ such that

$$\bar{\tau} \geqslant f(\delta_d, \overline{N}),$$

all the closed-loop signals are bounded, and

$$|x(t)|^2 \leqslant \bar{c}_1 \overline{X}_0^2 e^{-\lambda t} + \bar{c}_2(\bar{w}_x, \bar{w}_y, \bar{\delta})(1 + \bar{\gamma}_2 e^{-\lambda(t-t_s)}) \; \forall t \geqslant 0 \quad (22)$$

for some $\{\bar{c}_1, \bar{\gamma}_2\} > 0$ and for some function $\bar{c}_2 : [0,\infty)^3 \to [0,\infty)$ such that $\bar{c}_2 \to \gamma_\varepsilon \varepsilon$ as $\{\bar{w}_x, \bar{w}_y, \bar{\delta}\} \to 0$ for some constant $\gamma_\varepsilon > 0$ independent of ε, where ε is as in (6).

Roughly speaking, the theorem says that the supervisory control scheme is capable of stabilizing time-varying systems in the presence of unmodeled dynamics with bounded disturbances and bounded noise provided that the plant varies slowly enough in the sense of hybrid dwell-time, the unmodeled dynamics are small enough in the ISS sense, and the noise and disturbances are small enough in the $\mathcal{L}_\infty$ sense. The ultimate bound on the plant state x as $t \to \infty$ can be made arbitrarily close to the order of ε if the unmodeled dynamics, disturbances, and noise are sufficiently small. Note that the bounds depends on the bounds on the initial states (similarly in spirit to the result in [18]); the bounds given in the theorem are conservative (see Remark 3).

The supervisory control scheme for plants with time-varying parameters is the same as those for plants with constant parameters [14]. However, unlike the case of constant parameters where we have only one switching signal and hence, one switched system (*i.e.,* the switched injected system), here we have two switching signals in the closed loop: one is generated by the supervisory unit, the other comes from the plant itself. The switching times of these two signals, in general, do not coincide, leading to a more complex analysis than in the case of constant parameters.

The rest of the paper is devoted to the proof of the theorem and to the quantification of the class of switching signals $\mathcal{S}$ and the number $\bar{\delta}$ in the theorem's statement. We present the structure of the closed-loop system in Section V, followed by a formalism of interconnected asynchronous switched systems and a corresponding stability result in Section VI. In Section VII, we provide the proof of Theorem 1 using the result for interconnected switched systems in Section VI.

IV. Closed-Loop Structure

The closed loop consists of two switched systems:

1) **The switched system** Γ_s: The first switched system arises from the dynamics of the state estimation errors. Let $\tilde{x}_q := x - \hat{x}_q$ be the state estimation error of the q-th subsystem of the multi-estimator, $q \in \mathcal{P}$. Let $\zeta(t) := \tilde{x}_{s(t_s)}(t)$. Because s is constant in $[t_s, t)$ and $s(t_s)$ is the index of the nominal switched plant for time in $[t_s, t)$, in view of the linear observer dynamics (5), the dynamics of $\tilde{x}$ for time in $[t_s, t)$ are exponentially stable when $\delta_A, \delta_B, \delta_C, \Delta_x, \Delta_y, v$, and w are all zero. Further, because $u(t) = H_{\sigma(t)} x_{\mathbb{C}}(t)$ (in view of (4)), $x = \tilde{x}_{s(t_s)} + \hat{x}_{s(t_s)}$, and $\hat{x}_p$ and $x_{\mathbb{C}}$ are components of $x_{\mathbb{CE}}$ for all $p \in \mathcal{P}$, any term of the form Mx or Mu for some matrix M can be written as a linear combination of $\tilde{x}_{s(t_s)}$ and $x_{\mathbb{CE}}$. It follows that the dynamics of ζ are of the following form:

$$\dot{\zeta}(t) = \mathbf{E}_{s(t_s)}\zeta(t) + \delta_1(t)\zeta(t) + \delta_2(t)x_{\mathbb{CE}}(t) + \Delta_1(z, x, u, t) + v(t) \quad (23)$$

where $\mathbf{E}_p := A_p + L_p C_p$ are Hurwitz for all $p \in \mathcal{P}$, δ_1 and δ_2 are such that $\{\|\delta_1\|_\infty, \|\delta_2\|_\infty\} \to 0$ as $\delta_{\mathbf{P}} \to 0$, Δ_1 is such that if z, x, u are bounded, then $\|\Delta_1\| \to 0$ as $\delta_\Delta \to 0$, where δ_Δ is as in (17), and v is such that $\|v\|_\infty \to 0$ as $\{\|w_x\|_\infty, \|w_y\|_\infty\} \to 0$; see Appendix B for the formula and detailed derivation of $\delta_1, \delta_2, \Delta_1$, and v. For the purpose of analysis later, we will augment ζ with the variable $\xi(t) := \|(\tilde{y}_{s(t_s)})_{[0,t)}\|^2_{2,\lambda}$ (the variable ξ relates to μ_p as $\mu_{s(t_s)}(t) = \varepsilon + \gamma\xi(t)$) to arrive at the following switched system with jumps:

$$\Gamma_s : \begin{cases} \dot{\zeta} = \mathbf{E}_s\zeta + \delta_1\zeta + \delta_2 x_{\mathbb{CE}} + \Delta_1 + v, \\ \dot{\xi} = -\lambda\xi + \gamma|\tilde{y}_s|^2, \\ \begin{pmatrix} \zeta(t_s) \\ \xi(t_s) \end{pmatrix} = \varphi\left(\begin{pmatrix} \zeta(t_s^-) \\ \xi(t_s^-) \end{pmatrix}, t_s, x_{\mathbb{CE}}(t_s)\right), \ \forall t, \end{cases} \quad (24)$$

where $\tilde{y}_s(t) := \tilde{y}_{s(t_s)}(t)$, and for some jump map φ (recall that t_s is the latest switching time of s before t).

2) **The switched system** Π_σ: The second switched system is the switched injected system from (9) and (7)

$$\Pi_\sigma : \begin{cases} \dot{x}_{\mathbb{CE}} = \mathbf{A}_\sigma x_{\mathbb{CE}} + \mathbf{B}_\sigma \tilde{y}_\sigma \\ x_{\mathbb{CE}}(t_\sigma) = x_{\mathbb{CE}}(t_\sigma^-) \quad \forall t. \end{cases} \quad (25)$$

The second equation in (25) is to explicitly indicate that there is no state jump at switching times (cf. the system Γ_s which has jumps at switching times).

These two switched systems interact as follows:

1) **Constraint on** Γ_s: The following inequalities give a bound on the state jump of Γ_s at switching times: for all $t \geqslant 0$:

$$|\zeta(t_s)|^2 \leqslant 2|\zeta(t_s^-)|^2 + 4|x_{\mathbb{CE}}(t_s)|^2, \quad (26a)$$

$$\xi(t_s) \leqslant 2\xi(t_s^-) + 2\gamma_c\|(x_{\mathbb{CE}})_{[0,t_s)}\|^2_{2,\lambda} \quad (26b)$$

for some $\gamma_c > 0$. Also, $\tilde{y}_s$ has the following property:

$$\tilde{y}_s(t) = (C_{s(t_s)} + \delta_C(t))\zeta(t) + \Delta_2(z, x, x_{\mathbb{CE}}, u, t) + w_y(t) \quad (27)$$

where $\Delta_2 := \delta_3(t)x_{\mathbb{CE}} + \Delta_y(z, x, u, t)$, δ_3 is such that $\|\delta_3\|_\infty \to 0$ as $\delta_{\mathbf{P}} \to 0$, and Δ_y is as in (3). See Appendix C for the derivation of (26) and (27).

2) **Constraint on** Π_σ: This constraint tells how the input and the switching signal of Π_σ are bounded in terms of the state of Γ_s (see Appendix C for the derivation)

$$\|(\tilde{y}_\sigma)_{[0,t)}\|^2_{2,\lambda} \leqslant \frac{m(1+h)}{\gamma}(\varepsilon + \xi(t)), \quad (28a)$$

$$N_\sigma(t, t_0) \leqslant N_0(\xi, t) + \frac{t - t_0}{\tau_a} \quad (28b)$$

where

$$\tau_a := \frac{\ln(1+h)}{(m\lambda)}, \quad (29a)$$

$$N_0(\xi(t)) := m + \frac{m}{\ln(1+h)} \ln\left(\left(\frac{\xi(t)}{\varepsilon}\right) + 1\right). \quad (29b)$$

V. Interconnected Switched Systems

In order to make it easier to understand the closed-loop structure in the previous section, we consider the formalism of the closed loop described in the previous section and call it an interconnected switched system. The two switched systems (without unmodeled dynamics) are interconnected in the following way (see Fig. 2; the dash lines indicate that a subsystem constrains another subsystem or signal, and the solid lines are actual signals):

- The input $\tilde{y}_\sigma$ of the second switched system Π_σ is bounded in terms of the state ξ of the first switched system Γ_s by means of the relation (28a);
- The input $\tilde{y}_s$ of the first switched system Γ_s is bounded in terms of the state ζ of the first switched system Γ_s and the state $x_{\mathbb{CE}}$ of the second switched system Π_σ by means of the relation (27);
- The switching signal σ of the second switched system Π_σ is bounded in terms of the state of the first switched system Γ_s by means of the relation (28b);

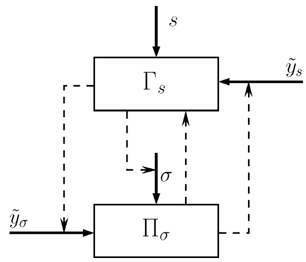


Fig. 2. Interconnected switched systems.

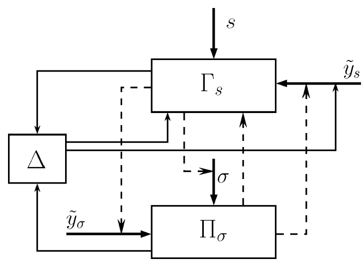


Fig. 3. Interconnected switched systems with unmodeled dynamics.

- The jump map φ of Γ_s as in (25) is bounded in terms of the state ζ of Γ_s and the state $x_{\mathbb{CE}}$ of Π_σ by means of the relation (26).

Interconnected switched systems with the presence of unmodeled dynamics are illustrated in Fig. 3.

Assuming that the subsystems of both the switched systems Γ_s and Π_σ are affine and zero-input exponentially stable, we want to study stability of the closed loop.

Stability of certain types of interconnected switched systems has been studied in [28]–[30]. In these works, the connection between the two switched systems in a loop is the usual feedback connection. In [28], a small-gain theorem for interconnected switched systems is provided. The works [29], [30] give passivity theorems for interconnected switched systems and hybrid systems. However, for the loop in Fig. 2, the small-gain theorem in [28] and the passivity theorem in [29], [30] are not directly applicable because it is difficult to quantify input-output relationship/input-output gains of the two switched systems, in which the first switched system's jump map is affected by the second switched system and the second switched system's switching signal is constrained by the first switched system. We provide here tools for analyzing such interconnected switched systems.

Lets look at the special case where s is a constant signal, and there are no unmodeled dynamics, no noise, and no disturbance ($\delta_{\mathbf{P}} = 0, v = 0, w = 0$). Because s is a constant signal, the jump constraint (26) for Γ_s does not come into effect, and Γ_s is a non-switched stable linear system. Then ζ is exponentially decaying to zero, and hence, $\tilde{y}_s$ goes to zero in view of (27), and also, ξ goes to zero. Then N_0 is bounded in view of (29), and σ is an average dwell-time switching signal. From (28a), (25), and the slow switching condition(14), it follows from the stability result for switched systems under average dwell-time [20] that $x_{\mathbb{CE}}$ is bounded. From there, stability of the plant state x can be concluded.

However, the situation is much more complicated when s is not a constant signal. The stability results [20], [31] for switched systems without jumps are not applicable here because Γ_s has jumps. The stability result for impulsive systems [32] is also not applicable here (we are not able to find a Lyapunov function as in [32]). But moreover, the issue here is that the jump map of Γ_s involves the state of the second switched system, while the input as well as the switching signal of the second switched system is affected by the state of the first system. This type of mutual interaction makes the analysis of the closed loop's behavior between switching times challenging. We observe that the switching signal σ is constrained by Γ_s but s is not constrained at all, so we use the following technique: we first eliminate the presence of σ by incorporating the properties (28a) and (28b) of σ into other inequalities, and after that we find the closed-loop behavior with respect to the switching signal s (without worrying about the switching signal σ).

Before going into details, we outline the steps for proving stability of the interconnected switched system $[\Gamma_s, \Pi_\sigma]$ with the interrelations (26), (27), (28a), and (28b).

1) We establish an ISS-like property of the switched system Γ_s in terms of the state of Π_σ and unmodeled dynamics, noise, and disturbances between consecutive switches of s (Lemma 1).
2) We establish an ISS-like property of the switched system Π_σ with respect to ξ for arbitrary time intervals using the property of N_σ (Lemma 2 and Lemma 3).
3) We define a Lyapunov-like function W which depends on the states of Γ_s and Π_σ and their norms, and analyze the behavior of W between consecutive switches of s (Lemma 4).
4) We establish boundedness of W using a hybrid dwell-time switching signal s and conclude boundedness of all continuous signals in the loop.

A. Switched System Γ_s

The lemma below says that the state of Γ_s is bounded by an exponentially decaying term with respect to the state ζ and $x_{\mathbb{CE}}$ at the last switching time t_s and by $\|\cdot\|_{2,\lambda}$ norms of the unmodeled dynamics, noise, and disturbances; see Appendix D for the proof.

