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Invertibility of switched linear systems ☆

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Abstract

We address the invertibility problem for switched systems, which is the problem of recovering the switching signal and the input uniquely given an output and an initial state. In the context of hybrid systems, this corresponds to recovering the discrete state and the input from partial measurements of the continuous state. In solving the invertibility problem, we introduce the concept of singular pairs for two systems. We give a necessary and sufficient condition for a switched system to be invertible, which says that the individual subsystems should be invertible and there should be no singular pairs. When the individual subsystems are invertible, we present an algorithm for finding switching signals and inputs that generate a given output in a finite interval when there is a finite number of such switching signals and inputs. Detailed examples are included.
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Keywords: Switched systems; Linear systems; Invertibility

1. Introduction

Switched systems—systems that comprise a family of dynamical subsystems together with a switching signal determining the active system at a current time—arise in many situations. Switching behaviors can come from controller design, such as in switching supervisory control (Morse, 1996) or gain scheduling control. Switching can also be inherent by nature, such as when a physical plant has the capability of undergoing several operational modes (e.g., an aircraft during different thrust modes (Lygeros et al., 1999), a walking robot during leg impact and leg swing modes (Westervelt et al., 2003), different formations of a group of vehicles, (Olfati-Saber and Murray, 2004). Also, switched systems may be viewed as higher-level abstractions of hybrid systems (see, e.g., Henzinger and Sastry, 1998 for hybrid system definitions), obtained by neglecting details of the discrete behavior and instead considering switching signals from a suitable class. Switched systems have attracted growing research interest recently and several important results for switched systems have been achieved, including various stability results (Agrachev and Liberzon, 2001; Branicky, 1998; Hespanha and Morse, 1999; Mancilla-Aguilar and Garcia, 2000), controllability and observability results (Sun et al., 2002; Xie and Wang, 2004; Vidal et al., 2002), and input-to-state properties (Hespanha, 2003; Xie et al., 2001; Vu et al., 2007) (see, e.g., Liberzon, 2003, for other references on switched systems and switching control).

Here we address the *invertibility problem* for switched systems, which concerns with the following question: *What is the condition on the subsystems of a switched system so that, given an initial state x_0 and the corresponding output y generated with some switching signal σ and input u, we can recover the switching signal σ and the input u uniquely*? The aforementioned problem is in the same vein with the classic invertibility problem for non-switched linear systems, where one wishes to recover the input uniquely knowing the initial state and the output. The invertibility problem for non-switched linear systems has been studied extensively, first by Brockett and Mesarovic (1965), then with other algebraic criteria by Silverman (1969) and Sain and Massey (1969); also, a geometric criterion is given

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by Morse and Wonham (1971) (see also the discussions and the references in Moylan, 1977). However, the invertibility problem for switched systems has not been investigated and it is the subject of this paper. As in the classic setting, we start with an output and an initial state, but here, the underlying process constitutes multiple models and we have an extra ingredient to recover apart from the input, namely, the switching signal.

We now briefly discuss how the invertibility problem for switched systems fits into the current literature. On the one hand, non-switched linear systems can be seen as switched systems with constant switching signals. In this regard, the invertibility problem for switched systems is an extension of the non-switched counterpart in the sense that we have to recover the switching signal in addition to the input, based on the output and the initial state. On the other hand, switched systems can be viewed as higher-level abstractions of hybrid systems. Recovering the switching signal for switched systems is equivalent to *mode identification* for hybrid systems. For autonomous hybrid systems, this mode identification task is a part of the observability problem for hybrid systems (Vidal et al., 2002; Vidal et al., 2003). Mode detection for non-autonomous hybrid systems using known inputs and outputs has been studied, for example, in De Santis et al. (2003), Babaali and Pappas (2005) (see also, Huang et al., 2004). Here, the difference is that we do not know the input and we wish to do both *mode detection* and *input recovery* at the same time using the outputs of switched systems (see also, Sundaram and Hadjicostis, 2006, for input recovery of switched systems with known switching signals).

1.1. A motivating example

Before going into detail in the next section, we present a motivating example to illustrate an interesting aspect of the invertibility problem for switched systems that does not have a counterpart in non-switched systems (details are in Example 2 in Section 5). Consider a switched system consisting of the two subsystems

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y = [0 \quad 1]\, x, \end{cases},$$

$$\Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u, \\ y = [1 \quad 2]\, x. \end{cases}$$

Both Γ_1 and Γ_2 are invertible, which means that for every output y in the output space of Γ_1 and every initial state x_0, there exists a unique input u such that the system Γ_1 with the initial state x_0 and the input u generates the output y, and similarly for Γ_2 (procedures for checking invertibility of non-switched linear systems are well-known as discussed in the introduction). One is given

$$y(t) = \begin{cases} 2e^{2t} - 3e^{t} & \text{if } t \in [0, t^*), \\ c_1 e^{t} + c_2 e^{2t} & \text{if } t \in [t^*, T), \end{cases}$$

where $t^* = \ln 3$, $T = \frac{6}{5}$, $c_1 = 15 + 18\ln(\frac{2}{3})$, $c_2 = \frac{-4}{3} - 4\ln(\frac{2}{3})$, and the initial state $x_0 = (-1, 0)^{\mathrm{T}}$, and is asked to find the switching signal σ and the input u that generate y. Because the output y has a discontinuity at time $t = t^*$ and is smooth everywhere else, one may guess that t^* must be a switching time and is the only one (assuming that there is no state jump at switching times and hence, x is always continuous; see Section 2.1). Following this reasoning, since there are two subsystems, there are only two possible switching signals:

$$\sigma_1(t) = \begin{cases} 1, & t \in [0, t^*) \\ 2, & t \in [t^*, T) \end{cases} \quad \text{and} \quad \sigma_2(t) = \begin{cases} 2, & t \in [0, t^*), \\ 1, & t \in [t^*, T). \end{cases}$$

For a fixed switching signal, by invertibility of Γ_1 and Γ_2, given the output and the initial state, the input can be reconstructed uniquely. One can then try both the switching signals σ_1 and σ_2 above to see which one gives an input that generates the output y. As it turns out, none of the switching signals σ_1 and σ_2 would give an input that generates y. Nonetheless, y is generated, *uniquely* in this case, by the following switching signal and input:

$$\sigma(t) = \begin{cases} 2, & t \in [0, t_1), \\ 1, & t \in [t_1, t^*), \\ 2, & t \in [t^*, T), \end{cases} \qquad u(t) = \begin{cases} 0, & t \in [0, t_1), \\ 6e^{2t} - 6e^{t}, & t \in [t_1, t^*), \\ 0, & t \in [t^*, T), \end{cases}$$

where $t_1 = \ln 2$ (details are in Example 2 in Section 5). In Section 4, we will show how a switch at a later time $t = t^*$ can be used to recover a switch at an earlier time $t = t_1$ even if the output is smooth at $t = t_1$.

1.2. Paper organization

In Section 2, we cover background on switched systems and invertibility of non-switched linear systems, including the structure algorithm; in Section 3, we define the invertibility problem for switched systems and provide a necessary and sufficient condition for invertibility, using the concept of singular pairs; in Section 4, we present an algorithm for finding inputs and switching signals that produce a given output with a given initial state; examples are in Section 5; in Section 6, we highlight the main results of the paper and discuss future work.

2. Notations and background

2.1. Notations and definitions

Denote by $C^n_{\mathscr{D}}$ the set of n times continuously differentiable functions on a domain $\mathscr{D}$; when the domain is not relevant, we write C^n. Denote by $\mathscr{F}^{\mathrm{pc}}_{\mathscr{D}}$ the set of *piecewise right-continuous functions* on a domain $\mathscr{D}$; when the domain is not relevant, we write $\mathscr{F}^{\mathrm{pc}}$. Denote by $\mathscr{F}^n$ the subset of $\mathscr{F}^{\mathrm{pc}}$ whose elements are C^n between two consecutive discontinuities. For $u : \mathscr{D} \to \mathbb{R}^n$, denote by $u_{\mathscr{Q}}$ the restriction of u onto $\mathscr{Q} \subseteq \mathscr{D}$.

A *switched linear system* is written as

$$\Gamma_\sigma : \begin{cases} \dot{x} = A_\sigma x + B_\sigma u, \\ y = C_\sigma x + D_\sigma u, \end{cases} \tag{1}$$

where $\sigma : [0, \infty) \to \mathscr{P}$ is a piecewise constant right-continuous *switching signal* that indicates the active subsystem at every time, $\mathscr{P}$ is some index set in a real vector space, and A_p, B_p, C_p, D_p, $p \in \mathscr{P}$, are the matrices of the individual subsystems. The discontinuities of σ are called *switching times*. Denote by σ^p the constant switching signal such that $\sigma^p(t) = p \forall t \geqslant 0$. For simplicity, we assume that the subsystems live in the same state space and there is no state jump at switching times. For any given initial state x_0, a switching signal σ, and a piecewise continuous input u on any time domain, a solution of (1) over the same domain always exists (in Carathéodory sense) and is unique and is denoted by $\Gamma_{x_0,\sigma}(u)$; denote by $\Gamma^{\mathrm{O}}_{x_0,\sigma}(u)$ the corresponding output. For the subsystem Γ_p, $p \in \mathscr{P}$, denote by $\Gamma_{p,x_0}(u)$ the trajectory of Γ_p with the initial state x_0 and the input $u \in \mathscr{F}^{\mathrm{pc}}$, and by $\Gamma^{\mathrm{O}}_{p,x_0}(u)$ the corresponding output.

To simplify presentation, we assume that the output dimensions of all the subsystems are the same, because detecting switchings between two subsystems with different output dimensions is trivial. However, the input dimensions of the subsystems can be different. To avoid further complications without compromising the content of the paper, we neglect to define input sets of switched systems rigourously with the understanding that inputs of the switched system (1) are concatenations of the corresponding inputs of the active subsystems. Characterizing the input set rigorously would require the concept of "hybrid functions" whose segments are functions not necessarily of the same dimensions, and also, the concept of compatible input functions and switching signals. We do not pursue these technicalities here and instead just write $\mathscr{U}$ for some input set of the switched system (1).

For the switched system (1), denote by n the state dimension, ℓ the output dimension, and m_p, $p \in \mathscr{P}$ the input dimensions of the subsystems. Denote by $\mathrm{H}_{x_0} : \mathscr{S} \times \mathscr{U} \to \mathscr{Y}$ the (switching signal $\times$ input)-output map for some input set $\mathscr{U}$, switching signal set $\mathscr{S}$, and the corresponding output set $\mathscr{Y}$. We seek conditions on the subsystem dynamics so that the map H_{x_0} is one-to-one for some sets $\mathscr{S}, \mathscr{U}$, and $\mathscr{Y}$ (precise problem formulation is in Section 3).

For $f_i \in \mathscr{F}^{\mathrm{pc}}_{[t_i,\tau_i)}$, $i = 1, 2$ (τ_i could be ∞), define the *concatenation map* $\oplus : \mathscr{F}^{\mathrm{pc}} \times \mathscr{F}^{\mathrm{pc}} \to \mathscr{F}^{\mathrm{pc}}$ as

$$(f_1 \oplus f_2)(t) := \begin{cases} f_1(t) & \text{if } t \in [t_1, \tau_1), \\ f_2(t_2 + t - \tau_1) & \text{if } t \in [\tau_1, \tau_1 + \tau_2 - t_2). \end{cases}$$

Note that if $\tau_1 = \infty$, $f_1 \oplus f_2 = f_1 \ \forall f_2$. The concatenation of vector functions is defined as pairwise concatenation of the corresponding elements. The concatenation of an element f and a set S is $f \oplus S := \{f \oplus g, g \in S\}$. By convention, $f \oplus \emptyset = \emptyset \ \forall f$. Finally, the concatenation of two sets S and T is $S \oplus T := \{f \oplus g, f \in S, g \in T\}$; by convention, $S \oplus \emptyset = \emptyset$ and $\emptyset \oplus S = \emptyset \ \forall S$.

2.2. Invertibility of non-switched linear systems

Consider a linear system:

$$\Gamma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du. \end{cases} \tag{2}$$

The invertibility problem for linear systems (from here onward, when we say invertibility we mean left invertibility) concerns with finding conditions on (2) so that for a given initial state x_0, the input–output map $\mathrm{H}_{x_0} : \mathscr{U} \to \mathscr{Y}$ is one-to-one (injective), where $\mathscr{U}$ is an input set and $\mathscr{Y}$ is the corresponding output set. There are two major approaches to the problem: one is the *algebraic approach* where conditions are obtained in terms of matrix rank equalities; the other is the *geometric approach* that is based on the invariant properties of subspaces.

An algebraic approach relies on the observation that differentiating y reveals extra information about u via $C\dot{x} = CAx + CBu$. If one keeps differentiating the output, one obtains more information about u from $y, \dot{y}, \ddot{y}, \ldots$. Along this approach, most well-known are the rank condition for invertibility (Sain and Massey, 1969) and the structure algorithm (Silverman, 1969), where the latter differs from the former in that it only differentiates parts of the output and not the entire output.

A geometric approach is different from an algebraic one in that it does not involve output differentiations. Instead, the invertibility property is realized from the fact that (i) H_{x_0} is invertible if and only if the kernel of H_0 is trivial, and (ii) the kernel of H_0 is the same as the set of inputs that yield the set $\mathscr{V}$ of states that are reachable from the origin while keeping the output zero. Left invertibility is equivalent to $\mathscr{V} \cap B \ker D = \{0\}$ and $\operatorname{Ker} B \cap \operatorname{Ker} D = \{0\}$ (see, e.g., Trentelman et al., 2001, Ch. 8.5; see also (Morse, 1971; Morse and Wonham, 1971)).

Remark 1. For non-switched linear systems, the following are equivalent: (i) There is a unique u such that $y = \mathrm{H}_{x_0}(u)$ for one particular pair (x_0, y) where y has a domain $\mathscr{D} \subseteq [0, \infty)$; (ii) There is a unique u such that $y = \mathrm{H}_{x_0}(u)$ for every y in the range of Γ_{x_0} for all possible domains and for all $x_0 \in \mathbb{R}^n$. As a consequence, the output set $\mathscr{Y}$ can be taken as the set of all output functions that are generated by continuous inputs on arbitrary time domain. As we will see later in Section 3.1, the map H_{x_0} for switched systems (as defined in Section 2.1) does not have this property and a careful consideration of the output set $\mathscr{Y}$ is needed.

2.3. The structure algorithm and the range theorem

In this paper, we pursue an algebraic approach (a geometric approach is equally interesting and could be the topic for future research). Particularly, we will employ the *structure algorithm* for non-switched linear systems (Silverman, 1969). This subsection covers the structure algorithm, closely following Silverman (1969), Silverman and Payne (1971). The reader is referred to Silverman and Payne (1971) for further technical details and proofs.

Consider the linear system (2). Let n be the state dimension, m the input dimension, and ℓ the output dimension. For the moment, assume that the input u is continuous (for piecewise continuous inputs, see Remark 2 below).