Lemma 1: Consider the switched system Γ_s in (24) with the constraints (26) and (27). For every $\lambda < \hat{\lambda}$, where $\hat{\lambda}$ is as in (12), for all $t \geqslant 0$, we have

$$|\zeta(t)|^2 \leqslant a_0(2|\zeta(t_s^-)|^2 + 4|x_{\mathbb{CE}}(t_s)|^2)\, e^{-\hat{\lambda}(t-t_s)} + \mathrm{U}_1(t) + \mathrm{V}_1(t) \tag{30a}$$

$$\xi(t) \leqslant f_1(t_s)\, e^{-\lambda(t-t_s)} + \mathrm{U}_2(t) + \mathrm{V}_2(t) \tag{30b}$$

where

$$f_1(t_s) := 2\xi(t_s^-) + 2\gamma_c\|(x_{\mathbb{CE}})_{[0,t_s)}\|_{2,\lambda}^2 + \frac{4c_1^2 a_0}{\hat{\lambda}-\lambda}|\zeta(t_s^-)|^2 + \frac{8c_1^2 a_0}{\hat{\lambda}-\lambda}|x_{\mathbb{CE}}(t_s)|^2 \tag{31a}$$

$$\mathrm{U}_1(t) := \hat{\gamma}_1\|(\Delta_{\tilde{x}})_{[t_s,t)}\|_{2,\hat{\lambda}}^2, \tag{31b}$$

$$\mathrm{V}_1(t) := \hat{\gamma}_2\|(v)_{[t_s,t)}\|_{2,\hat{\lambda}}^2, \tag{31c}$$

$$\mathrm{U}_2(t) := \frac{2c_1^2\hat{\gamma}_1}{\hat{\lambda}-\lambda}\|(\Delta_{\tilde{x}})_{[t_s,t)}\|_{2,\lambda}^2 + 4\gamma\|(\Delta_2)_{[t_s,t)}\|_{2,\lambda}^2, \tag{31d}$$

$$\mathrm{V}_2(t) := \frac{2c_1^2\hat{\gamma}_2}{\hat{\lambda}-\lambda}\|(v)_{[t_s,t)}\|_{2,\lambda}^2 + 4\gamma\|(w_y)_{[t_s,t)}\|_{2,\lambda}^2, \tag{31e}$$

$$\Delta_{\tilde{x}} := \delta_1\zeta + \delta_2 x_{\mathbb{CE}} + \Delta_1 \tag{31f}$$

$\delta_1, \delta_2, \Delta_1$, and v are as in (24), Δ_2 is as in (27), and $\{a_0, c_1, \hat{\gamma}_1, \hat{\gamma}_2\} > 0$ are constants.

The first terms in (30a) and (30b) are those involving the states at and up to the time t_s and they are multiplied by exponentially decaying functions of $t - t_s$. The terms U_1 and U_2 are due to the unmodeled dynamics and switched plant approximation, and the terms V_1 and V_2 are due to disturbances and noise. These four terms are not multiplied by exponentially decaying functions.

B. Switched System Π_σ

We now characterize the property of the second switched system Π_σ. See Appendix E for the proof of Lemma 2 and Appendix F for the proof of Lemma 3.

Lemma 2: Consider the switched system Π_σ in (25) with the constraints (28a) and (28b). Suppose that (14) and (15) hold. For every $\lambda < \lambda_0$, where λ_0 is as in (10b), we have

$$|x_{\mathbb{CE}}(t)|^2 \leqslant \gamma_1(\varepsilon + \xi(t))^\kappa e^{-(\lambda_0 - \lambda\kappa)(t-t_0)}|x_{\mathbb{CE}}(t_0)|^2 + \gamma_2(\varepsilon + \xi(t))^{\kappa+1} \quad \forall t \geqslant t_0 \quad (32)$$

for some constants $\{\gamma_1, \gamma_2\} > 0$ and κ as in (14).

Define

$$\bar{\lambda} := \min\{\lambda_0 - \lambda\kappa, (\kappa+1)\lambda\} \quad (33)$$

where λ_0 is as in (10b), $\hat{\lambda}$ is as in (12), and κ is as in (14).

Lemma 3: Consider the switched system Π_σ in (25) with the constraints (28a) and (28b) and the switched system Γ_s in (24) with the constraints (26) and (27). Suppose that λ satisfies (13), (14), and (15). We have

$$|x_{\mathbb{CE}}(t)|^2 \leqslant g(t)\, e^{-\bar{\lambda}(t-t_s)} + \gamma_2 2^\kappa \Delta_{x_{\mathbb{CE}}}^{\kappa+1}(t) \quad (34a)$$

$$\|(x_{\mathbb{CE}})_{[0,t)}\|_{2,\lambda}^2 \leqslant \|(x_{\mathbb{CE}})_{[0,t_s)}\|_{2,\lambda}^2\, e^{-\lambda(t-t_s)} + \frac{g(t)}{\bar{\lambda}-\lambda}\, e^{-\lambda(t-t_s)} + \gamma_2 2^\kappa \|(\Delta_{x_{\mathbb{CE}}}^{(\kappa+1)/2})_{t_s,t}\|_{2,\lambda}^2 \quad (34b)$$

for all $t \geqslant 0$, where $\bar{\lambda}$ is as in (33)

$$\Delta_{x_{\mathbb{CE}}}(t) := \varepsilon + \mathrm{U}_2(t) + \mathrm{V}_2(t), \quad (35a)$$

$$g(t) := (\gamma_1 2^{\kappa-1} f_1^\kappa(t_s) + \gamma_1 2^{\kappa-1}\|(\Delta_{x_{\mathbb{CE}}})_{[t_s,t]}\|^\kappa)|x_{\mathbb{CE}}(t_s)|^2 + \gamma_2 2^\kappa f_1^{\kappa+1}(t_s) \quad (35b)$$

f_1, U_2, and V_2 are as in Lemma 1, κ is as in (14), and γ_1 and γ_2 are as in Lemma 2.

C. Lyapunov-Like Function

We now introduce a Lyapunov-like function for the closed loop. Let

$$W(t) := 2\xi(t) + 2\gamma_c\|(x_{\mathbb{CE}})_{0,t}\|_{2,\lambda}^2 + \frac{4c_1^2\gamma a_0}{\hat{\lambda}-\lambda}|\zeta(t)|^2 + \frac{8c_1^2\gamma a_0}{\hat{\lambda}-\lambda}|x_{\mathbb{CE}}(t)|^2. \quad (36)$$

By convention, $W(0^-) = W(0)$. Note that $W(t_s^-) = f(t_s)$, where $f(t_s)$ is as in (31a). The following lemma gives a characterization of W with respect to the switching signal s; see Appendix G for the proof.

Lemma 4: Consider the switched system Π_σ in (25) with the constraints (28a) and (28b) and the switched system Γ_s in (24) with the constraints (26) and (27). Suppose that (13), (14), and (15) hold. Let W be as in (36). Let $T > 0$ and suppose that for all $t \in [0, T)$

$$\|(v)_{[t_s,t)}\| \leqslant \bar{v}, \; \|(w_y)_{[t_s,t)}\| \leqslant \bar{w}_y \quad (37a)$$

$$\|(\Delta_{\tilde{x}})_{[t_s,t)}\| \leqslant \bar{\delta}_{\tilde{x}} \quad (37b)$$

$$\|(\Delta_{x_{\mathbb{CE}}})_{[t_s,t)}\| \leqslant \bar{\delta}_{x_{\mathbb{CE}}} \quad (37c)$$

for some positive constants $\bar{\delta}_{x_{\mathbb{CE}}} > \varepsilon$ and $\{\bar{\delta}_{\tilde{x}}, \bar{v}, \bar{w}_y\} > 0$, where v is as in (24), w_y is as in (27), $\Delta_{\tilde{x}}$ is as in(31f), and $\Delta_{x_{\mathbb{CE}}}$ is as in (35a). We have

$$W(t) \leqslant (\alpha_1 W^\kappa(t_s^-) + \alpha_2)W(t_s^-)\, e^{-\lambda(t-t_s)} + \alpha_3(\bar{\delta}_{x_{\mathbb{CE}}}, \bar{\delta}_{\tilde{x}}, \bar{v}, \bar{w}_y) \quad \forall t \in [0, T] \quad (38)$$

where $\bar{\lambda}$ is as in (33), κ is as in (14), $\{\alpha_1, \alpha_2\} > 0$ are some constants, and α_3 is such that $\alpha_3 \to \gamma_\varepsilon\varepsilon$ as $\{\bar{\delta}_{x_{\mathbb{CE}}}, \bar{\delta}_{\tilde{x}}, \bar{v}, \bar{w}_y\} \to 0$.

D. Stability Property of the Function W

From (38), the function $W : [0, \infty) \to [0, \infty)$ satisfies an inequality of the following form:

$$W(t) \leqslant \rho(W(t_s^-))W(t_s^-)e^{-\lambda(t-t_s)} + \alpha_3 \quad (39)$$

for all $t_s \geqslant 0$ for some $\rho \in \mathcal{J}$ and $\alpha_3 \geqslant 0$. If there is no switching or s has finitely many switches (i.e., t_s is bounded), then it can seen from (39) that $W(t) \to \alpha_3$ as $t \to \infty$. However, the situation is more complicated when s has infinitely many switchings. We want to find a condition on the switching signal s to guarantee that W is bounded, and goes to zero when $\alpha_3 \to 0$. Before presenting such a result (Lemma 6 below), we need a preliminary result on hybrid average dwell-time switching signals.

Define the function H, which is parameterized by $\rho \in \mathcal{J}$, $\{W_0, \alpha_3, \lambda, \tau_{\mathbf{a}}, \tau_{\mathbf{d}}\} > 0$, and $\overline{N} \geqslant 1$, as follows:

$$\mathrm{H}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}^{\rho,W_0,\alpha_3,\lambda}(M) := \bar{\mu}^{\overline{N}+1}W_0 L + \alpha_3 + \alpha_3\bar{\mu}^{\overline{N}}\frac{L}{1-L} - M,$$
$$\bar{\mu} := \rho(M), \quad L := e^{-(\lambda - \ln\bar{\mu}/\tau_{\mathbf{a}})\tau_{\mathbf{d}}}. \quad (40)$$

The constant W_0 plays the role of a bound on the initial state $W(0)$. This function H stems from stability analysis of W. In particular, we can guarantee boundedness of W if there exists M such that $\mathrm{H}(M) \leqslant 0$ (see the proof of Lemma 6 in Appendix I). This leads us to find conditions on $\tau_{\mathbf{a}}, \tau_{\mathbf{d}}$, and $\overline{N}$ to guarantee that there exists $M > 0$ such that $\mathrm{H}(M) < 0$. Formally, let

$$\mathcal{S}^\rho(W_0, \alpha_3, \lambda) := \{(\tau_{\mathbf{d}}, \overline{N}, \tau_{\mathbf{a}}) \text{ such that } \tau_{\mathbf{d}} \geqslant 0, \overline{N} \geqslant 1, \text{ and } \exists M > W_0 : \lambda\tau_{\mathbf{a}} > \ln\bar{\mu},\ \mathrm{H}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}^{\rho,W_0,\alpha_3,\lambda}(M) \leqslant 0\}. \quad (41)$$

The set $\mathcal{S}^\rho(W_0, \alpha_3, \lambda)$ is always nonempty. To see this, pick any $M > \max\{\alpha_3, W_0\}$. Because $L \to 0$ as $\tau_{\mathbf{d}} \to \infty$, for every W_0 and $\overline{N}$, for a large enough $\tau_{\mathbf{d}}$, we will have $\mathrm{H}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}^{\rho,W_0,\alpha_3,\lambda}(M) \leqslant 0$ and hence, $\mathcal{S}^\rho(W_0, \alpha_3, \lambda)$ is nonempty. We have the following lemma to characterize the set $\mathcal{S}^\rho(W_0, \alpha_3, \lambda)$ (see Appendix H for the proof).

Lemma 5: Consider the set $\mathcal{S}^\rho(W_0, \alpha_3, \lambda)$ defined as in (41). For every $\{W_0, \alpha_3, \lambda\} > 0$, there exist $\underline{\tau}_{\mathbf{d}} \geqslant 0$ and a function $f : [\underline{\tau}_{\mathbf{d}}, \infty) \times [1, \infty) \to [0, \infty)$ such that

$$\mathcal{S}^\rho(W_0, \alpha_3, \lambda) = \{(\tau_{\mathbf{d}}, \overline{N}, \tau_{\mathbf{a}}) \big| \tau_{\mathbf{d}} \geqslant \underline{\tau}_{\mathbf{d}}, \overline{N} \geqslant 1, \tau_{\mathbf{a}} \geqslant f(\tau_{\mathbf{d}}, \overline{N})\}.$$

Furthermore, the function f can be characterized by two functions, $\phi_{\rho,\tau_{\mathbf{d}},W_0}:[1,\infty)\to[0,\infty)$ and $\psi_{\rho,\overline{N},W_0}:[\underline{\tau}_{\mathbf{d}},\infty)\to[0,\infty)$, such that

$$f(\tau_{\mathbf{d}},\overline{N}) := \max\{\phi_{\rho,\tau_{\mathbf{d}},W_0}(N),\psi_{\rho,\overline{N},W_0}(\tau_{\mathbf{d}})\}. \tag{42}$$

Using Lemma 5, we have the following stability result for the function W (see Appendix I for the proof).

Lemma 6: Consider a scalar function $W:[0,\infty)\to[0,\infty)$ which satisfies (39) for some class $\mathcal{J}$ function ρ and some constant $\{W_0,\alpha_3\}>0$. Suppose that $s\in\mathcal{S}_{hybrid}[\tau_{\mathbf{d}},\tau_{\mathbf{a}},\overline{N}]$ and $\tau_{\mathbf{a}}\geqslant f(\tau_{\mathbf{d}},\overline{N})$, where f is as in Lemma 5. Then for all $W(0)\leqslant W_0$, we have

$$W(t)\leqslant\bar{\gamma}_1 W(0)e^{-\underline{\lambda}t}+\alpha_3+\alpha_3\bar{\gamma}_2e^{-\underline{\lambda}(t-t_s)}\quad\forall t\geqslant 0 \tag{43}$$

for some constants $\{\bar{\gamma}_1,\bar{\gamma}_2,\underline{\lambda}\}>0$.