Let $q_0 := \operatorname{Rank} D$. $\exists$ a nonsingular $S_0 \in \mathbb{R}^{\ell \times \ell}$ such that $D_0 := S_0 D = [\overline{D}_0^{\mathrm{T}} \ 0]^{\mathrm{T}}$, where $\overline{D}_0$ has q_0 rows and rank q_0. We have $y_0 := S_0 y := C_0 x + D_0 u$ where $C_0 := S_0 C$. Suppose that at step k, we have $y_k = C_k x + D_k u$, where D_k has the form $[\overline{D}_k^{\mathrm{T}} \ 0]^{\mathrm{T}}$; $\overline{D}_k$ has q_k rows and is full rank. Define $\overline{I}_k :=$

$[I_{q_k}\ 0_{q_k\times\ell-q_k}]$, where I_{q_k} is the identity matrix of dimension q_k, and $\underline{I}_k := [0_{\ell-q_k\times q_k}\ I_{\ell-q_k}]$. Let $\overline{C}_k := \overline{I}_k C_k$, $\widetilde{C}_k := \underline{I}_k C_k$, $\bar{y}_k := \overline{I}_k y_k$, and $\tilde{y}_k := \underline{I}_k y_k$. Define the differential operator $M_k := \begin{bmatrix} \overline{I}_k \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{I}_k \end{bmatrix}$. Then $M_k y_k = \begin{bmatrix} \overline{C}_k \\ \widetilde{C}_k A \end{bmatrix} x + \begin{bmatrix} \overline{D}_k \\ \widetilde{C}_k B \end{bmatrix} u =: \widehat{C}_k x + \widehat{D}_k u$. Let $q_{k+1} := \operatorname{Rank}\widehat{D}_k$. $\exists$ a nonsingular $S_{k+1} \in \mathbb{R}^{\ell\times\ell}$ such that $D_{k+1} := S_{k+1}\widehat{D}_k = [\overline{D}_{k+1}^{\mathrm{T}}\ 0]^{\mathrm{T}}$, where $\overline{D}_{k+1}$ has q_{k+1} rows and rank q_{k+1}. Let $y_{k+1} := S_{k+1} M_k y_k$ and $C_{k+1} := S_{k+1}\widehat{C}_k$. Then $y_{k+1} = C_{k+1}x + D_{k+1}u$ and we can repeat the procedure. Let $N_k := \prod_{i=0}^{k} S_{k-i} M_{k-i-1}$, $k = 0, 1, \ldots$ $(M_{-1} := I)$, $\overline{N}_k := \overline{I}_k N_k$, and $\widetilde{N}_k := \underline{I}_k N_k$. Then $y_k = N_k y$, $\bar{y}_k = \overline{N}_k y$, and $\tilde{y}_k = \widetilde{N}_k y$. Notice that since D_k has ℓ rows and m columns, $q_k \leqslant \min\{\ell, m\}$ for all k and since $q_{k+1} \geqslant q_k$, it was shown in Silverman and Payne (1971) that $\exists$ a smallest integer $\alpha \leqslant n$ such that $q_k = q_\alpha$ $\forall k \geqslant \alpha$.

If $q_\alpha = m$, the system is (left) invertible and an inverse is

$$\Gamma^{-1}: \begin{cases} \bar{y}_\alpha = \overline{N}_\alpha y, \\ \dot{z} = (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)z + B\overline{D}_\alpha^{-1}\bar{y}_\alpha, \\ u = -\overline{D}_\alpha^{-1}\overline{C}_\alpha z + \overline{D}_\alpha^{-1} y_\alpha \end{cases} \tag{3}$$

with the initial state $z(0) = x_0$. If $q_\alpha < m$, the system is not invertible, and then a generalized inverse is

$$\Gamma^{-1}: \begin{cases} \bar{y}_\alpha = \overline{N}_\alpha y, \\ \dot{z} = (A - B\overline{D}_\alpha^{\dagger}\overline{C}_\alpha)z + B\overline{D}_\alpha^{\dagger}\bar{y}_\alpha + BKv, \\ u = -\overline{D}_\alpha^{\dagger}\overline{C}_\alpha z + \overline{D}_\alpha^{\dagger}\bar{y}_\alpha + Kv \end{cases} \tag{4}$$

with $z(0) = x_0$, where $\overline{D}_\alpha^{\dagger} := \overline{D}_\alpha^{\mathrm{T}}(\overline{D}_\alpha\overline{D}_\alpha^{\mathrm{T}})^{-1}$ is a right-inverse of $\overline{D}_\alpha$ and $K :=$ Null $\overline{D}_\alpha$. The system Γ^{-1} in (4) is called a generalized inverse because $y = \Gamma_{x_0}^{\mathrm{O}}(u)$ if and only if $u = \Gamma_{x_0}^{-1,\mathrm{O}}(y_\alpha, v)$ for some v.

Let $L_k := [\widetilde{C}_0^{\mathrm{T}}\ \ldots\ \widetilde{C}_k^{\mathrm{T}}]^{\mathrm{T}}$. Silverman and Payne (1971) had shown that $\exists$ a smallest number β, $\alpha \leqslant \beta \leqslant n$, such that $\operatorname{Rank}L_k = \operatorname{Rank}L_\beta$ $\forall k \geqslant \beta$. Also, $\exists$ a number δ, $\beta \leqslant \delta \leqslant n$ such that $\widetilde{C}_\delta = \sum_{i=0}^{\delta-1} P_i(\prod_{j=i+1}^{\delta}\widetilde{R}_j)\widetilde{C}_i$ for some matrices $\widetilde{R}_j$ and P_i (see, Silverman and Payne, 1971, p. 205, for details). The number δ is not easily determined as α and β. The significance of β and δ is that they can be used to characterize the set of all outputs of a linear system as in the range theorem. We include the range theorem (Silverman and Payne, 1971, Theorem 4.3) below. Definethe differential operators $\mathbf{M}_1 := \left(\frac{\mathrm{d}^\delta}{\mathrm{d}t^\delta} - \sum_{i=0}^{\delta-1} P_i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\right)\prod_{j=0}^{\alpha}\widetilde{R}_j$ and $\mathbf{M}_2 := \sum_{j=0}^{\delta}\left(\prod_{\ell=j+1}^{\alpha}\widetilde{R}_\ell\right) K_j \frac{\mathrm{d}^{\delta-1}}{\mathrm{d}t^{\delta-j}} - \sum_{j=0}^{\delta-1} P_j \times \sum_{k=0}^{j}\left(\prod_{\ell=k+1}^{\alpha}\widetilde{R}_\ell\right) K_k \frac{\mathrm{d}^{j-k}}{\mathrm{d}t^{j-k}}$ for some matrices K_i from the structure algorithm (see, Silverman and Payne, 1971, for details).

Theorem 1 (*Silverman and Payne, 1971*). *A function $f : [t_0, T) \to \mathbb{R}^\ell$ is in the range of Γ_{x_0} if and only if*

(i) *f is such that $N_\delta f$ is defined and continuous*;
(ii) $\widetilde{N}_k f|_{t_0} = \widetilde{C}_k x_0$, $k = 0, \ldots, \beta - 1$;
(iii) $(\mathbf{M}_1 - \mathbf{M}_2\overline{N}_\alpha) f \equiv 0$.

To simplify the presentation, we paraphrase the range theorem into Lemma 1 below that is easier to understand at the expense of having to define extra notations. Define

- $\mathbf{N} := [\widetilde{N}_0^{\mathrm{T}} \ldots \widetilde{N}_{\beta-1}^{\mathrm{T}}]^{\mathrm{T}}$, $L := L_{\beta-1}$,
- $\widehat{\mathscr{Y}}$ be the set of functions $f : \mathscr{D} \to \mathbb{R}^\ell$ for all $\mathscr{D} \subseteq [0, \infty)$ which satisfy (i) and (iii) of Theorem 1.

Lemma 1. *For a linear system Γ, using the structure algorithm on the system matrices, construct the set $\widehat{\mathscr{Y}}$, the differential operator $\mathbf{N} : \widehat{\mathscr{Y}} \to \mathscr{C}^0$, and the matrix L. There exists $u \in C^0$ such that $y = \Gamma_{x_0}(u)$ if and only if $y \in \widehat{\mathscr{Y}}$ and $\mathbf{N}y|_{t_0^+} = Lx_0$ where t_0 is the initial time of y.*

Roughly speaking, the set $\widehat{\mathscr{Y}}$ characterizes functions that can be generated by the system from all initial positions (in some sense, the condition (iii) capture the relationship among the output components regardless of the input). The condition $\mathbf{N}y|_{t_0^+} = Lx_0$ guarantees that a particular y can be generated starting from the particular initial state x_0 at time t_0. We evaluate $\mathbf{N}y$ at t_0^+, which means $\lim_{t\to t_0^+}\mathbf{N}y|_t$, to reflect that y does not need to be defined for $t < t_0$. This is especially useful later when we consider switched systems where inputs and outputs can be piecewise right-continuous.

Remark 2. If we allow piecewise continuous inputs, the invertibility condition for Γ is still the same ($q_\alpha = m$) but v in (4) can be piecewise continuous functions. The inverse, the generalized inverse, and Theorem 1 are applicable in between every two consecutive discontinuities of the input (on every output segment $y_{[t,\tau)}$ in which $N_\delta y_{[t,\tau)}$ exists and is continuous).

Remark 3. When $q_\alpha = \ell$, $\mathbf{M}_1 = \mathbf{M}_2 = 0$ and the condition (iii) becomes trivial. Also, in this case, $\alpha = \beta = \delta$. The set $\widehat{\mathscr{Y}}$ is simplified to the set of functions f for which $N_\delta f$ is defined and continuous. In particular, any C^n function will be in $\widehat{\mathscr{Y}}$. For an invertible system, $q_\alpha = \ell$ if the input and output dimensions are the same.

The differential operator $\mathbf{N}$ is used in Lemma 1 to deal with a general output y that may not be differentiable but $\mathbf{N}y$ exists (an example is that $\dot{y}$ is not differentiable so $\ddot{y}$ does not exist but $\mathbf{N}y = \frac{\mathrm{d}}{\mathrm{d}t}(y - \dot{y})$ exists and is continuous). Let's take a closer look at $\mathbf{N}$. We have

$$M_0 y_0 = \begin{pmatrix} \bar{y}_0 \\ \dot{\tilde{y}}_0 \end{pmatrix} = \begin{pmatrix} \overline{S}_0 \\ 0 \end{pmatrix} y + \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 0 \\ \widetilde{S}_0 \end{pmatrix} y =: K_{0,0}y + \frac{\mathrm{d}}{\mathrm{d}t}K_{0,1}y. \tag{5}$$

Let $M_i y_i := K_{i,0}y + \frac{\mathrm{d}}{\mathrm{d}t}(K_{i,1}y + \cdots + \frac{\mathrm{d}}{\mathrm{d}t}K_{i,i+1}y)$. Then in view of $M_{i+1}y_{i+1} = \begin{bmatrix} \overline{S}_{i+1} \\ 0 \end{bmatrix} M_i y_i + \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} 0 \\ \widetilde{S}_{i+1} \end{bmatrix} M_i y_i$, we have $K_{i,j} \in \mathbb{R}^{\ell\times\ell}$ defined recursively as

$$K_{i+1,j} = \begin{bmatrix} \overline{S}_{i+1} \\ 0 \end{bmatrix} K_{i,j} + \begin{bmatrix} 0 \\ \widetilde{S}_{i+1} \end{bmatrix} K_{i,j-1} \tag{6}$$

for $0 \leqslant j \leqslant i+2$, $i \geqslant 0$, where $K_{i,-1} = 0$ $\forall i$ by convention and $K_{0,0}$, $K_{0,1}$ are as in (5). In view of the definition of $K_{i,j}$ and the fact $\widetilde{N}_k y = \widetilde{S}_k M_{i-1} y_{i-1}$, we have

$$\mathbf{N}y =: N_0 y + \frac{\mathrm{d}}{\mathrm{d}t}(N_1 y + \cdots + \frac{\mathrm{d}}{\mathrm{d}t} N_{\beta-1} y), \tag{7}$$

where $N_i := \begin{bmatrix} \widetilde{S}_0 K_{-1,i} \\ \vdots \\ \widetilde{S}_{\beta-1} K_{\beta-2,i} \end{bmatrix}$, $0 \leqslant i \leqslant \beta-1$, $K_{-1,0} = I$ and $K_{j,k} = 0\ \forall k \geqslant j+2, \forall j$ by convention. We then have the following lemma.

Lemma 2. *Consider the linear system* (2). *Let β be the number and $\mathbf{N}$ be the differential operator as described in the structure algorithm. For any $\kappa \geqslant \beta$, for every y such that $\mathbf{N}y$ exists and is continuous, we have $\mathbf{N}y = N_0 y + \frac{\mathrm{d}}{\mathrm{d}t}(N_1 + \cdots + \frac{\mathrm{d}}{\mathrm{d}t}(N_{\kappa-2} y + \frac{\mathrm{d}}{\mathrm{d}t} N_{\kappa-1} y))$, where $N_i, 0 \leqslant i \leqslant \beta-1$, are as in* (7) *and $N_i = 0, i \geqslant \beta$.*

In general, $\mathbf{N}y$ is calculated by a chain of differentiations and additions as in the lemma. However, whenever $y \in C^{\beta-1}$, calculating $\mathbf{N}y$ can be simplified to the matrix multiplication $[N_0 \ \ldots \ N_{\beta-1}]$ by $(y, \ldots, y^{(\beta-1)})^{\mathrm{T}}$.

3. Invertibility of switched systems

The Invertibility Problem. *Consider a* (*switching signal $\times$ input*)*-output map $\mathrm{H}_{x_0} : \mathscr{S} \times \mathscr{U} \to \mathscr{Y}$ for the switched system* (1). *Find a set $\mathscr{Y}$ and a condition on the subsystems, independent of x_0, such that the map H_{x_0} is one-to-one.*

3.1. Invertibility and singular pairs

We say that H_{x_0} is *invertible at y* if $\mathrm{H}_{x_0}(\sigma_1, u_1) = \mathrm{H}_{x_0}(\sigma_2, u_2) = y \Rightarrow \sigma_1 = \sigma_2, u_1 = u_2$ (similarly for non-switched systems, $\mathrm{H}_{x_0} : \mathscr{U} \to \mathscr{Y}$ is invertible at y if $\mathrm{H}_{x_0}(u_1) = \mathrm{H}_{x_0}(u_2) = y \Rightarrow u_1 = u_2$). We say that H_{x_0} is *invertible on $\mathscr{Y}$* if it is invertible at y, $\forall y \in \mathscr{Y}$. We say that the switched system is *invertible on $\mathscr{Y}$* if H_{x_0} is invertible on $\mathscr{Y}$ for all x_0.