Remark 2: The result in Lemma 6 can also be applied to stability analysis of switched nonlinear systems in which a constant gain among the ISS-Lyapunov functions of the subsystems either does not exist or is not available (cf. [31] where such a constant gain is assumed). Consider the switched nonlinear system

$$\dot{x}=f_\sigma(x,u) \tag{44}$$

where $\sigma:[0,\infty)\to\mathcal{P}$ is a switching signal. Assume that every subsystem $\dot{x}=f_p(x,u)$ is ISS. We want to find classes of switching signals that guarantee ISS of the switched system. Since every subsystem is ISS, there exists a family of positive definite functions $V_p(x)$, $p\in\mathcal{P}$, such that $\forall x$

$$\alpha_1(|x|)\leqslant V_p(x)\leqslant\alpha_2(|x|) \tag{45a}$$

$$\frac{\partial V_p(x)}{\partial x}f_p(x)\leqslant-\lambda V_p(x)+\gamma(|u|) \tag{45b}$$

for some functions $\alpha_1,\alpha_2,\gamma\in\mathcal{K}_\infty$ and $\lambda>0$. The existence of such a common α_1,α_2,γ, and λ is guaranteed if the set $\mathcal{P}$ is finite or if the set $\mathcal{P}$ is compact and suitable continuity assumptions with respect to p hold (see [31, Remark 1]). Let ρ be a class $\mathcal{J}$ function such that

$$V_p(x)\leqslant\rho\big(V_q(x)\big)V_q(x)\quad\forall x\neq 0,\forall p,q\in\mathcal{P}. \tag{46}$$

Such ρ always exists, e.g.,

$$\rho(r):=\sup_{x:V_p(x)=r}(\alpha_2(|x|)/\alpha_1(|x|)).$$

From (45) and (46), we get

$$V_{\sigma(t)}(x(t))\leqslant\rho\big(V_{\sigma(t_\sigma^-)}(x(t_\sigma^-))\big)V_{\sigma(t_\sigma^-)}(x(t_\sigma^-))e^{-\lambda(t-t_\sigma)}+\int_{\tau_k}^{t}e^{-\lambda(t-\tau)}\gamma(|u(\tau)|)d\tau. \tag{47}$$

We then can apply Lemma 6 with $W(t)=V_{\sigma(t)}(x(t))$ and $\alpha_3:=(1/\lambda)\gamma(\|u\|_{[0,t)})$ to conclude ISS of the switched system (44) with bounded inputs (with a known bound) under a class of hybrid dwell-time switching signals.

E. Stability of Interconnected Switched System

We first state a result for interconnected switched systems with disturbances and noise but without unmodeled dynamics and without switched system approximation (*i.e.,* the plant is a switched system). Basically, the following theorem says that if the noise and disturbances are small enough (the condition (49)), the switching signal s is a hybrid dwell-time switching signal, and s satisfies the condition (50), then the states have the ISS-like property (51) with respect to the bounds on the noise and disturbances. Let

$$Z(t):=\max\{|\xi(t)|,|\zeta(t)|^2,|x_{\mathbb{CE}}(t)|^2\}. \tag{48}$$

Theorem 2: Consider the interconnected switched system of Π_σ in (25) and Γ_s in (24) with the constraints (26), (27), and (28). Suppose that $\Delta_1=0,\delta_1=0,\delta_2=0,\delta_C=0,\Delta_2=0$, where Δ_1,δ_1, and δ_2 are as in (24), and δ_C and Δ_2 are as in (27). Suppose that (13), (14), and (15) hold. For every $\{\bar{v},\bar{w}_y,Z_0\}>0$, there exists $f:[\underline{\tau}_{\mathbf{d}},\infty)\times[1,N_{\max})\to[0,\infty)$ for some $\underline{\tau}_{\mathbf{d}}>0,N_{\max}>1$ such that if

$$\|v\|_\infty\leqslant\bar{v},\ \|w_y\|_\infty\leqslant\bar{w}_y \tag{49}$$

where v is as in (24) and w_y is as in (27), and $s\in\mathcal{S}_{hybrid}[\tau_{\mathbf{d}},\tau_{\mathbf{a}},\overline{N}]$ and

$$\tau_{\mathbf{a}}\geqslant f(\tau_{\mathbf{d}},\overline{N}) \tag{50}$$

then for all $Z(0)\leqslant Z_0$, we have

$$Z(t)\leqslant\bar{c}_1Z(0)e^{-\underline{\lambda}t}+\bar{c}_2(\bar{v},\bar{w}_y)(1+\bar{\gamma}_2e^{-\underline{\lambda}(t-t_s)})\quad\forall t\geqslant 0 \tag{51}$$

for some $\{\bar{c}_1,\bar{\gamma}_2\}>0$, and a function $\bar{c}_2:[0,\infty)^2\to[0,\infty)$ such that $\bar{c}_2\to\gamma_\varepsilon\varepsilon$ as $\{\bar{v},\bar{w}_y\}\to 0$ for some $\gamma_\varepsilon>0$ independent of ε.

Proof: Consider the function W defined as in (36). If $\{\xi(0),|\zeta(0)|^2,|x_{\mathbb{CE}}(0)|^2\}\leqslant Z_0$, then

$$W(0)\leqslant 2Z_0+\frac{12c_1^2\gamma a_0}{\hat{\lambda}-\lambda}Z_0=:W_0. \tag{52}$$

Because $\delta_1=\delta_2=0$, $\Delta_1=0$, $\Delta_2=0$, we have $\mathrm{U}_2=0$ and $\Delta_{\tilde{x}}=0$ (where U_2 and $\Delta_{\tilde{x}}$ are as in (31)), which implies (37b) is true. Also, $\Delta_{x_{\mathbb{CE}}}\leqslant\varepsilon+(2c_1^2\gamma a_0/(\hat{\lambda}-\lambda)\lambda)\bar{v}^2+4\gamma\bar{w}_y^2=:\bar{\delta}_{x_{\mathbb{CE}}}$, and so (37c) is true. Thus, (37b) and (37c) hold for all $t\geqslant 0$ (i.e., $T=\infty$ in Lemma 4). Because(13)–(15) hold, the function W satisfies the inequality (38) by Lemma 4. Because $\alpha_3\to 0$ as $\{\bar{v},\bar{w}_y\}\to 0$, where α_3 is as in Lemma 4, by Lemma 5, there exist $\bar{v},\bar{w}_y$, $\bar{v}>\bar{w}_y$, small enough such that $\mathcal{S}^\rho(W_0,\alpha_3,\lambda)\neq\emptyset$, and a function as in Lemma 5 exists. Because the switching signal $s\in\mathcal{S}_{hybrid}[\tau_{\mathbf{d}},\tau_{\mathbf{a}},\overline{N}]$, and $(\tau_{\mathbf{d}},\tau_{\mathbf{a}},\overline{N})$ satisfies the condition (50), it follows from Lemma 6 that W has the ISS-like property (43) for all $t\geqslant 0$. From (43) and the definition of W as in (36), we obtain(51), where $\bar{c}_1:=\max\{1/2,(\hat{\lambda}-\lambda)/4c_1^2\gamma a_0\}\bar{\gamma}_1(2+12c_1^2\gamma a_0/(\hat{\lambda}-\lambda))$ and $\bar{c}_2:=\max\{1/2,(\hat{\lambda}-\lambda)/4c_1^2\gamma a_0\}\alpha_3$, in view of the fact that $\{\xi(t),|\zeta(t)|^2,|x_{\mathbb{CE}}(t)|^2\}\leqslant\max\{1/2,(\hat{\lambda}-\lambda)/4c_1^2\gamma a_0\}W(t)$ (from the definition of W as in (36)). The property of $\bar{c}_2$ asserted in the theorem follows from the property of α_3 as in Lemma 4. ■

When unmodeled dynamics are present, we have the following result, which essentially says that we still have the ISS-like property if the unmodeled dynamics are small enough in a certain sense. Suppose that Δ_1 and Δ_2, where Δ_1 is as in (24) and Δ_2 is as in (27), satisfy

$$|\Delta_1(t)|\leqslant\delta_g(\bar{\gamma}_1(\bar{x}(t))+\alpha), \tag{53a}$$

$$|\Delta_2(t)| \leqslant \delta_3(t)x_{\mathbb{CE}}(t) + \delta_g(\bar{\gamma}_2(\bar{x}(t)) + \alpha) \quad \forall t \quad (53b)$$

where $\bar{x}(t) := \max\{\|(x_{\mathbb{CE}})_{[0,t]}\|^2, \|(\zeta)_{[0,t]}\|^2, \|(\xi)_{[0,t]}\|\}$ for some $\{\delta_g, \delta_3\} \geqslant 0$ with respect to some $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathcal{K}_\infty$ and constant $\alpha > 0$.

Theorem 3: Consider the interconnected switched system of Π_σ *in (25) and* Γ_s *in (24) with the constraints (26), (27), and (28). Suppose that (53) holds for some* $\{\delta_g, \delta_3\} \geqslant 0$*. Suppose that (13), (14), and (15) hold. Suppose that* $Z(0) \leqslant Z_0$*, where* Z *is as in (48). There exist* $\{\varepsilon, \bar{v}, \bar{w}_y, \bar{\delta}, \bar{\delta}_g\} > 0$ *such that for all* v*,* w_y*,* $\delta_1, \delta_2, \Delta_1$*, and* Δ_2 *such that*

$$\|v\|_\infty \leqslant \bar{v},\ \|w_y\|_\infty \leqslant \bar{w}_y, \quad (54a)$$

$$\{\|\delta_1\|_\infty, \|\delta_2\|_\infty, \|\delta_3\|_\infty\} \leqslant \bar{\delta}, \quad (54b)$$

$$\delta_g \leqslant \bar{\Delta} \quad (54c)$$

where v *is as in (24),* w_y *is as in (27),* δ_1, δ_2*, and* Δ_1 *are as in (24),* Δ_2 *is as in (27), and* δ_g is as in *(53a), there exists* $f : [\underline{\tau}_{\mathbf{d}}, \infty) \times [1, N_{\max}) \to [0, \infty)$ *for some* $\underline{\tau}_{\mathbf{d}} > 0$ *such that if* $s \in \mathcal{S}_{hybrid}[\tau_{\mathbf{d}}, \tau_{\mathbf{a}}, \overline{N}]$ *and*

$$\tau_{\mathbf{a}} \geqslant f(\tau_{\mathbf{d}}, \overline{N}) \quad (55)$$

then

$$Z(t) \leqslant \bar{c}_1 Z(0)e^{-\lambda t} + \bar{c}_2(\bar{v}, \bar{w}_y, \bar{\delta}, \bar{\Delta}) \quad \forall t \geqslant 0 \quad (56)$$

for some constant $\bar{c}_1 > 0$ *and a function* $\bar{c}_2$ *such that* $\bar{c}_2 \to \gamma_\varepsilon \varepsilon$ *as* $\{\bar{v}, \bar{w}_y, \bar{\delta}, \bar{\Delta}\} \to 0$ *for some* $\gamma_\varepsilon > 0$.

Proof: The basic idea behind the proof is that if the switched plant is stable with disturbances and noise (Theorem 2), then the supervisory control scheme is able to handle unmodeled dynamics with small enough δ_g and smaller noise and disturbances bounds.

From the definition of $\Delta_{\tilde{x}}$ as in (31f) and (53), we get

$$\begin{aligned}\|(\Delta_{\tilde{x}})_{[0,t]}\|_\infty &\leqslant b_2\bar{\delta}\bar{x}(t) + \delta_g(\bar{\gamma}_1(\bar{x}(t)) + \alpha) \\ &=: b_3(\bar{\delta}, \delta_g, \bar{x}(t), \alpha)\end{aligned} \quad (57)$$

for some constant b_2, where $\bar{x}$ and α are as in (31f). The function b_3 has the property that for a fixed $\bar{x}(t)$ and α, $b_3 \to 0$ as $\{\bar{\delta}, \delta_g\} \to 0$. From the definition of $\Delta_{x_{\mathbb{CE}}}$ as in (35a), (53), and (57), we have

$$\begin{aligned}\|(\Delta_{x_{\mathbb{CE}}})_{[0,t]}\|_\infty &\leqslant \varepsilon + \frac{2c_1^2\gamma\hat{\gamma}_1}{(\hat{\lambda} - \lambda)\lambda}c_3^2(\bar{\delta}, \delta_g, \bar{x}(t), \alpha) \\ &\quad + 4\left(\frac{\gamma}{\lambda}\right)\delta_g^2(\bar{\gamma}_2(\bar{x}(t)) + \alpha)^2 \\ &\quad + \frac{2c_1^2\gamma\hat{\gamma}_2}{(\hat{\lambda} - \lambda)\lambda}\bar{v}^2 + \left(\frac{4\gamma}{\lambda}\right)\bar{w}_y^2 \\ &=: \varepsilon + b_4(\bar{\delta}, \delta_g, \bar{x}(t), \alpha, \bar{v}, \bar{w}_y)\end{aligned} \quad (58)$$

where b_4 has the property that $b_4 \to 0$ as $\{\bar{\delta}, \delta_g, \bar{c}, \bar{w}_y\} \to 0$ for fixed $\bar{x}(t)$ and α. From the definition of W as in (39), we have

$$\{|x_{\mathbb{CE}}(t)|^2, |\zeta(t)|^2, \xi(t)\} \leqslant cW(t) \quad (59)$$

for some constant $c > 0$.

Let $\bar{\alpha}_3$ be as in Lemma 5 and α_3 as in Lemma 4. From the properties $\alpha_3 \to 0$ as $\{\bar{\delta}_{x_{\mathbb{CE}}}, \bar{\delta}_{\tilde{x}}, \bar{v}, \bar{w}_y\} \to 0$, $b_3 \to 0$ as $\{\bar{\delta}, \delta_g\} \to 0$, and $b_4 \to 0$ as $\{\bar{\delta}, \delta_g, \bar{c}, \bar{w}_y\} \to 0$, we have that for given $\bar{\delta}_{x_{\mathbb{CE}}} > \varepsilon$ and $\bar{\delta}_{\tilde{x}} > 0$, there exist $\epsilon_0, \varepsilon, \bar{\delta}, \bar{\Delta}, \bar{v}, \bar{w}_y > 0$, $\bar{v} > \bar{w}_y$, such that

$$\alpha_3(\bar{\delta}_{x_{\mathbb{CE}}}, \bar{\delta}_{\tilde{x}}, \bar{v}, \bar{w}_y) \leqslant \bar{\alpha}_3 \quad (60a)$$

$$\varepsilon + b_4(\bar{\delta}, \bar{\Delta}, c(\overline{W} + \epsilon_0), \alpha, \bar{v}, \bar{w}_y) + \epsilon_0 < \bar{\delta}_{x_{\mathbb{CE}}} \quad (60b)$$

$$b_3(\bar{\delta}, \bar{\Delta}, c(\overline{W} + \epsilon_0), \alpha) + \epsilon_0 < \bar{\delta}_{\tilde{x}} \quad (60c)$$

$$\overline{W} := \gamma_1 W_0 + \gamma_2\alpha_3(\bar{\delta}_{x_{\mathbb{CE}}}, \bar{\delta}_{\tilde{x}}, \bar{v}, \bar{w}_y) \quad (60d)$$

where W_0 is as in (52) and c is as in (59).

Now, let f be the function as in Lemma 5 with the parameter W_0, α_3, λ. Consider the interconnected switched system in the theorem with this function f. Let $T := \sup\{t \geqslant 0 \mid \forall s \in [0,t] : W(t) \leqslant \overline{W} + \epsilon_0\}$. From (60d), we have $W(0) \leqslant W_0 < \overline{W}$, and so $T > 0$. Suppose that $T < \infty$.

Because $W(t) \leqslant \overline{W} + \epsilon_0$ for all $t \in [0, T)$, we have $\bar{x}(t) \leqslant c(\overline{W} + \epsilon_0)$ for all $t \in [0, T)$ in view of (59) and the definition of $\bar{x}$. Then for all $t \in [0, T)$, we have $\|(\Delta_{x_{\mathbb{CE}}})_{[0,t)}\| < \bar{\delta}_{x_{\mathbb{CE}}}$ in view of (58) and (60b), and $\|(\Delta_{\tilde{x}})_{[0,t)}\| < \bar{\delta}_{\tilde{x}}$ in view of (57) and (60c), and in particular, $\|(\Delta_{x_{\mathbb{CE}}})_{[t_s,t)}\| < \bar{\delta}_{x_{\mathbb{CE}}}$ and $\|(\Delta_{\tilde{x}})_{[t_s,t)}\| < \bar{\delta}_{\tilde{x}}$ for all $t \in [0, T)$. From Lemma 4, we have that W satisfies (38) up to the time T. Because $s \in \mathcal{S}_{hybrid}[\tau_{\mathbf{d}}, \tau_{\mathbf{a}}, \overline{N}]$ and s satisfies the condition (55), by Lemma 6, $W(t) \leqslant \gamma_1 W_0 + \gamma_2\alpha_3 = \overline{W}$ for all $t \in [0, T]$. No matter whether T is a switching time of s or not, there exists $\epsilon_1 > 0$ such that there is no switching of s in the open interval $(T, T + \epsilon_1)$. Because W is continuous on $[T, T+\epsilon_1)$, there exists $\epsilon_2 > 0$ such that $W(t) \leqslant \overline{W} + \epsilon_0$ for all $t \in [T, T + \epsilon_2]$, and hence $W(t) \leqslant \overline{W} + \epsilon_0$ for all $t \in [0, T + \epsilon_2]$. This contradicts the definition of T, and therefore, we must have $T = \infty$. We then have $\|(\Delta_{x_{\mathbb{CE}}})_{[0,t)}\| < \bar{\delta}_{x_{\mathbb{CE}}}$ and $\|(\Delta_{\tilde{x}})_{[0,t)}\| < \bar{\delta}_{\tilde{x}}$ for all $t \geqslant 0$. Because $s \in \mathcal{S}_{hybrid}[\tau_{\mathbf{d}}, \tau_{\mathbf{a}}, \overline{N}]$ and s satisfies the condition (55), by Lemma 6, we obtain $W(t) \leqslant \gamma_1 W(0) + \gamma_2\alpha_3$ for all $t \geqslant 0$. In view of the definition of W as in (36), the foregoing inequality implies(56) for some constant $\bar{c}_1$ and some function $\bar{c}_2$ (see the proof of Theorem 2 for the formula). We have $c_4 \to 0$ as $\{\bar{v}, \bar{w}_y, \bar{\delta}, \bar{\Delta}\} \to 0$ so one can take $\bar{\delta}_{x_{\mathbb{CE}}} \to \varepsilon$ as $\{\bar{v}, \bar{w}_y, \bar{\delta}, \bar{\Delta}\} \to 0$. The limiting property of $\bar{c}_2$ follows from the property of α_3 as in Lemma 4. ■

PROOF OF THEOREM 1

F. Bounds of the Unmodeled Dynamics

Note that $u = K_{\sigma(t)}\hat{x}_{\sigma(t)} = \overline{K}_{\sigma(t)}x_{\mathbb{CE}}(t)$. Also, $x = \tilde{x}_{s(t)} + \hat{x}_{s(t)}$, so $|x(t)| \leqslant |\tilde{x}_{s(t)}(t)| + |\hat{x}_{s(t)}(t)|$ and $\|x\|_{[0,t)} \leqslant \|\zeta\|_{[0,t)} + \|x_{\mathbb{CE}}\|_{[0,t)}$. From (17a), using the separation property of class $\mathcal{K}_\infty$ functions (that for every $\gamma \in \mathcal{K}_\infty, \exists\gamma_1, \gamma_2 \in \mathcal{K}_\infty : \gamma(r + t) \leqslant \gamma_1(r) + \gamma_2(t)\ \forall r, t$), we have

$$\begin{aligned}|\Delta_x(z, x, u, t)| &\leqslant \delta_g\gamma_x^0(|z(0)|)e^{-\lambda_\Delta t} + \delta_g\tilde{\gamma}_x^1(\|\zeta\|_{[0,t]}) \\ &\quad + \delta_g\tilde{\gamma}_x^2(\|x_{\mathbb{CE}}\|_{[0,t]})\end{aligned} \quad (61)$$

for some $\tilde{\gamma}_x^1, \tilde{\gamma}_x^2 \in \mathcal{K}_\infty$ and $\delta_g > 0$ such that $\delta_g \to 0$ as $\delta_\Delta \to 0$. Similarly, we have

$$\begin{aligned}|\Delta_y(z, x, u, t)| &\leqslant \delta_g\gamma_y^0(|z(0)|)e^{-\lambda_\Delta t} + \delta_g\tilde{\gamma}_y^1(\|\zeta\|_{[0,t]}) \\ &\quad + \delta_g\tilde{\gamma}_y^2(\|x_{\mathbb{CE}}\|_{[0,t]})\end{aligned} \quad (62)$$

for some $\tilde{\gamma}_y^1, \tilde{\gamma}_y^2 \in \mathcal{K}_\infty$.

From the definition of $\Delta_1 = \Delta_x + L_{s(t_s)}\Delta_y$, (61), and (62), we have

$$|\Delta_1(t)| \leqslant \delta_g(\bar{\gamma}_1(\max\{\|x_{\mathbb{CE}}\|_{[0,t]}, \|\zeta\|_{[0,t]}\}) + \bar{\gamma}_2(|z(0)|)) \quad (63)$$

for some $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathcal{K}_\infty$. Because $|z(0)| \leqslant \overline{X}_0$, it follows from (63) that Δ_1 has the property (53a) where $\alpha = \bar{\gamma}_2(\overline{X}_0)$. From the definition of $\Delta_2 = \delta_3 x_{\mathbb{CE}} + \Delta_y$ as in (27), (62), and the fact $|z(0)| \leqslant \overline{X}_0$, we have that Δ_2 has the property (53b) for some constant α. Because of the property of $\mathcal{K}_\infty$ functions, we can always have the same α in both (53a) and (53b).

G. Stability

From (20) and (48), we have that if $X(0) \leqslant \overline{X}_0$, then $Z(0) \leqslant 2\overline{X}_0^2 =: Z_0$. We have shown that Δ_1 and Δ_2 have the properties (53). Because $\{\|\delta_1\|_\infty, \|\delta_2\|_\infty, \|\delta_3\|_\infty\} \to 0$ as $\delta_{\mathbf{P}} \to 0$, for every $\bar{\delta}$ in (54b), the condition (54b) is satisfied if $\delta_{\mathbf{P}}$ is small enough. Also, because $\delta_g \to 0$ as $\delta_\Delta \to 0$, the condition (54c) is satisfied if δ_Δ is small enough. The foregoing facts and the fact $Z(0) \leqslant Z_0$, when applied to Theorem 3, imply that for small enough $\bar{v}, \bar{w}_y, \delta_{\mathbf{P}}, \delta_\Delta, \varepsilon$, there exists a function f such that if $s \in \mathcal{S}_{hybrid}[\tau_{\mathbf{d}}, \tau_{\mathbf{a}}, \overline{N}]$ and $\tau_{\mathbf{a}} \geqslant f(\tau_{\mathbf{d}}, \overline{N})$, then for all $Z(0) \leqslant Z_0$, we have (56). From (56) and the fact that $x = \tilde{x}_p + \hat{x}_p$ for all p, it follows that x has the property (22). Since the unmodeled dynamics is input-to-state stable, from boundedness of x, u, we also have z bounded. From the fact that $\bar{v} \to 0$ as $\bar{w}_x, \bar{w}_y \to 0$ and $\delta_{x_{\mathbb{CE}}} \to \varepsilon$ as $\{\bar{w}_x, \bar{w}_y, \delta_{\mathbf{P}}\} \to 0$, the limiting property of $\bar{c}_2$ follows as in Theorem 3, such that $\bar{c}_2 \to \gamma_\varepsilon \varepsilon$ as $\{\bar{w}_x, \bar{w}_y, \bar{\delta}\} \to 0$.

VI. Numerical Example

Consider the following uncertain system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} u$$
$$y = 10^{-4} [1 \quad 0] x$$

where $b(t)$ are unknown. We know that $b(t) \in (-3, 3) \setminus \{0\} =: \Omega$. The stabilization of the foregoing uncertain system is challenging because the sign of b is not known. The previously reported result for supervisory control with constant unknown parameters is not applicable here if b has large variation such that the system cannot be approximated by a system with a constant b and small unmodeled dynamics to a good degree. An example of such b is a periodic square signal alternating equally between two values -1 and 1 with period $2T$, $T > 0$.

The design procedure is as follows. Pick $\Omega_1 := (-3, 0)$, $\Omega_2 := (0, 3)$, $q_1 = -2 \in \Omega_1$, and $q_2 = 2 \in \Omega_2$. The set $\mathcal{P} = \{1, 2\}$. We have $A_p = \begin{bmatrix} 0 & 1 \\ 2 & -6 \end{bmatrix}$, $B_p = \begin{bmatrix} 0 \\ q_p \end{bmatrix}$, $C_p = 10^{-4}[1 \quad 0]$, $p \in \mathcal{P}$. Design feedback gains K_p such that $A_p + B_p K_p$ have poles at $-2, -1$ for all $p \in \mathcal{P}$, and design observer gains L_p such that $A_p + L_p C$ have poles at $-6, -2$ for all $p \in \mathcal{P}$.

For these K_p, L_p, we have the constants $\mu = 3.5, \lambda_0 = 0.1, \gamma_0 = 180$, $a_1 = 10^{-8}$, and $a_2 = 2 \times 10^{-7}$ (see Appendix J for the procedure of how to calculate theses constants using LMIs). Pick $h = 0.5$, $\varepsilon = 10^{-2}$, and $\gamma = \gamma_0 \times 10^3 = 1.8 \times 10^5$. Calculate $\kappa = 6.1794$. Pick $\lambda = \lambda_0/(\kappa + 1) - 0.001$. Calculate $\bar{\lambda} = \min\{\lambda_0 - \lambda\kappa, (\kappa + 1)\lambda\} = 0.0201$. Calculate $\gamma_c = 1.4142^{-4}$, $\bar{v} = \bar{w} = 10^{-6}$, $\bar{\delta}_{x_{\mathbb{CE}}} = 0.0101$, $\bar{\delta}_x = 2 \times 10^{-5}$. Calculate $\alpha_1 = 1.1146 \times 10^{17}$, $\alpha_2 = 6.1759 \times 10^4$, and $\alpha_3 = 0.0170$. For $W_0 = 10^{-3}$, the curves $\phi_{\rho,\tau_{\mathbf{d}},W_0}$ with

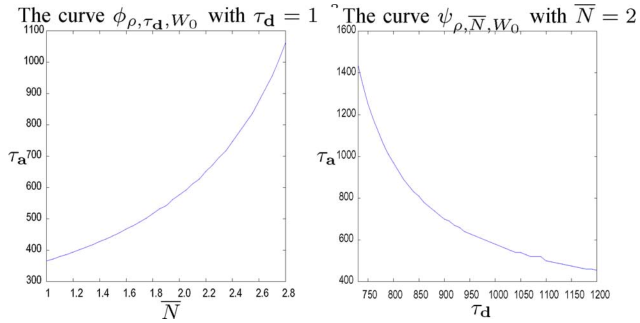


Fig. 4. Curves $\phi_{\rho,\tau_{\mathbf{d}},W_0}$ and $\psi_{\rho,\overline{N},W_0}$.

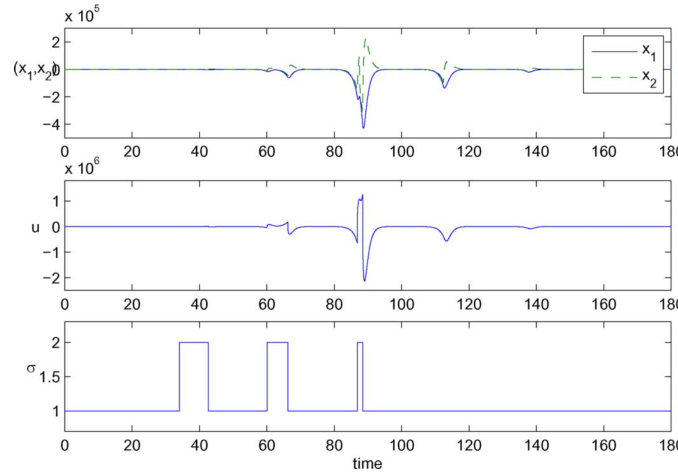


Fig. 5. Simulation result.

$\tau_{\mathbf{d}} = 10^3$ and the curve $\psi_{\rho,\overline{N},W_0}$ with $\overline{N} = 2$ are plotted in Fig. 4.

For $W_0 = 10^{-3}$, it is calculated that $\overline{X}_0 = 0.0158$. For $\tau_{\mathbf{a}} = 10^3$, $\tau_{\mathbf{d}} = 300$, $\overline{N} = 2$, we get $M = 0.01$ and then, $\bar{\delta} = 1.8686 \times 10^{-24}$. Then for all the initial state less than $\overline{X}_0$, for all noise and disturbances less than $\bar{w}_x$ and $\bar{w}_y$, for all unmodeled dynamics less than $\bar{\delta}$, the state will satisfy (22) with $\bar{c}_1 = 7.5795 \times 10^{17}$, $\bar{c}_2 = 4.7360$, and $\bar{\gamma}_2 = 3.7479 \times 10^{10}$.

We simulate the control system with the following noise, disturbances, and unmodeled dynamics:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} u + w_x + z$$
$$y = 10^{-4} [1 \quad 0] x + w_y$$
$$\dot{z} = -z + 10^{-8} x$$

where $w_x = w_y = \text{Rand}(-10^{-2}, 10^{-2})$ are uniform random sequences in $[-10^{-2}, 10^{-2}]$, and b is a periodic square signal alternating equally between two values -1 and 1 with period $2T$, $T = 12.5$. The simulation result is plotted in Fig. 5.