There is a major difference between the maps H_{x_0} for non-switched systems and for switched systems. The former is a linear map on vector spaces (i.e. the input functions). The latter is a nonlinear map on the domain $\mathscr{S} \times \mathscr{U}$, of which $\mathscr{S}$ is not a vector space. For switched systems, uniqueness of (σ, u) for one pair (x_0, y) does not imply uniqueness for other pairs, and thus, a switched system may be invertible on one output set $\mathscr{Y}_1$ but not invertible on another set $\mathscr{Y}_2$ (which is not the case for non-switched systems; see Remark 1). This situation prompts a more delicate definition of the output set $\mathscr{Y}$ for switched systems, instead of letting $\mathscr{Y}$ be the set generated by all possible combinations of piecewise continuous inputs and switching signals. For the invertibility problem, we look for a suitable set $\mathscr{Y}$ and a condition on the subsystems so that unique recovery of (σ, u) is guaranteed for all $y \in \mathscr{Y}$ and all $x_0 \in \mathbb{R}^n$.

It is obvious that $\mathrm{H}_0(\sigma, 0) = 0\ \forall \sigma$ regardless of the subsystem dynamics and therefore, the map H_0 is not one-to-one if the function $0 \in \mathscr{Y}$. Note that the available information is the same for both non-switched systems and switched systems, namely, the pair (x_0, y), but the domain in the switched system case has been enlarged to $\mathscr{S} \times \mathscr{U}$, compared to $\mathscr{U}$ in the non-switched system case. For non-switched systems, under a certain condition on the system dynamics (i.e. when the system is invertible), the information $(x_0, y) = (0, 0)$ is sufficient to determine u uniquely while for switched systems, that information is insufficient to determine (σ, u) uniquely, regardless of what the subsystems are. This illustrates the issue of why we cannot take the output set $\mathscr{Y}$ to be all the possible outputs. We call those pairs (x_0, y) for which H_{x_0} is not invertible at y *singular pairs*. Fortunately, $x_0 = 0$ and $y_{[0,\varepsilon)} \equiv 0$ for some $\varepsilon > 0$ are the only type of singular pairs that are independent of the subsystems and for other pairs (x_0, y), the invertibility of H_{x_0} at y depends on the subsystem dynamics and properties of y.

Definition 1. Let $x_0 \in \mathbb{R}^n$ and $y \in C^0$ be a function in $\mathbb{R}^\ell$ on some time interval. The pair (x_0, y) is a *singular pair* of the two subsystems Γ_p, Γ_q if there exist u_1, u_2 such that $\Gamma^{\mathrm{O}}_{p,x_0}(u_1) = \Gamma^{\mathrm{O}}_{q,x_0}(u_2) = y$.

We proceed to develop a formula for checking if (x_0, y) is a singular pair of Γ_p, Γ_q, utilizing the range theorem (Theorem 1 in this paper). For the subsystem indexed by p, denote by $\mathbf{N}_p$, L_p, and $\widehat{\mathscr{Y}}_p$ the corresponding objects of interest as in Lemma 1. It follows from Definition 1 and Lemma 1 that (x_0, y) is a singular pair if and only if $y \in \widehat{\mathscr{Y}}_p \cap \widehat{\mathscr{Y}}_q$ and

$$\mathbf{N}_{pq} y|_{t_0^+} := \begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y|_{t_0^+} = \begin{bmatrix} L_p \\ L_q \end{bmatrix} x_0 =: L_{pq} x_0, \tag{8}$$

where t_0 is the initial time of y. For a given (x_0, y), the condition (8) can be directly verified since all $\widehat{\mathscr{Y}}_p, \widehat{\mathscr{Y}}_q, \mathbf{N}_p, \mathbf{N}_q, L_p, L_q$ are known. Observe that $0 \in \operatorname{Im} L_{pq}$ and we can always have (8) with $x_0 = 0$ and y such that $\mathbf{N}_{pq} y|_{t_0^+} = 0$. If $y_{[t_0, t_0+\varepsilon)} \equiv 0$ and $x_0 = 0$, then (8) holds regardless of $\mathbf{N}_p, \mathbf{N}_q, L_p, L_q$. Apart from this case, in general, $\mathbf{N}_{pq} y|_{t_0^+} = 0$ depends on $\mathbf{N}_p, \mathbf{N}_q$, and y and it is possible to find conditions on $\mathbf{N}_p, \mathbf{N}_q, L_p, L_q$ and y so that there is no x_0 satisfying (8) if $\mathbf{N}_{pq} y|_{t_0^+} \neq 0$.

Remark 4. The singular pair notion relates to the scenario where there is a switch in the underlying dynamical system yet the output is still smooth at the switching time. A similar scenario but with a different objective can be found in the context of bumpless switching (Arehart and Wolovich, 1996), in which the objective is to design the subsystems so that the output of the switched system is continuous. In the invertibility problem considered here, the subsystems are fixed and the objective is to recover the switching signal and the input.

3.2. A solution of the invertibility problem

We now return to the invertibility problem. Let $\mathscr{Y}^{\mathrm{all}}$ be the set of outputs of the system (1) generated by all possible piecewise continuous inputs and switching signals from all possible initial states for all possible durations (the set $\mathscr{Y}^{\mathrm{all}}$ can be seen as all the possible concatenations of all elements of $\widehat{\mathscr{Y}}_p, \forall p \in \mathscr{P}$). Let $\overline{\mathscr{Y}} \subset \mathscr{Y}^{\mathrm{all}}$ be the largest subset of $\mathscr{Y}^{\mathrm{all}}$ such that if $y \in \overline{\mathscr{Y}}$

and $y_{[t_0,t_0+\varepsilon)} \in \widehat{\mathscr{Y}}_p \cap \widehat{\mathscr{Y}}_q$ for some $p \neq q$, $p, q \in \mathscr{P}$, $\varepsilon > 0$, $t_0 \geqslant 0$, then $\mathbf{N}_{pq} y|_{t_0^+} \neq 0$. In other words, we avoid functions whose segments can form singular pairs with $x_0 = 0$ (so that even if $x_0 = 0$, there is no $y \in \overline{\mathscr{Y}}$ that can form a singular pair with x_0). Excluding such functions from our output set, we can impose conditions on the subsystems to eliminate the possibility of singular pairs for all $x_0 \in \mathbb{R}^n$ and all $y \in \overline{\mathscr{Y}}$. Note that the singular pair concept in Definition 1 is defined for continuous functions. Applying to switched systems, we check for singular pairs for the continuous output segments in between consecutive discontinuities at the output (note that $\widehat{\mathscr{Y}}_p \cap \widehat{\mathscr{Y}}_q \subseteq C^0$ if the intersection is non-empty).

Theorem 2. *Consider the switched system* (1) *and the output set* $\overline{\mathscr{Y}}$. *The switched system is invertible on* $\overline{\mathscr{Y}}$ *if and only if all the subsystems are invertible and the subsystem dynamics are such that for all* $x_0 \in \mathbb{R}^n$ *and* $y \in \overline{\mathscr{Y}} \cap C^0$, the pairs (x_0, y) *are not singular pairs of* Γ_p, Γ_q *for all* $p \neq q$, $p, q \in \mathscr{P}$.