Remark 3: The bounds provided in this paper are very conservative: the bounds in the simulation are much smaller than the calculated bounds, and the calculated stability margin is extremely small (simulation shows the state bound of about 5 $\times 10^5$ with the unmodeled dynamics bound of 10^{-8}, compared with the calculated values of 1.1952 $\times 10^{16}$ and 1.8686 $\times 10^{-24}$, respectively). It is observed that in simulations with larger initial states, noise, disturbances, and unmodeled dynamics, the control system in the example still remains stable. The conservativeness of the bounds comes from three main sources: i) the *bounds on the switching signal's properties in (28) ii) the bound on the state $x_{\mathbb{CE}}$ under the switching signal σ in (32), and iii) the bounds on the state $x_{\mathbb{CE}}$ under the switching signal s in (34), not to mention the conservativeness of the numerical calculation of the constants used in these bounds (e.g., μ, a_1, a_2, λ_0.) While the difference between the simulation bounds and the calculated bounds is large, the conservativeness of the bounds is not entirely surprising because even in the case of constant unknown parameters, the reported bounds on the closed-loop states under supervisory control are also conservative, several of which are the starting points for the analysis of the time-varying case in this paper. It is noted that performance of supervisory control, even in the case of constant unknown parameters, is still an open question. The use of LMIs to improve the numerical bounds as in this work (see Appendix J) is a step forward in addressing performance issues in supervisory control.*

Remark 4: We have also performed simulations of the supervisory control scheme in this paper applied to the output tracking problem, where we make the output of the uncertain plant $\dot{x} = a(t)x + b(t)u, y = x$ with unknown a and b to track the output of the reference model $P_m = 1/(1+s)$. Simulation results show that the supervisory control scheme also works well in this output tracking example and is the motivation for further theoretical research on supervisory control.

VII. Conclusion

We addressed the stabilization problem for time-varying uncertain systems using supervisory control. We introduced a new class of switching signals called hybrid dwell-time switching signals, which are characterized by both dwell-time and average dwell-time. We showed that in the presence of bounded disturbances and noise, all the closed-loop signals are bounded provided that the plant varies slowly enough in the hybrid dwell-time sense and the unmodeled dynamics are small enough. In the absence of unmodeled dynamics, disturbances, and noise, the closed-loop plant state can be made as small as desired. In proving closed-loop stability, we also studied stability of interconnected switched systems in which the jump map of the first switched system is bounded by the state of the second switched system, and the switching signal of the second switched system is bounded in terms of the state of the first switched system. We provided an ISS-like stability result for such interconnected switched systems. This stability result can also be applied to stability analysis of switched nonlinear systems. Numerical simulations and discussions are provided to illuminate the utility and drawback of the proposed control scheme.

This work has provided a theoretical foundation for applying supervisory control to time-varying systems. This contribution can be viewed in two ways: at the qualitative level, it says that the supervisory control design provides a margin of robustness against noise, disturbances, and unmodeled dynamics; at the quantitative level, it provides a description of this robustness margin. Future work aims to address performance issues such as how to obtain tighter bounds and how to choose the design parameters to improve transient response. Another potential direction is to address the issue of fast-switching plants by finding the design parameters that yield the largest class of hybrid dwell-time signals. Also, supervisory control for output tracking of uncertain time-varying plants deserves further exploration.

Appendix A
The Injected Systems

Because the linear controller $\mathbf{C}_p$ stabilizes the plant $\dot{x} = A_p x + B_p u,\ y = C_p x$, it follows that the system:

$$\begin{cases} \dot{\hat{x}}_p = A_p\hat{x}_p + B_p u + L_p(C_p\hat{x}_p - y) \\ \hat{y}_p = C_p\hat{x}_p \\ \dot{x}_{\mathbb{C}} = F_p(x_{\mathbb{C}}, u, y) = F_p(x_{\mathbb{C}}, u, \hat{y}_p - (C_p\hat{x}_p - y)) \\ u = H_p x_{\mathbb{C}} \end{cases} \tag{64}$$

is exponentially stable when $C_p\hat{x}_p - y = 0$. Also, for $q \neq p$, we can write

$$\begin{aligned} &A_q\hat{x}_q + B_q u + L_q(C_q\hat{x}_q - y) \\ &\quad = (A_q + L_qC_q)\hat{x}_q + (B_q u - L_qC_p\hat{x}_p) + L_q(C_p\hat{x}_p - y) \end{aligned}$$

and so the system

$$\begin{cases} \dot{\hat{x}}_q = A_q\hat{x}_q + B_q u + L_q(C_q\hat{x}_q - y) \\ \dot{x}_{\mathbb{C}} = F_p(x_{\mathbb{C}}, u, y) \\ u = H_p x_{\mathbb{C}} \end{cases} \tag{65}$$

is exponentially stable if $C_p\hat{x}_p - y = 0$ and $B_qH_px_{\mathbb{C}} - L_qC_p\hat{x}_p = 0$. Therefore, if $C_p\hat{x}_p - y = 0$, then $x_{\mathbb{C}}$ and $\hat{x}_p$ go to zero exponentially in view of (64), and hence, $B_qH_px_{\mathbb{C}} - L_qC_p\hat{x}_p$ goes to zero exponentially, and thus, $\hat{x}_q$ goes to zero exponentially for all $q \neq p$. The foregoing reasoning shows that for a fixed controller, the injected system is exponentially stable when $y - \tilde{y}_p = 0$. Because the injected system is linear, it follows that each injected system must be of the form $\dot{x}_{\mathbb{CE}} = \mathbf{A}_p x_{\mathbb{CE}} + \mathbf{B}_p(\hat{y}_p - y)$, where $x_{\mathbb{CE}} := (x_{\mathbb{E}}, x_{\mathbb{C}})$ is the state of the injected system, and $\mathbf{A}_p$ is Hurwitz $\forall p \in \mathcal{P}$.

Appendix B
The Error Dynamics

Because s is constant in $[t_s, t)$ and $s(t_s)$ is the index of the nominal switched plant for time in $[t_s, t)$, in view of the linear observer dynamics (5), we have

$$\begin{aligned} \dot{\tilde{x}}_{s(t_s)}(t) = \mathbf{E}_{s(t_s)}\tilde{x}_{s(t_s)}(t) + \delta_1(t)x(t) + \delta_B(t)u \\ + \Delta_x^e(z, x, u, t) + v(t) \end{aligned}$$

where $\mathbf{E}_p := A_p + L_pC_p$, $\delta_1(t) := \delta_A(t) + L_{s(t_s)}\delta_C(t)$, $\Delta_x^e(z, x, u, t) := \Delta_x(z, x, u, t) + L_{s(t_s)}\Delta_y(z, x, u, t)$, and $v(t) = w_x(t) + L_{s(t_s)}w_y(t)$; $\mathbf{A}_p$ are Hurwitz for all $p \in \mathcal{P}$ by construction. In view of $u(t) = H_{\sigma(t)}x_{\mathbb{C}}(t)$ (from (4)) and $x = \tilde{x}_{s(t_s)} + \hat{x}_{s(t_s)}$, we then obtain

$$\begin{aligned} \dot{\tilde{x}}_{s(t_s)}(t) = \mathbf{E}_{s(t_s)}\tilde{x}_{s(t_s)}(t) + \delta_1(t)\tilde{x}_{s(t_s)}(t) \\ + \delta_2(t)x_{\mathbb{CE}}(t) + \Delta_x^e(z, x, u, t) + v(t) \end{aligned} \tag{66}$$

where $\delta_2(t)x_{\mathbb{CE}} := \delta_1(t)\hat{x}_{s(t_s)} + \delta_B(t)H_{\sigma(t_s)}x_{\mathbb{C}}$ (recall that $\hat{x}_p$ and $x_{\mathbb{C}}$ are components of $x_{\mathbb{CE}}$ for all $p \in \mathcal{P}$ so δ_2 is a linear combination of δ_1 and δ_B). Equation (66) is rewritten as the switched system

$$\dot{\zeta} = \mathbf{E}_s\zeta + \delta_1\zeta + \delta_2 x_{\mathbb{CE}} + \Delta_1 + v \tag{67}$$

where $\zeta(t) := \tilde{x}_{s(t_s)}(t)$ and $\Delta_1 := \Delta_x^e(z, x, u, t)$. It is clear from the definitions that δ_1 is such that $\{\|\delta_1\|_\infty, \|\delta_2\|_\infty\} \to 0$ as $\delta_{\mathbf{P}} \to 0$, Δ_1 is such that $\|\Delta_1\| \to 0$ as $\delta_\Delta \to 0$, and $\|v\|_\infty \to 0$ as $\{\|w_x\|_\infty, \|w_y\|_\infty\} \to 0$.

Appendix C
Constraints Between Γ_s and Π_σ

Constraint for Γ_s: Because $\tilde{x}_p(t)+\hat{x}_p(t)=\tilde{x}_q(t)+\hat{x}_q(t)=x(t)\ \forall t, \forall p,q\in\mathcal{P}$, we have $\tilde{x}_p=\tilde{x}_q+\hat{x}_p-\hat{x}_q$ and so

$$\begin{aligned}|\tilde{x}_p(t)|^2 &\leqslant 2|\tilde{x}_q(t)|^2+2|\hat{x}_p(t)-\hat{x}_q(t)|^2\\ &\leqslant 2|\tilde{x}_q(t)|^2+4|x_{\mathbb{CE}}(t)|^2\ \forall t\end{aligned}\tag{68}$$

for all $p,q\in\mathcal{P}$ in view of the fact that $\hat{x}_p$ is a component of $x_{\mathbb{CE}}$. Therefore, in view of $\zeta(t)=\tilde{x}_{s(t_s)}(t)$, we have (26a).

We have $\tilde{y}_p(t)=y(t)-\hat{y}_p(t)=\tilde{y}_q(t)+[C_q-C_p](x_q(t),x_p(t))^T$ for all $t\geqslant 0$, and hence, $|\tilde{y}_p(t)|\leqslant|\tilde{y}_q(t)|+\gamma_c|x_{\mathbb{CE}}|$, where $\gamma_c:=\max_{p,q}\|[C_q-C_p]\|$, and in view of the fact that x_p,x_q are components of $x_{\mathbb{CE}}$.

Note that the $e^{-\lambda t}$-weighted $\mathcal{L}_2$ norm has the following properties. From the definition of $\|\cdot\|_{2,\lambda}$ as in Section II, we have

$$\frac{d}{dt}\|(f)_{[t_0,t)}\|^2_{2,\lambda}=-\lambda\|(f)_{[t_0,t)}\|^2_{2,\lambda}+|f(t)|^2.\tag{69}$$

The $e^{-\lambda t}$-weighted $\mathcal{L}_2$ norm is decreasing in λ and additionally, has the following properties:

$$\int_{t_0}^{t}e^{-\lambda(t-\tau)}e^{-\bar{\lambda}\tau}d\tau\leqslant\frac{1}{\bar{\lambda}-\lambda}e^{-\lambda(t-t_0)}\tag{70a}$$

$$\left\|\left(\|(f)_{[t_0,*)}\|_{2,\bar{\lambda}}\right)_{[t_0,t)}\right\|^2_{2,\lambda}\leqslant\frac{1}{\bar{\lambda}-\lambda}\|(f)_{[t_0,t)}\|^2_{2,\lambda}\tag{70b}$$

for all $\bar{\lambda}>\lambda>0$ (note that the left-hand side of (70a) is the $e^{-\lambda t}$-weighted $\mathcal{L}_2$ norm of the exponentially decaying function with the rate $\bar{\lambda}$).

Taking $e^{-\lambda t}$-weighted $\mathcal{L}_2$ norm of both sides of the foregoing inequality, we get $\|(\tilde{y}_p)_{[0,t)}\|^2_{2,\lambda}\leqslant 2\|(\tilde{y}_q)_{[0,t)}\|^2_{2,\lambda}+2\gamma_c\|(x_{\mathbb{CE}})_{[0,t)}\|^2_{2,\lambda}$. Letting $q=s(t_s)$, $p=s(t_s^-)$, and $t=t_s$, we get (26b) in view of $\xi(t)=\|(\tilde{y}_{s(t_s)})_{[0,t)}\|^2_{2,\lambda}$.

We have

$$\begin{aligned}\tilde{y}_{s(t_s)}(t)=&\,C_{s(t_s)}x(t)+\delta_C(t)x(t)+\Delta_y(z,x,u,t)+w(t)\\ &-C_{s(t_s)}\hat{x}_{s(t_s)}(t)\\ =&\,(C_{s(t_s)}+\delta_C(t))\tilde{x}_p(t)+\delta_C(t)\hat{x}_p(t)\\ &+\Delta_y(z,x,u,t)+w(t).\end{aligned}$$

Therefore

$$\begin{aligned}\tilde{y}_{s(t_s)}(t)=(C_{s(t_s)}+\delta_C(t))\tilde{x}_{s(t_s)}(t)+\delta_3(t)x_{\mathbb{CE}}\\ +\Delta_y(z,x,u,t)+w(t)\end{aligned}$$

where $\delta_3:=\|\delta_C\|_\infty$, and we get (27).

Constraint for Π_σ: The hysteresis switching logic has the following properties ([31, Lemma 4.2],[14, Lemma 1]): for every index $q\in\mathcal{P}$ and arbitrary $t\geqslant t_0\geqslant 0$, we have $\forall t\geqslant t_0$

$$N_\sigma(t,t_0)\leqslant m+\frac{m}{\ln(1+h)}\ln\left(\frac{\mu_q(t)}{\varepsilon}\right)+\frac{m\lambda(t-t_0)}{\ln(1+h)},\tag{71}$$

$$\|(\tilde{y}_\sigma)_{t_0,t}\|^2_{2\lambda}\leqslant\frac{m(1+h)}{\gamma}\mu_q(t).\tag{72}$$

Note that the above inequalities hold for any $q\in\mathcal{P}$. In view of $\mu_{s(t_s)}(t)=\varepsilon+\xi(t)$, we obtain (28b) and (28a).

Appendix D
Proof of Lemma 1

Since s is constant in $[t_s,t)$ and $\mathbf{E}_{s(t_s)}$ is Hurwitz, from (24), we have

$$\begin{aligned}|\zeta(t)|^2\leqslant a_0e^{-\hat{\lambda}(t-t_s)}|\zeta(t_s)|^2+\hat{\gamma}_1\|(\Delta_{\tilde{x}})_{[t_s,t)}\|^2_{2,\hat{\lambda}}\\ +\hat{\gamma}_2\|(v)_{[t_s,t)}\|^2_{2,\hat{\lambda}}\end{aligned}$$

for some $a_0,\hat{\gamma}$. The foregoing inequality and (26a) give (30a).

From (27), we have $|\tilde{y}_{s(t)}(t)|\leqslant c_1|\zeta(t)|+|\Delta_2(t)|+|w_y(t)|$, where $c_1:=\max_p\|C_p\|+\delta_{\mathbf{P}}$, and so $|\tilde{y}_{s(t)}(t)|^2\leqslant 2c_1^2|\zeta(t)|^2+4|\Delta_2(t)|^2+4|w_y(t)|^2$. Then

$$\begin{aligned}\|(\tilde{y}_s)_{[t_s,t)}\|^2_{2,\lambda}\leqslant 2c_1^2\|(\zeta)_{[t_s,t)}\|^2_{2,\lambda}+4\|(\Delta_2)_{[t_s,t)}\|^2_{2,\lambda}\\ +4\|(w_y)_{[t_s,t)}\|^2_{2,\lambda}.\end{aligned}\tag{73}$$

Taking $e^{-\lambda t}$-weighted $\mathcal{L}_2$ norm on both sides of (30a), in view of (70a) and (70b), we get

$$\begin{aligned}\|(\zeta)_{[t_s,t)}\|^2_{2,\lambda}\leqslant&\,(2|\zeta(t_s^-)|^2+4|x_{\mathbb{CE}}(t_s)|^2)\frac{a_0}{\hat{\lambda}-\lambda}e^{-\lambda(t-t_s)}\\ &+\frac{\hat{\gamma}_1}{\hat{\lambda}-\lambda}\|(\Delta_{\tilde{x}})_{[t_s,t)}\|^2_{2,\lambda}+\frac{\hat{\gamma}_2}{\hat{\lambda}-\lambda}\|(v)_{[t_s,t)}\|^2_{2,\lambda}.\end{aligned}$$

Combining the foregoing inequality with (73), the fact that $\xi(t)=\xi(t_s)e^{-\lambda(t-t_s)}+\gamma\|(\tilde{y}_s)_{t_s,t}\|^2_{2\lambda}$, and (26b), we get (30b).

Appendix E
Proof of Lemma 2

Let $T\geqslant t_0$ be an arbitrary time. Let $\tau_k, k=1,\ldots,N_\sigma(T,t_0)$ be the switching times of σ in $[t_0,T)$; by convention, $\tau_0=t_0$ and $\tau_{N_\sigma(T,t_0)+1}=T$. Let $V_\sigma(t):=V_{\sigma(t)}(x_{\mathbb{CE}}(t))$ where V_p are as in (10). From (10b), we have

$$V_\sigma(t)\leqslant e^{-\lambda_0(t-\tau_k)}V_\sigma(\tau_k)+\gamma_0\|(y_\sigma)_{[\tau_k,t)}\|^2_{2,\lambda_0}\tag{74}$$

for all $t\in[\tau_k,\tau_{k+1})$. The foregoing inequality and (11) give $V_\sigma(t)\leqslant e^{-\lambda_0(t-\tau_k)}\mu V_\sigma(\tau_k^-)+\gamma_0\|(y_\sigma)_{[\tau_k,t)}\|^2_{2,\lambda_0}$ for all $t\in[\tau_k,\tau_{k+1})$. Letting $t=\tau_{k+1}$ and iterating the foregoing inequality for $k=1$ to N_σ, together with (74) with $t=\tau_1$ and $\tau_k=\tau_0$, we then have

$$\begin{aligned}V_\sigma(T^-)\leqslant&\,\mu^{N_\sigma(T,t_0)}e^{-\lambda_0(T-t_0)}V_\sigma(t_0)\\ &+\gamma_0\sum_{i=0}^{N_\sigma(T,t_0)}g_i\|(y_\sigma)_{[\tau_i,\tau_{i+1})}\|^2_{2,\lambda_0}\end{aligned}\tag{75}$$

where $g_i:=\mu^{N_\sigma(T,t_0)-i}e^{-\lambda_0(T-\tau_{i+1})}$. Because there are no more than $N_\sigma(T,t_0)-i$ switches in the interval $[\tau_{i+1},T)$, from (28b), we have

$$\begin{aligned}\mu^{N_\sigma(T,t_0)-i}&\leqslant\mu^{d_1+d_2\ln(\varepsilon+\xi(t))}\mu^{(T-t_{i+1})/\tau_a}\\ &=\bar{\gamma}_0(\varepsilon+\xi(t))^\kappa e^{\lambda\kappa(T-t_{i+1})}\end{aligned}\tag{76}$$

where τ_a is as in (29), $d_1:=m-(m/\ln(1+h))\ln\varepsilon$, $d_2:=m/\ln(1+h)$, $\bar{\gamma}_0=\mu^{d_1}=\mu^m\varepsilon^{-\kappa}$, and $\kappa=d_2\ln\mu$, and the equality follows from the fact that $\ln \mu/\tau_a = \lambda\kappa$. Note that we have $\kappa > 1$ in view of (13). Because $\lambda < \lambda_0$, we have

$$\|(y_\sigma)_{\tau_i,\tau_{i+1}}\|^2_{2,\lambda_0} \leqslant \|(y_\sigma)_{\tau_i,\tau_{i+1}}\|^2_{2,\lambda}.$$

The foregoing inequality and (76) give:

$$\begin{aligned}\sum_{i=0}^{N_\sigma(T,t_0)} & g_i\|(y_\sigma)_{\tau_i,\tau_{i+1}}\|^2_{2,\lambda_0} \\ &\leqslant \bar{\gamma}_0(\varepsilon+\xi(t))^\kappa \\ &\quad\cdot \sum_{i=0}^{N_\sigma(T,t_0)} e^{-(\lambda_0-\lambda\kappa)(T-\tau_{i+1})}\|(y_\sigma)_{\tau_i,\tau_{i+1}}\|^2_{2,\lambda} \\ &\leqslant \bar{\gamma}_0(\varepsilon+\xi(t))^\kappa\|(y_\sigma)_{t_0,T}\|^2_{2,\lambda} \qquad (77)\end{aligned}$$

where the last inequality holds because $0 < \lambda < \lambda_0 - \lambda\kappa$ in view of (14). From (75), (77), and (28a), we obtain

$$\begin{aligned}V_\sigma(T^-) \leqslant \bar{\gamma}_0(\varepsilon+\xi(T))^\kappa e^{-(\lambda_0-\lambda\kappa)(T-t_0)}V_\sigma(t_0) \\ +\bar{\gamma}_0\gamma_0\frac{m(1+h)}{\gamma}(\varepsilon+\xi(T))^{\kappa+1}. \quad (78)\end{aligned}$$

The foregoing inequality and (10a), in view of the fact that $x_{\mathbb{CE}}$ is continuous, give (32) (replacing T in (78) with t) where $\gamma_1 := \bar{\gamma}_0 a_2/a_1$ and $\gamma_2 := \bar{\gamma}_0 m(1+h)(\gamma_0/\gamma)(a_2/a_1)$.

Appendix F
Proof of Lemma 3

For the convex function $a \mapsto a^r, r \geqslant 1, a > 0$ we have $(a/2+b/2)^r \leqslant (1/2)(a^r+b^r)$ and so, $(a+b)^r \leqslant 2^{r-1}(a^r+b^r)\ \forall a,b \geqslant 0$. Using the foregoing identity with (30a), for all $r > 1$, we have

$$\begin{aligned}(\varepsilon+\xi(t))^r \leqslant 2^{r-1}f_1^r(t_s)e^{-r\lambda(\tau-t_s)} \\ +2^{r-1}(\varepsilon+\mathrm{U}_2(t)+\mathrm{V}_2(t))^r. \quad (79)\end{aligned}$$

Applying (79) with $r=\kappa$ and $r=\kappa+1$ to (32), we get

$$\begin{aligned}&|x_{\mathbb{CE}}(t)|^2 \\ &\leqslant \gamma_1 2^{\kappa-1}f_1^\kappa(t_s)e^{-\lambda_0(t-t_s)}|x_{\mathbb{CE}}(t_s)|^2 \\ &\quad+\gamma_1 2^{\kappa-1}(\varepsilon+\mathrm{U}_2(t)+\mathrm{V}_2(t))^\kappa e^{-(\lambda_0-\lambda\kappa)(t-t_s)}|x_{\mathbb{CE}}(t_s)|^2 \\ &\quad+\gamma_2 2^\kappa f_1^{\kappa+1}(t_s)e^{-(\kappa+1)\lambda(t-t_s)} \\ &\quad+\gamma_2 2^\kappa(\varepsilon+\mathrm{U}_2(t)+\mathrm{V}_2(t))^{\kappa+1}.\end{aligned}$$

The foregoing inequality leads to (34a) in view of the definition of g and the fact $\nu(\tau) \leqslant \|\nu\|_{[t_s,t]}\ \forall\tau \in [t_s,t)$. Taking $e^{-\lambda t}$-weighted $\mathcal{L}_2$ norm on both sides of (34a), we get

$$\|(x_{\mathbb{CE}})_{[t_s,t)}\|^2_{2,\lambda}\leqslant\frac{g(t)}{\bar{\lambda}-\lambda}e^{-\lambda(t-t_s)}+\gamma_2 2^\kappa\|(\nu^{(\kappa+1)/2})_{t_s,t}\|^2_{2\lambda}.$$

From (69), we have $\|(x_{\mathbb{CE}})_{[0,t)}\|^2_{2,\lambda} = e^{-\lambda(t-t_s)}\|(x_{\mathbb{CE}})_{[0,t)}\|^2_{2,\lambda} + \|(x_{\mathbb{CE}})_{[t_s,t)}\|^2_{2,\lambda}$. The preceding inequality, in view of the fact that $\|(x_{\mathbb{CE}})_{[0,t)}\|^2_{2\lambda} = \|(x_{\mathbb{CE}})_{[0,t_s)}\|^2_{2,\lambda}e^{-\lambda(t-t_s)}+\|(x_{\mathbb{CE}})_{[t_s,t)}\|^2_{2,\lambda}$ [which follows directly from (69)], gives (34b).

Appendix G
Proof of Lemma 4

From the definition of W, we have

$$|x_{\mathbb{CE}}(t)|^2 \leqslant \frac{\hat{\lambda}-\lambda}{8c_1^2\gamma a_0}W(t) \quad \forall t. \tag{80}$$

From the definition of f_1 as in Lemma 1 and g as in (35), in view of (80), we have

$$\begin{aligned}g(t) &\leqslant \gamma_1 2^{\kappa-1}\frac{\hat{\lambda}-\lambda}{8c_1^2\gamma a_0}W^{\kappa+1}(t_s^-)+\gamma_1 2^{\kappa-1}\bar{\delta}^\kappa_{x_{\mathbb{CE}}}|x_{\mathbb{CE}}(t_s)|^2 \\ &\quad+\gamma_2 2^\kappa W^{\kappa+1}(t_s^-) \\ &=: \gamma_3 W^{\kappa+1}(t_s^-)+\gamma_4|x_{\mathbb{CE}}(t_s)|^2 \quad \forall t \in [0,T] \qquad (81)\end{aligned}$$

where $\gamma_3 := \gamma_1 2^{\kappa-1}((\hat{\lambda}-\lambda)/8c_1^2\gamma a_0)+\gamma_2 2^\kappa$ and $\gamma_4 := \gamma_1 2^{\kappa-1}\bar{\delta}^\kappa_{x_{\mathbb{CE}}}$, in view of (37b) and (37c). From (34a) and (81), we get

$$|x_{\mathbb{CE}}(t)|^2\leqslant(\gamma_3 W^{\kappa+1}(t_s^-)+\gamma_4|x_{\mathbb{CE}}(t_s)|^2)e^{-\lambda(t-t_s)}+\gamma_5 \tag{82}$$

where $\gamma_5 := \gamma_2 2^\kappa\bar{\delta}^{\kappa+1}_{x_{\mathbb{CE}}}$. From (34b) and (81), we have

$$\begin{aligned}&\|(x_{\mathbb{CE}})_{[0,t)}\|^2_{2,\lambda} \leqslant \|(x_{\mathbb{CE}})_{[0,t_s)}\|^2_{2,\lambda}e^{-\lambda(t-t_s)} \\ &+\left(\frac{1}{\bar{\lambda}-\lambda}\gamma_3 W^{\kappa+1}(t_s^-)+\frac{1}{\bar{\lambda}-\lambda}\gamma_4|x_{\mathbb{CE}}(t_s)|^2\right)e^{-\lambda(t-t_s)}+\frac{\gamma_5}{\lambda}. \qquad (83)\end{aligned}$$

From (30b), we get

$$\xi(t) \leqslant W(t_s^-)e^{-\lambda(t-t_s)}+\gamma_6 \tag{84}$$

where $\gamma_6 := \bar{\delta}_{x_{\mathbb{CE}}}-\varepsilon$. From (30a), we get

$$|\zeta(t)|^2 \leqslant a_0(2|\zeta(t_s^-)|^2+4|x_{\mathbb{CE}}(t_s)|^2)e^{-\lambda(t-t_s)}+\gamma_7 \tag{85}$$

in view of $\lambda < \hat{\lambda}$, where $\gamma_7 := (\hat{\gamma}_1/\lambda)\bar{\delta}^2_{\tilde{x}}+(\hat{\gamma}_2/\lambda)\bar{v}^2$. From (82), (83), (84), (85), and the definition of W as in (36), we get

$$\begin{aligned}W(t) &\leqslant 2W(t_s^-)e^{-\lambda(t-t_s)}+c_3W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} \\ &\quad+(c_4\|(x_{\mathbb{CE}})_{0,t_s}\|^2_{2,\lambda}+c_5|\zeta(t_s^-)|^2 \\ &\quad+c_6|x_{\mathbb{CE}}(t_s^-)|^2)e^{-\lambda(t-t_s)}+c_7 \qquad (86)\end{aligned}$$

where $c_3 := 2\gamma_c\gamma_3/(\bar{\lambda}-\lambda)+4c_1^2\gamma a_0\gamma_c/(\hat{\lambda}-\lambda)$, $c_4 := 2\gamma_c$, $c_5 := 8c_1^2\gamma a_0^2/(\hat{\lambda}-\lambda)$, $c_6 := 2\gamma_c\gamma_4/(\bar{\lambda}-\lambda)+16c_1^2\gamma a_0^2/(\hat{\lambda}-\lambda)+8c_1^2\gamma a_0\gamma_4/(\hat{\lambda}-\lambda)$, and $c_7 := 2\gamma_6+2\gamma_c\gamma_5/\lambda+4c_1^2\gamma a_0\gamma_7/(\bar{\lambda}-\lambda)+8c_1^2\gamma a_0\gamma_5/(\hat{\lambda}-\lambda)$. Let

$$\begin{aligned}c_8 &:= \max\left\{\frac{c_4}{(2\gamma_c)},\frac{c_5}{\left(\frac{4c_1^2\gamma a_0}{(\hat{\lambda}-\lambda)}\right)},\frac{c_6}{\left(\frac{8c_1^2\gamma a_0}{(\hat{\lambda}-\lambda)}\right)}\right\} \\ &= c_6\frac{(\hat{\lambda}-\lambda)}{(8c_1^2\gamma a_0)}.\end{aligned}$$

Then from (86), we get

$$\begin{aligned}W(t) \leqslant 2W(t_s^-)e^{-\lambda(t-t_s)}+c_3W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} \\ +c_8W(t_s^-)e^{-\lambda(t-t_s)}+c_7.\end{aligned}$$

We then have (38) where $\alpha_1 := c_3, \alpha_2 := 2+c_8$, and $\alpha_3 := c_7$.