Proof (*Sufficiency*). Suppose that $\mathrm{H}_{x_0}(\sigma_1, u_1) = \mathrm{H}_{x_0}(\sigma_2, u_2) = y \in \overline{\mathscr{Y}}$. Let $t_1 := \min\{t > 0 : u_1$ or u_2 or σ_1 or σ_2 is discontinuous at $t\}$. Let $p = \sigma_1(0)$ and $q = \sigma_2(0)$. From Lemma 1, we have $\mathbf{N}_p y|_{0+} = L_p x_0$ and $\mathbf{N}_q y|_{0+} = L_q x_0$, and $y_{[0,t_1)} \in \widehat{\mathscr{Y}}_p \cap \widehat{\mathscr{Y}}_q$. By definition, $(x_0, y_{[0,t_1)})$ is a singular pair of Γ_p, Γ_q if $p \neq q$. Also by definition, $y_{[0,t_1)} \in \overline{\mathscr{Y}} \cap C^0$. Since there is no singular pair for Γ_p, Γ_q, we must have $p = q$, i.e. $\sigma_1(t) = \sigma_2(t) = p$ $\forall t \in [0, t_1)$. Since Γ_p is invertible, $u_{1[0,t_1)} = u_{2[0,t_1)} = u_{[0,t_1)} = \Gamma^{-1,\mathrm{O}}_{p,x_0}(y_{[0,t_1)})$ is uniquely recovered on $[0, t_1)$ (recall from Section 2.1 that $\Gamma^{-1,\mathrm{O}}_{p,x_0}(y_{[0,t_1)})$ is the output of the inverse of Γ_p starting at x_0 with input $y_{[0,t_1)}$). Let $x_1 = x(t_1^-)$. By continuity of the trajectory, we have $x(t_1) = x(t_1^-) = x_1$. If $t_1 = \infty$, we then have $\sigma_1(t) = \sigma_2(t)$ and $u_1(t) = u_2(t)$ $\forall t \in [0, \infty)$.

Suppose that $t_1 < \infty$. We have $\mathrm{H}_{x_1}(\sigma_{1[t_1,\infty)}, u_{1[t_1,\infty)}) = \mathrm{H}_{x_1}(\sigma_{2[t_1,\infty)}, u_{2[t_1,\infty)}) = y_{[t_1,\infty)}$. Let $t_2 := \min\{t > t_1 :$ u_1 or u_2 or σ_1 or σ_2 is discontinuous at $t\}$. Repeating the arguments in the previous paragraph, we must have $\sigma_1(t) = \sigma_2(t) = q$ $\forall t \in [t_1, t_2)$ for some $q \in \mathscr{P}$, and $u_{1[t_1,t_2)} = u_{2[t_1,t_2)} = u_{[t_1,t_2)} = \Gamma^{-1,\mathrm{O}}_{q,x_0}(y_{[t_1,t_2)})$ is uniquely recovered on $[t_1, t_2)$.

If $t_2 = \infty$, we then have $\sigma_1(t) = \sigma_2(t)$ and $u_1(t) = u_2(t)$ for all $t \in [0, \infty)$; otherwise, repeat the procedure with $y_{[t_2,\infty)}$. Since there cannot be infinitely many discontinuities in a finite interval (in other words, if there are infinitely many discontinuities, the interval must be $[0, \infty)$), we conclude that $\sigma_1(t) = \sigma_2(t)$ and $u_1(t) = u_2(t)$ $\forall t \in [0, \infty)$.

Necessity: Suppose that Γ_p is not invertible for some $p \in \mathscr{P}$. Pick some x_0 and $y \in \overline{\mathscr{Y}} \cap \widehat{\mathscr{Y}}_p$; this is always possible from the definition of the set $\overline{\mathscr{Y}}$. Since $y \in \widehat{\mathscr{Y}}_p$, there exists u such that $y = \Gamma_{p,x_0}(u)$. Since Γ_p is not invertible, there exists $\tilde{u} \neq u$ such that $\Gamma_{p,x_0}(\tilde{u}) = y$ (see Remark 1). Then $\mathrm{H}_{x_0}(\sigma^p, u) = \mathrm{H}_{x_0}(\sigma^p, \tilde{u}) = y$. That means the map H_{x_0} is not invertible at y and thus, the switched system is not invertible on $\overline{\mathscr{Y}}$, a contradiction. Therefore, Γ_p must be invertible for all $p \in \mathscr{P}$.

Suppose that there are some $x_0 \in \mathbb{R}^n$, $y \in \overline{\mathscr{Y}} \cap C^0$ and $p \neq q, p, q \in \mathscr{P}$ such that (x_0, y) is a singular pair of Γ_p, Γ_q. This means $\exists u_1, u_2 \in C^0$ (not necessarily different) such that $\Gamma_{p,x_0}(u_1) = \Gamma_{q,x_0}(u_2) = y$ and therefore, $\mathrm{H}_{x_0}(\sigma^p, u_1) = \mathrm{H}_{x_0}(\sigma^q, u_2) = y$. Since $\sigma^p \neq \sigma^q$, the foregoing equality implies that H_{x_0} is not invertible at y, and thus, the switched system is not invertible on $\overline{\mathscr{Y}}$, a contradiction. Therefore, for all $x_0 \in \mathbb{R}^n$ and $y \in \overline{\mathscr{Y}} \cap C^0$, (x_0, y) are not singular pairs of Γ_p, Γ_q for all $p \neq q$, $p, q \in \mathscr{P}$. □

While checking for singularity for given x_0 and y is feasible using (8), in general, checking for singularity for all x_0 and all $y \in \overline{\mathscr{Y}}$ (and hence, checking invertibility) is not an easy task. We now develop a rank condition for checking invertibility of switched systems, which is more computationally friendly. In the case when the subsystem input and output dimensions are equal, the rank condition is also necessary. We first have the following lemma. For an index p, let $W_p := [N_{p,0}\ \ N_{p,1}\ \ \ldots\ \ N_{p,n-1}]$ where $N_{p,0}, \ldots, N_{p,n-1}$ are the matrices as in Lemma 2 for the subsystem with index p. Define $W_{pq} := [W_p^{\mathrm{T}}\ \ W_q^{\mathrm{T}}]^{\mathrm{T}}$.

Lemma 3. *Consider the switched system* (1) *and the output set* $\overline{\mathscr{Y}}$. *Consider the following two statements*:

(**S1**) *The subsystem dynamics are such that for all* $x_0 \in \mathbb{R}^n$ *and* $y \in \overline{\mathscr{Y}} \cap C^0$, *the pairs* (x_0, y) *are not singular pairs of* Γ_p, Γ_q *for all* $p \neq q$, $p, q \in \mathscr{P}$;

(**S2**) *The subsystem dynamics are such that*

$$Rank[W_{pq}\ \ L_{pq}] = Rank W_{pq} + Rank L_{pq} \tag{9}$$

for all $p \neq q$, $p, q \in \mathscr{P}$ *such that* $\widehat{\mathscr{Y}}_p \cap \widehat{\mathscr{Y}}_q \neq \{0\}$.

Then S2 always implies S1. If the subsystems are invertible and the input and output dimensions are the same ($m_p = \ell$ $\forall p \in \mathscr{P}$), *then S1 also implies S2.*

Proof. (**S2**) $\Rightarrow$ (**S1**): Suppose that $\exists\, x_0 \in \mathbb{R}^n$ and $y \in \overline{\mathscr{Y}} \cap C^0$ such that (x_0, y) is a singular pair of Γ_p, Γ_q for some $p \neq q$. Let $N_{p,q,k} := [N_{p,k}^{\mathrm{T}}\ \ N_{q,k}^{\mathrm{T}}]^{\mathrm{T}}$. From (8) and Lemma 2, we have $\mathbf{N}_{pq} y = N_{p,q,0} y + \frac{\mathrm{d}}{\mathrm{d}t}(N_{p,q,1} y + \cdots + \frac{\mathrm{d}}{\mathrm{d}t} N_{p,q,n-1} y)$ in view of the definition of $N_{p,i}, N_{q,i}, 0 \leqslant i \leqslant n-1$. Since $N_{p,q,n-1} y \in \mathrm{Range} N_{p,q,n-1}$, we also have $\frac{\mathrm{d}}{\mathrm{d}t} N_{p,q,n-1} y \in \mathrm{Range} N_{p,q,n-1}$. It follows that $\mathbf{N}_{pq} y \in \mathrm{Range} W_{pq}$. The rank condition (9) implies that $\mathrm{Range} W_{pq} \cap \mathrm{Range} L_{pq} = \{0\}$. Therefore, from (8), we must have $\mathbf{N}_{pq} y|_{t_0^+} = 0$ if $p \neq q$. But this equality contradicts the fact that $y \in \overline{\mathscr{Y}}$, and hence, there are no x_0, y that can form a singular pair for some Γ_p, Γ_q, $p \neq q$.

(**S1**) + **invertible subsystems** + ($m_p = \ell$ $\forall p \in \mathscr{P}$) $\Rightarrow$ (**S2**): Suppose that the rank condition is violated for some $p \neq q$, $p, q \in \mathscr{P}$. That implies that $\mathrm{Range} W_{pq} \cap \mathrm{Range} L_{pq} \neq \{0\}$ and hence, there exist ξ and x_0 such that $W_{pq}\xi = L_{pq} x_0 \neq 0$. If the subsystems are invertible and the input and output dimensions are the same, then C^n functions are always in $\widehat{\mathscr{Y}}_p$, $\widehat{\mathscr{Y}}_q$ (see Remark 3). There always exists a C^n function y on an interval $[0, \varepsilon)$ for some $\varepsilon > 0$ such that $(y, \ldots, y^{(n-1)})^{\mathrm{T}}|_0 = \xi$ and $(y, \ldots, y^{(n-1)})^{\mathrm{T}}|_t \notin \mathrm{Ker}\ W_{pq} \forall t \in [0, \varepsilon)$. Since $y \in C^n$, $\mathbf{N}_{pq} y|_{0+} = W_{pq}(y, \ldots, y^{(n-1)})^{\mathrm{T}}|_{0+} = W_{pq}\xi$ and $\mathbf{N}_{pq} y|_t = W_{pq}(y, \ldots, y^{(n-1)})^{\mathrm{T}}|_t \neq 0$ $\forall t \in [0, \varepsilon)$, i.e. $y \in \widehat{\mathscr{Y}}_p \cap \widehat{\mathscr{Y}}_q \cap \overline{\mathscr{Y}}$.

It follows that x_0, y forms a singular pair for Γ_p, Γ_q, a contradiction. □

From Theorem 2 and Lemma 3, we have the following result.

Theorem 3. *Consider the switched system* (1) *and the output set* $\overline{\mathscr{Y}}$. *The switched system is invertible on* $\overline{\mathscr{Y}}$ *if all the subsystems are invertible and the rank condition* (9) *holds. If the input and output dimensions of all the subsystems are the same, then invertibility of all the subsystems together with the rank condition* (9) *is also necessary for invertibility of the switched system.*

Remark 5. If a subsystem has more inputs than outputs, then it cannot be (left) invertible. On the other hand, if it has more outputs than inputs, then some outputs are redundant (as far as the task of recovering the input is concerned). Thus, the case of input and output dimensions being equal is, perhaps, the most interesting case.

When a switched system is invertible *i.e.* it satisfies the conditions of Theorem 2, a *switched inverse system* can be constructed as follows. Define the *index inversion function* $\overline{\Sigma}^{-1}$: $\mathbb{R}^n \times \overline{\mathscr{Y}} \to \mathscr{P}$ as

$$\overline{\Sigma}^{-1} : (x_0, y) \mapsto p : y \in \widehat{\mathscr{Y}}_p \quad \text{and} \quad \mathbf{N}_p y|_{t_0^+} = L_p x_0, \tag{10}$$

where t_0 is the initial time of y. The function $\overline{\Sigma}^{-1}$ is well-defined since there is no singular pair, and so, the index p at every time t_0 is uniquely determined from the output y and state x. In the invertibility problem, it is assumed that $y \in \overline{\mathscr{Y}}$ is an output so the existence of p in (10) is guaranteed. Then a switched inverse system is

$$\Gamma_\sigma^{-1} : \begin{cases} \sigma(t) = \overline{\Sigma}^{-1}(z(t), y_{[t,t+\varepsilon)}), \\ \dot{z} = (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)} z + (B\overline{D}_\alpha^{-1} N_\alpha)_{\sigma(t)} y, \\ u = -(\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)} z + (\overline{D}_\alpha^{-1} N_\alpha)_{\sigma(t)} y \end{cases} \tag{11}$$

with $z(0) = x_0$ where $\varepsilon > 0$ is sufficiently small. The notation $(\cdot)_{\sigma(t)}$ denotes the object in the parentheses calculated for the subsystem with index $\sigma(t)$. The initial condition $z(0) = x_0$ helps determine the initial active subsystem, $\sigma(0) = \overline{\Sigma}^{-1}(x_0, y_{[0,\varepsilon)})$ at $t = 0$, from which time onwards, the switching signal and the input as well as the state are determined uniquely and simultaneously via (11). In (11), we use $\sigma(t)$ in the right-hand side of $\dot{z}$ and $z(t)$ in the formula of $\sigma(t)$ for notational convenience. Indeed, if t is a switching time, $\overline{\Sigma}^{-1}$ helps recover σ at the time t using small enough ε such that $t + \varepsilon$ is less than the next switching time. If t is not a switching time, σ is constant between t and the next switching time and is equal to σ at the last switching time.

Remark 6. In (11), since we use a full order inverse for each subsystem, the state z is exactly the same as the state x of the switched system, and hence, we can use z in the index inversion function $\overline{\Sigma}^{-1}$. If we use a reduced-order inverse for each subsystem (see, e.g., Silverman, 1969), we still get u but then need to plug this u into the switched system to get x to use in $\overline{\Sigma}^{-1}$.

Remark 7. Let $\bar{\beta} := \max_{p \in \mathscr{P}}\{\beta_p\}$, where β_p, $p \in \mathscr{P}$, are the β as in Theorem 1 for the subsystems. In Lemma 3, instead of W_p, we can work with $\overline{W}_p$ where $\overline{W}_p y^{\bar{\beta}} := [N_{p,0} \ldots N_{p,\bar{\beta}-1}](y, \ldots, y^{(\bar{\beta}-1)})^{\mathrm{T}}$. In general, $\overline{W}_p$ have fewer columns than W_p, which make checking the rank condition for systems with large dimensions simpler.

Remark 8. Our results can be extended to include the case where output dimensions are different. If subsystem output dimensions are different, one needs to use "hybrid functions" to describe the output set $\overline{\mathscr{Y}}$ of the switched system (which are now not functions but concatenations of functions of different dimensions). Other than that, Definition 1 is unchanged (since it implies that the output dimensions of the two systems must be the same in order for (x_0, y) to be a singular pair) and the statements of Theorems 2 and 3 remain the same.