From the definition of γ_5, we have $\gamma_5 \to c_{\gamma_5}\varepsilon$ for some constant $c_{\gamma_5} > 0$ as $\bar{\delta}_{x\mathbb{CE}} \to \varepsilon$. From the definitions of γ_6, $\gamma_6 \to 0$ as $\bar{\delta}_{x\mathbb{CE}} \to \varepsilon$. From the definition of γ_7, we have $\gamma_7 \to 0$ as $\{\bar{\delta}_x, \bar{v}\} \to 0$. It follows from the definition of c_8 that $c_8 \to c_{\gamma_8}\varepsilon$ for some constant $c_{\gamma_8} > 0$ as $\{\bar{\delta}_{\tilde{x}}, \bar{v}, \bar{w}_y\} \to 0$ and $\bar{\delta}_{x\mathbb{CE}} \to \varepsilon$.

Appendix H
Proof of Lemma 5

The function f has two parameters, $\tau_{\mathbf{d}}$ and $\overline{N}$, and thus, is a mesh in 3-D if we plot $\bar{\tau}_a$ versus $\tau_{\mathbf{d}}$ and $\overline{N}$. It is difficult to solve for the function f analytically, but for given W_0, α_3, and ρ, it can be numerically calculated as follows.

Start from $\tau_{\mathbf{d}} = 0$ and $\overline{N} = 1$. For each pair of $\tau_{\mathbf{d}}$ and $\overline{N}$, starting from $\bar{\tau} = \tau_{\mathbf{d}}$ and in small increment of $\delta_a > 0$, sequentially check M from 0 in increment of δ_M until $M_{\max}$ for some $M_{\max} > 0$ whether that M satisfies $\mathrm{H}^{\rho,W_0,\alpha_3,\lambda}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}(M) \leqslant 0$. The first $\bar{\tau}$ that gives the existence of M such that $\mathrm{H}^{\rho,W_0,\alpha_3,\lambda}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}(M) \leqslant 0$ is an approximated value of $f(\tau_{\mathbf{d}}, \overline{N})$. Repeat the procedure for a new $\overline{N}$ with a small increment δ_N or a new $\tau_{\mathbf{d}}$ with a small increment δ_d; Stop when both $\tau_{\mathbf{d}}$ and $\overline{N}$ reach some chosen upper bounds. This procedure will produce a granular mesh for the function f. The smaller δ_a, δ_N, and δ_d are, the closer the numerically calculated mesh is to the function f, at the expense of the computational time.

To help better understand the function f, we can also have following characterization of cuts of f when we fixed either $\overline{N}$ or $\tau_{\mathbf{d}}$.

Average Dwell-Time versus Chatter Bound Curves:

- Fix a $\tau_{\mathbf{d}}$. Define the set $\mathcal{A}_{\rho,\tau_{\mathbf{d}}}$ parameterized by W_0 as

$$\mathcal{A}_{\rho,\tau_{\mathbf{d}}}(W_0) := \Bigg\{ (\overline{N}, \tau_{\mathbf{a}}) \big| \overline{N} \geqslant 1, \tau_{\mathbf{a}} > 0, \text{ and } \exists M > W_0 : \tau_{\mathbf{a}} > \frac{\ln \rho(M)}{\lambda} \text{ and } \mathrm{H}^{\rho,W_0,\alpha_3,\lambda}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}(M) \leqslant 0 \Bigg\}. \quad (87)$$

Since the function H is increasing in $\overline{N}$ and decreasing in $\tau_{\mathbf{a}}$, in view of (87), there exists a function $\tau_{\mathbf{a}} = \phi_{\rho,\tau_{\mathbf{d}},W_0}(\overline{N})$ as the lower boundary of $\mathcal{A}_{\rho,\tau_{\mathbf{d}}}(W_0)$ such that $\mathcal{A}_{\rho,\tau_{\mathbf{d}}}(W_0) := \{(n,t) : 1 \leqslant n \leqslant N_{\max}, t > \phi_{\rho,\tau_{\mathbf{d}},W_0}(n)\}$ for some $N_{\max}$ ($N_{\max}$ can be ∞). We will call $\phi_{\rho,\tau_{\mathbf{d}},W_0}$ an *average dwell-time versus chatter bound curve*. It is not easy to characterize the function $\phi_{\rho,\tau_{\mathbf{d}},W_0}$ analytically, but the function can be calculated numerically for given $\rho, \alpha_3, \lambda, \kappa$, $\tau_{\mathbf{d}}$, and W_0 (up to approximation errors). The algorithm is as follows: Start from $\overline{N} = 1$. For each $\overline{N}$, starting from $\bar{\tau} = \tau_{\mathbf{d}}$ and in small increment of $\delta_\tau > 0$, sequentially check M from 0 in increment of δ_M until $M_{\max}$ for some $M_{\max} > 0$ whether that M satisfies $\mathrm{H}^{\rho,W_0,\alpha_3,\lambda}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}(M) \leqslant 0$. The first $\bar{\tau}$ that gives the existence of M such that $\mathrm{H}^{\rho,W_0,\alpha_3,\lambda}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}(M) \leqslant 0$ is an approximated value of $\phi_{\rho,\tau_{\mathbf{d}},W_0}(\overline{N})$. Repeat the procedure for new $\overline{N}$ with a small increment δ_N. The smaller δ_τ and δ_N are, the closer the numerically calculated curve is to the curve $\phi_{\rho,\tau_{\mathbf{d}},W_0}$, at the expense of the computational time.

Average Dwell-Time versus Dwell-Time Curves:

- Fix a $\overline{N}$. Define the set $\mathcal{B}_{\rho,\overline{N}}$ parameterized by W_0 as

$$\mathcal{B}_{\rho,\overline{N}}(W_0) := \Bigg\{ (\tau_{\mathbf{a}}, \tau_{\mathbf{d}}) : \tau_{\mathbf{a}} > 0, \text{ and } \exists M > W_0 : \tau_{\mathbf{a}} > \frac{\ln \rho(M)}{\lambda} \text{ and } \mathrm{H}^{\rho,W_0,\alpha_3,\lambda}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}(M) \leqslant 0 \Bigg\}. \quad (88)$$

Since the function H is decreasing in $\tau_{\mathbf{d}}$ and also decreasing in $\tau_{\mathbf{a}}$, in view of (88), there exists a function $\tau_{\mathbf{a}} = \psi_{\rho,\overline{N},W_0}(\tau_{\mathbf{d}})$ as the lower boundary of $\mathcal{B}_{\rho,\overline{N}}(W_0)$ such that $\mathcal{B}_{\rho,\overline{N}}(W_0) := \{(t,d) : \tau_{\mathbf{d}}^{\min} \leqslant d \leqslant \tau_{\mathbf{d}}^{\max}, t > \psi_{\rho,\overline{N},W_0}(d)\}$ for some $\tau_{\mathbf{d}}^{\max} > \tau_{\mathbf{d}}^{\min}$. We call $\psi_{\rho,\overline{N},W_0}$ an *average dwell-time versus dwell-time curve*. The function $\psi_{\rho,\overline{N},W_0}$ is not easy to characterize analytically but can be calculated numerically for given $\rho, \alpha_3, \lambda, \kappa$, $\overline{N}$, and W_0. Similarly to the case with fixed $\tau_{\mathbf{d}}$, the algorithm for a fixed $\overline{N}$ is as follows: Start from $\tau_{\mathbf{d}} = 0$. For each $\tau_{\mathbf{d}}$, starting from $\bar{\tau} = \tau_{\mathbf{d}}$ and in small increment of $\delta_\tau > 0$, sequentially check M from 0 in increment of δ_M until $M_{\max}$ for some $M_{\max} > 0$ whether that M satisfies $\mathrm{H}^{\rho,W_0,\alpha_3,\lambda}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}(M) \leqslant 0$. The first $\bar{\tau}$ that gives the existence of M such that $\mathrm{H}^{\rho,W_0,\alpha_3,\lambda}_{\tau_{\mathbf{a}},\tau_{\mathbf{d}},\overline{N}}(M) \leqslant 0$ is an approximated value of $\psi_{\rho,\overline{N},W_0}(\tau_{\mathbf{d}})$. Repeat the procedure for new $\tau_{\mathbf{d}}$ with a small increment δ_d. The smaller δ_τ and δ_d are, the closer the numerically calculated curve is to the curve $\psi_{\rho,\overline{N},W_0}$, at the expense of the computational time.

Using the average dwell-time versus chatter bound curve and the average dwell-time versus dwell-time curve curve, we obtain the (42).

Remark 5: When ρ is bounded, say $\rho(M) \leqslant \bar{\rho}$ $\forall M$, for every $W_0 \geqslant 0$, $\tau_{\mathbf{d}} > 0$, $\overline{N} \geqslant 1$, and $\tau \geqslant \ln \rho/\lambda$, we can always choose M large enough so that $h_\rho(M, \overline{N}, \tau_{\mathbf{a}}, \tau_{\mathbf{d}}, W_0) < 0$. Then, $\mathcal{A}_{\rho,\tau_{\mathbf{d}}}(W_0) = \{(\overline{N}, \tau_{\mathbf{a}}) : \overline{N} \geqslant 1, \tau_{\mathbf{a}} > \ln \bar{\rho}/\lambda\}$ and $\mathcal{B}_{\rho,\overline{N}}(W_0) = \{(t,d) : d > 0, t > \ln \bar{\rho}/\lambda, t \geqslant d\}$, and therefore, both the sets $\mathcal{A}$ and $\mathcal{B}$ can be characterized by a single number $\ln \bar{\rho}/\lambda$, which is exactly the lower bound on average dwell-time for stability of W (as reported in [20], [31]; see also [32]). In general, when ρ is an unbounded function, the set $\mathcal{A}$ is characterized by an average dwell-time versus chatter bound curve, and the set $\mathcal{B}$ is characterized by an average dwell-time versus dwell-time curve (these curves are horizontal lines when ρ is constant).

Appendix I
Proof of Lemma 6

Because $(\tau_{\mathbf{d}}, \overline{N}, \tau_{\mathbf{a}}) \in \mathcal{S}^\rho(W_0, \alpha_3, \lambda)$, from the definition of $\mathcal{S}^\rho(W_0, \alpha_3, \lambda)$ as in (41), there exists $M > W_0$ such that

$$\bar{\mu}^{\overline{N}+1} W_0 e^{-\underline{\lambda}\tau_{\mathbf{d}}} + \alpha_3 + \alpha_3 \bar{\mu}^{\overline{N}} \frac{e^{-\underline{\lambda}\tau_{\mathbf{d}}}}{1 - e^{-\underline{\lambda}\tau_{\mathbf{d}}}} \leqslant M \quad (89)$$

where $\underline{\lambda} := \lambda - \ln \bar{\mu}/\tau_{\mathbf{a}} > 0$ and $\bar{\mu} := \rho(M)$.

If the switching signal s is a constant (no switching in s), then clearly W satisfies (43), in view of (39).

Suppose that s has at least one switch. Denote by $\mathcal{T}_{[t_0,t)}$ the set of switching times of s in $[t_0,t)$. Define

$$T := \sup\{t \geqslant 0 \mid W(\tau) \leqslant M \quad \forall \tau \in \mathcal{T}_{[0,t)}\}.$$

Let τ_1 be the first switching time of s. We have

$$W(\tau_1^-) \leqslant \rho(W_0)W_0 e^{-\lambda\tau_1} + \alpha_3 \leqslant \rho(W_0)W_0 e^{-\lambda\tau_1} + \alpha_3 \leqslant \rho(M)W_0 e^{-\lambda\tau_{\mathbf{d}}} + \alpha_3 \leqslant$$

in view of $\tau_1 \geqslant \tau_{\mathbf{d}}$ and $W_0 < M$. The foregoing inequality of W together with (89) imply that $W(\tau_1^-) \leqslant M$ in view of $\overline{N} \geqslant 1$ and $\underline{\lambda} < \lambda$. From the definition of T, we have that $T \geqslant \tau_1$ (and thus, T is not empty). We will next show that $T = \infty$ by contradiction.

Suppose that $T < \infty$. This means there is a switching time τ after T such that $W(\tau^-) > M$. Let $\tau_1, \ldots, \tau_{N_s(\tau,0)}$ be the switching times of s in $[0,\tau)$; by convention $\tau_0 = 0$ and $\tau_{N_s(\tau,0)+1} = \tau$. From the definition of T and τ_i, we have

$$\rho(W(\tau_i^-)) \leqslant \rho(M) = \bar{\mu} \quad \forall i = 0, \ldots, N_s(t,0). \tag{90}$$

From (39), we have $W(\tau_{i+1}^-) \leqslant \bar{\mu}W(\tau_i^-)e^{-\lambda(\tau_{i+1}-\tau_i)} + \alpha_3$, $i = 0, \ldots, N_s(t,0)$ in view of (90). Iterating the foregoing inequality for $i = 0$ to $i = N_s(t,0)$, in view of $W(0^-) = W(0)$, we obtain

$$W(\tau^-) \leqslant \bar{\mu}^{N_s(\tau,0)+1}W(0)e^{-\lambda\tau} + \alpha_3 + \alpha_3 \sum_{i=0}^{N_s(\tau,0)} \bar{\mu}^{N_s(\tau,0)-i} e^{-\lambda(\tau-\tau_{i+1})}. \tag{91}$$

Because $s \in \mathcal{S}_{hybrid}[\tau_{\mathbf{d}}, \tau_{\mathbf{a}}, \overline{N}]$, in view of the fact that there are $N_s(\tau,0)$ switches in $[0,t)$, we have $\bar{\mu}^{N_s(\tau,0)} \leqslant \bar{\mu}^{\overline{N}_s(\tau,0)+t/\tau_{\mathbf{a}}}$. Also, because there are $N_s(\tau,0) - i$ switches in $[\tau_{i+1},\tau)$, we have $\bar{\mu}^{N_s(\tau,0)-i} \leqslant \bar{\mu}^{\overline{N}+(\tau-\tau_{i+1})/\tau_{\mathbf{a}}}$. The foregoing inequality and (91) yield

$$W(\tau^-) \leqslant \bar{\mu}^{\overline{N}+1}W(t_0)e^{-\underline{\lambda}\tau} + \alpha_3 + \alpha_3\bar{\mu}^{\overline{N}} \sum_{i=0}^{N_s(\tau,0)} e^{-\underline{\lambda}(\tau-\tau_{i+1})} \tag{92}$$

where $\underline{\lambda} := \lambda - \ln\bar{\mu}/\tau_{\mathbf{a}}$. Also because $s \in \mathcal{S}_{hybrid}[\tau_{\mathbf{d}}, \tau_{\mathbf{a}}, \overline{N}]$, we have $\tau_{i+1} - \tau_i > \tau_{\mathbf{d}}$, and then $\tau - \tau_{i+1} > (N_s(\tau,0) - i - 1)\tau_{\mathbf{d}} + (\tau - \tau_{N_s(\tau,0)})$, $i = 0, \ldots, N_s(\tau,0) - 1$.

We then have

$$\sum_{i=0}^{N_s(\tau,0)} e^{-\underline{\lambda}(\tau-\tau_{i+1})} \leqslant e^{-\underline{\lambda}\tau_{\mathbf{d}}} \sum_{i=0}^{N_s(\tau,0)} e^{-\underline{\lambda}\tau_{\mathbf{d}}(N_s(\tau,0)-i-1)} \leqslant \frac{e^{-\underline{\lambda}(\tau-\tau_{N_s(\tau,0)})}}{(1-e^{-\underline{\lambda}\tau_{\mathbf{d}}})}.$$

The foregoing inequality and (92) yield

$$W(\tau^-) \leqslant \bar{\mu}^{\overline{N}+1}W(t_0)e^{-\underline{\lambda}\tau} + \alpha_3 + \alpha_3\bar{\mu}^{\overline{N}}e^{-\underline{\lambda}(\tau-\tau_{N_s(\tau,0)})}(1-e^{-\underline{\lambda}\tau_{\mathbf{d}}})^{-1}. \tag{93}$$

Because s has a dwell-time $\tau_{\mathbf{d}}$, we have $\tau - \tau_{N_s(t,0)} \geqslant \tau_{\mathbf{d}}$. Then the right-hand side of (93) is bounded by $\bar{\mu}^{\overline{N}+1}W(t_0)e^{-\underline{\lambda}\tau_{\mathbf{d}}} + \alpha_3 + \alpha_3\bar{\mu}^{\overline{N}}e^{-\underline{\lambda}\tau_{\mathbf{d}}}(1-e^{-\underline{\lambda}\tau_{\mathbf{d}}})^{-1} \leqslant M$, in view of (89). Thus, $W(\tau^-) \leqslant M$, in which $\tau > T$, a contradiction with the definition of T.

Therefore, $T = \infty$. In view of (93), we have

$$W(t) \leqslant \bar{\mu}^{\overline{N}+1}W(t_0)e^{-\underline{\lambda}t} + \alpha_3 + \alpha_3\bar{\mu}^{\overline{N}}e^{-\underline{\lambda}(t-t_s)}(1-e^{-\underline{\lambda}\tau_{\mathbf{d}}})^{-1} \tag{94}$$

for all $t \geqslant 0$. We then get (43) with $\bar{\gamma}_1 := \bar{\mu}^{\overline{N}+1}$ and $\bar{\gamma}_2 := \bar{\mu}^{\overline{N}}(1-e^{-\underline{\lambda}\tau_{\mathbf{d}}})^{-1}$.

APPENDIX J
LMIs FOR NUMERICAL CALCULATION

The explicit forms for the matrices $\mathbf{A}_p$ and $\mathbf{B}_p$ of the injected system in (9) are

$$\mathbf{A}_p := \mathrm{diag}(A_k + L_kC_k, k = 1, \ldots, m) + \begin{bmatrix} 0_{n\times n(p-1)} & B_1K_p - L_1C_p & 0_{n\times(n-p)} \\ \vdots & B_kK_p - L_kC_p & \vdots \\ 0_{n\times n(p-1)} & B_mK_p - L_mC_p & 0_{n\times(n-p)} \end{bmatrix}, \tag{95}$$

$$\mathbf{B}_p := \begin{bmatrix} L_1 \\ \vdots \\ L_m \end{bmatrix}. \tag{96}$$

Because A_p are Hurwitz, the functions V_p as in (10) can be obtained by solving the Lyapunov equation $A_p^T P_p + P_p A_p = -Q$ for some $Q > 0$. However, this approach will likely lead to a big Lyapunov gain μ, which is undesirable. Instead, we will employ LMIs to find P_p, $p \in \mathcal{P}$ that yield the smallest μ possible. To do so, we write (10b) as

$$\dot{V}_p = x_{\mathbb{CE}}^T(\mathbf{A}_p^T P_p + P_p\mathbf{A}_p)x_{\mathbb{CE}} + x_{\mathbb{CE}}^T P_p\mathbf{B}_p\tilde{y}_p + \tilde{y}_p^T\mathbf{B}_p^T P_p$$
$$\leqslant -\lambda_0 x_{\mathbb{CE}}^T P_p x_{\mathbb{CE}} + \gamma_0\tilde{y}_p^T\tilde{y}_p$$
$$\Leftrightarrow [x_{\mathbb{CE}} \quad \tilde{y}_p] \begin{bmatrix} \mathbf{A}_p^T P_p + P_p\mathbf{A}_p + \lambda_0 P_p & P_p\mathbf{B}_p \\ \mathbf{B}_p^T P_p & -\gamma_0 I \end{bmatrix} \begin{bmatrix} x_{\mathbb{CE}} \\ \tilde{y}_p \end{bmatrix} \leqslant 0.$$

Because the foregoing inequality is true for all $x_{\mathbb{CE}}$ and all $\tilde{y}_p$, it is implied by

$$\begin{bmatrix} \mathbf{A}_p^T P_p + P_p\mathbf{A}_p + \lambda_0 P_p & P_p\mathbf{B}_p \\ \mathbf{B}_p^T P_p & -\gamma_0 I \end{bmatrix} \prec 0$$

where $\prec$ reads "negative definite" (similarly, $\succ$ reads "positive definite"). We then have the following LMIs:

$$\begin{bmatrix} \mathbf{A}_p^T P_p + P_p\mathbf{A}_p + \lambda_0 P_p & P_p\mathbf{B}_p \\ \mathbf{B}_p^T P_p & -\gamma_0 I \end{bmatrix} \prec 0$$
$$P_p \succ 0 \quad p = 1, \ldots, m. \tag{97}$$

The above set of LMIs can be solved numerically for given $\mathbf{A}_p, \mathbf{B}_p, \lambda_0, \gamma$. For our analysis here, we also want the ratio a_2/a_1 small, where a_1 and a_2 are as in (10). This can be achieved by adding the following LMIs into (97):

$$P_p \succ a_1 I, \quad P_p \prec a_2 I. \tag{98}$$

To find small μ, a_1, and a_2, the algorithm is as follows. First, pick a small λ_0 such that $0 < \lambda_0 < -\max_{p\in\mathcal{P}} \mathrm{Re}(\mathrm{eig}\mathbf{A}_p)$ (for example, $\lambda_0 = 0.001$). Pick $a_1 > 0$ and $a_2 > 0$ such that a_2/a_1 is large (for example, $a_2/a_1 = 10^6$). Start with some small μ and large γ (such as $\mu = 1$ and $\gamma = 1000$), check for feasibility of the set of (97) and (98). If the LMIs do not have a solution, then increase μ until the LMIs have a solution. For that μ, try to reduce γ while the LMIs still have a solution. Then try to increase λ_0 and to reduce a_2/a_1 while the LMIs still have a solution. The increment and the decrement of those constants can be fully automated by programming the above algorithm into computer codes.

REFERENCES

[1] K. Tsakalis and P. Ioannou, "Adaptive control of linear time-varying plants: A new model reference controller structure," *IEEE Trans. Autom. Control*, vol. AC-34, no. 10, pp. 1038–1046, Oct. 1989.

[2] R. H. Middleton and G. C. Goodwin, "Adaptive control of time-varying linear systems," *IEEE Trans. Autom. Control*, vol. AC-33, no. 2, pp. 150–155, Feb. 1988.

[3] K. Tsakalis and P. Ioannou, "A new indirect adaptive control scheme for time-varying plants," *IEEE Trans. Autom. Control*, vol. 35, no. 6, pp. 697–705, Jun. 1990.

[4] P. G. Voulgaris, M. A. Dahleh, and L. S. Valavani, "Robust adaptive control: A slowly varying systems approach," *Automatica*, vol. 30, no. 9, pp. 1455–1461, 1994.

[5] D. Dimogianopoulos and R. Lozano, "Adaptive control for linear slowly time-varying systems using direct least-squares estimation," *Automatica*, vol. 37, no. 2, pp. 251–256, 2001.

[6] Y. Zhang, B. Fidan, and P. Ioannou, "Backstepping control of linear time-varying systems with known and unknown parameters," *IEEE Trans. Autom. Control*, vol. 48, no. 11, pp. 1908–1925, Nov. 2003.

[7] B. Fidan, Y. Zhang, and P. Ioannou, "Adaptive control of a class of slowly time varying systems with unmodeling uncertainties," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 915–920, Jun. 2005.

[8] G. Kreisselmeier, "Adaptive control of a class of slowly time-varying plants," *Syst. Control Lett.*, vol. 8, no. 2, pp. 97–103, 1986.

[9] F. Giri, M. M'Saad, L. Dugard, and J. Dion, "Robust adaptive regulation with minimal prior knowledge," *IEEE Trans. Autom. Control*, vol. 37, no. 3, pp. 305–315, Mar. 1992.

[10] R. Marino and P. Tomei, "Adaptive control of linear time-varying systems," *Automatica*, vol. 39, pp. 651–659, 2003.

[11] P. Rosa, M. Athans, S. Fekri, and C. Silvestre, "Further evaluation of the RMMAC method with time-varying parameters," in *Proc. Mediterranean Conf. Control Autom.*, 2007, pp. 1–6.

[12] D. E. Miller, "A new approach to model reference adaptive control," *IEEE Trans. Autom. Control*, vol. 48, no. 5, pp. 743–757, May 2003.

[13] A. S. Morse, "Supervisory control of families of linear set-point controllers, Part 2: Robustness," *IEEE Trans. Autom. Control*, vol. 42, no. 11, pp. 1500–1515, Nov. 1997.

[14] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Hysteresis-based switching algorithms for supervisory control of uncertain systems," *Automatica*, vol. 39, no. 2, pp. 263–272, 2003.

[15] D. Liberzon, *Switching in Systems and Control*. Boston, MA: Birkhäuser, 2003.

[16] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Overcoming the limitations of adaptive control by means of logic-based switching," *Syst. Control Lett.*, vol. 49, no. 1, pp. 49–65, 2003.

[17] A. S. Morse, "Supervisory control of families of linear set-point controllers, Part 1: Exact matching," *IEEE Trans. Autom. Control*, vol. 41, no. 10, pp. 1413–1431, Oct. 1996.

[18] J. P. Hespanha, D. Liberzon, A. S. Morse, B. D. O. Anderson, T. S. Brinsmead, and F. Bruyne, "Multiple model adaptive control, Part 2: Switching," *Int. J. Robust Nonlin. Control*, vol. 11, pp. 479–496, 2001.

[19] M. J. Feiler and K. S. Narendra, "Simultaneous identification and control of time-varying systems," in *Proc. 45th IEEE Conf. Decision Control*, 2006, pp. 1093–1098.

[20] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. 38th IEEE Conf. Decision Control*, 1999, pp. 2655–2660.

[21] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[22] H. Khalil, *Nonlinear Systems*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 2002.

[23] B. D. O. Anderson, T. S. Brinsmead, F. D. Bruyne, J. P. Hespanha, D. Liberzon, and A. S. Morse, "Multiple model adaptive control, Part 1: Finite controller coverings," *Int. J. Robust Nonlin. Control*, vol. 10, pp. 909–929, 2000.

[24] J. P. Hespanha and A. S. Morse, "Stabilization of nonholonomic integrators via logic-based switching," *Automatica*, vol. 35, pp. 385–393, 1999.

[25] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Supervision of integral-input-to-state stabilizing controllers," *Automatica*, vol. 38, no. 8, pp. 1327–1335, 2002.

[26] E. D. Sontag, "Input to state stability: Basic concepts and results," in *Nonlinear and Optimal Control Theory*, P. Nistri and G. Stefani, Eds. Berlin, Germany: Springer-Verlag, 2007, pp. 166–220.

[27] J. P. Hespanha, "Uniform stability of switched linear systems: Extensions of LaSalle's invariance principle," *IEEE Trans. Autom. Control*, vol. 49, no. 4, pp. 470–482, Apr. 2004.

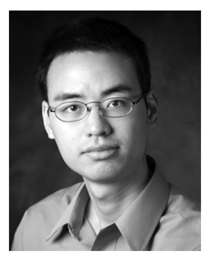
[28] D. Nesic and D. Liberzon, "A small-gain approach to stability analysis of hybrid systems," in *Proc. 44th IEEE Conf. Decision Control*, 2005, pp. 5409–5414.

[29] J. Zhao and D. Hill, "Dissipativity theory for switched systems," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 941–953, May 2008.

[30] M. Zefran, F. Bullo, and M. Stein, "A notion of passivity for hybrid systems," in *Proc. 40th IEEE Conf. Decision Control*, 2001, pp. 771–773.

[31] L. Vu, D. Chatterjee, and D. Liberzon, "Input-to-state stability of switched systems and switching adaptive control," *Automatica*, vol. 43, no. 4, pp. 639–646, 2007.

[32] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, pp. 2735–2744, 2005.



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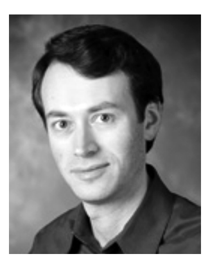

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