Remark 9. The results in this section can also be extended to include the case when there are state jumps at switching times. Denote by $f_{p,q} : \mathbb{R}^n \to \mathbb{R}^n$ the *jump map* (also called reset map) i.e. if τ is a switching time, $x(\tau) = x(\tau^+) = f_{\sigma(\tau^-),\sigma(\tau)}(x(\tau^-))$. Note that the case of identity jump maps $f_{p,q}(x) = x \; \forall p, q \in \mathscr{P}, \forall x \in \mathbb{R}^n$ is the case considered in this paper. For non-identity jump maps, the concept of singular pairs changes to "(x_0, y) is a singular pair of Γ_p, Γ_q if $\exists\, u_1, u_2$ such that $\Gamma^{\mathrm{O}}_{p,x_0}(u_1) = \Gamma^{\mathrm{O}}_{q,f_{p,q}(x_0)}(u_2) = y$ or $\Gamma^{\mathrm{O}}_{p,f_{q,p}(x_0)}(u_1) = \Gamma^{\mathrm{O}}_{q,x_0}(u_2) = y$". The Eq. (8) becomes:

$$\mathbf{N}_{pq} y|_{t_0} = \begin{bmatrix} L_p x_0 \\ L_q f_{p,q}(x_0) \end{bmatrix} \quad \text{or} \quad \mathbf{N}_{pq} y|_{t_0} = \begin{bmatrix} L_p f_{q,p}(x_0) \\ L_q x_0 \end{bmatrix}.$$

There will also be a distinction between identifying the initial switching mode and subsequent switching modes. Another generalization is to include switching mechanisms, such as *switching surfaces*. Denote by $S_{p,q}$ the switching surface for system p changing to system q such that $x(t) = f_{p,q}(x(t^-))$ if $x(t^-) \in S_{p,q}$ and $\sigma(t^-) = p$. Then we only need to check for singularity for $x_0 \in S_{p,q}$ and $x_0 \in S_{q,p}$ instead of $x_0 \in \mathbb{R}^n$ for Γ_p, Γ_q.

4. Output generation

In the previous section, we considered the invertibility question of whether one can recover (σ, u) uniquely for all y in some output set $\overline{\mathscr{Y}}$. In this section, we address a different but closely related problem which concerns with finding (σ, u) (there may be more than one) such that $\mathrm{H}_{x_0}(\sigma, u) = y$ for given y and x_0.

For the switched system (1), denote by $\mathrm{H}_{x_0}^{-1}(y)$ the *preimage* of an output y under H_{x_0},

$$\mathrm{H}_{x_0}^{-1}(y) := \{(\sigma, u) : \mathrm{H}_{x_0}(\sigma, u) = y\}. \tag{12}$$

By convention, $\mathrm{H}_{x_0}^{-1}(y) = \emptyset$ if y is not in the image set of H_{x_0}. In general, $\mathrm{H}_{x_0}^{-1}(y)$ is a set for a given y (when $\mathrm{H}_{x_0}^{-1}(y)$ is a *singleton*, the map H_{x_0} is invertible at y). We want to find conditions and an algorithm to generate $\mathrm{H}_{x_0}^{-1}(y)$ when $\mathrm{H}_{x_0}^{-1}(y)$ is a finite set.

We need the individual subsystems to be invertible because if this is not the case, then the set $\mathrm{H}_{x_0}^{-1}(y)$ will be infinite by virtue of the following lemma.

Lemma 4. *Consider the non-switched linear system Γ in* (2) *with piecewise continuous input. Suppose that Γ is not invertible. Let $[a,b]$ be an arbitrary interval. For every $u \in \mathscr{F}^{\mathrm{pc}}_{[a,b]}$ and $x_a \in \mathbb{R}^n$, there are infinitely many different $v \in \mathscr{F}^{\mathrm{pc}}_{[a,b]}$, $v \neq u$, such that $\Gamma^{\mathrm{O}}_{x_a}(v) = \Gamma^{\mathrm{O}}_{x_a}(u)$ and $\Gamma_{x_a}(v)|_b = \Gamma_{x_a}(u)|_b$.*

Proof. If Γ is not invertible, then $K \neq 0$ in the generalized inverse (4). Then the controllable subspace $\bar{\mathscr{C}}$ of $(A - B\overline{D}_\alpha^\dagger \overline{C}_\alpha, BK)$ is non-trivial. Pick any $\xi \in \bar{\mathscr{C}}, \xi \neq 0$ and $T_1, T_2 > 0$ such that $T_1 + T_2 = b - a =: T$. Since $0, \xi \in \bar{\mathscr{C}}$,$\exists$ a nonzero $w_1 \in \mathscr{C}^0_{[0,T_1]}$ such that $\Gamma_0^{-1}(0, w_1)|_{T_1} = \xi$ and $w_2 \in \mathscr{C}^0_{[0,T_2]}$ such that $\Gamma_\xi^{-1}(0, w_2)|_{T_2} = 0$. By the time-invariant property, we then have $\Gamma_0^{-1}(0, w)|_T = 0$ where $w = w_1 \oplus w_2$. Let $\bar{u} = \Gamma_0^{-1,\mathrm{O}}(0, w)$; then $\Gamma_0(\bar{u})|_T = 0$ and $\Gamma_0^{\mathrm{O}}(\bar{u}) \equiv 0$. Also by the time-invariant property, if $\hat{u}(t) := \bar{u}(t-a)$, $u \in \mathscr{F}^{\mathrm{pc}}_{[a,b]}$, then $\Gamma_0(\hat{u})|_b = 0$ and $\Gamma_0^{\mathrm{O}}(\hat{u}) \equiv 0$. Let $v = u + \hat{u}$. Linearity of Γ implies $\Gamma_{x_a}(v) = \Gamma_{x_a}(u) + \Gamma_0(\hat{u})$ and $\Gamma^{\mathrm{O}}_{x_a}(v) = \Gamma^{\mathrm{O}}_{x_a}(u) + \Gamma_0^{\mathrm{O}}(\hat{u})$, and hence, $\Gamma_{x_a}(v)|_b = \Gamma_{x_a}(u)|_b$ and $\Gamma^{\mathrm{O}}_{x_a}(v) = \Gamma^{\mathrm{O}}_{x_a}(u)$. Since there are infinitely many $\xi \in \bar{\mathscr{C}}$, we have infinitely many such v. $\square$

Corollary 1. *Consider a function $y = \mathrm{H}_{x_0}(\sigma, u)$ for some σ, u. Let $\mathscr{Q} \subseteq \mathscr{P}$ be the set of values of σ. If there exists $q \in \mathscr{Q}$ such that Γ_q is not invertible, then there exist infinitely many u such that $y = \mathrm{H}_{x_0}(\sigma, u)$.*

The previous discussion motivates us to introduce the following assumption.

Assumption 1. The individual subsystems Γ_p are invertible for all $p \in \mathscr{P}$.

However, we have no other assumption on the subsystem dynamics and the switched system may not be invertible as the subsystems may not satisfy the invertibility condition in the previous section. Since we look for an algorithm to find $\mathrm{H}_{x_0}^{-1}(y)$, we only consider the functions y of finite intervals (and hence, there is a finite number of switches) to avoid infinite loop reasoning when there are infinitely many switchings.

As we shall see, it is possible to use a switch at a later time to recover a "hidden switch" earlier (e.g. a switch at which the output is smooth). We now present a switching inversion algorithm for switched systems that returns $\mathrm{H}_{x_0}^{-1}(y)$ for a function $y \in \mathscr{F}^{\mathrm{pc}}_{\mathscr{D}}$ where $\mathscr{D}$ is a finite interval, when $\mathrm{H}_{x_0}^{-1}(y)$ is a finite set. The parameters to the algorithm are $x_0 \in \mathbb{R}^n$ and $y \in \mathscr{F}^{\mathrm{pc}}_{\mathscr{D}}$, and the return is $\mathrm{H}_{x_0}^{-1}(y)$ as in (12). Define the *index-matching map*[1] $\Sigma^{-1} : \mathbb{R}^n \times \mathscr{F}^{\mathrm{pc}} \to 2^{\mathscr{P}}$ as

$$\Sigma^{-1}(x_0, y) := \{p : y \in \widehat{\mathscr{Y}}_p \text{ and } \mathbf{N}_p y|_{t_0^+} = L_p x_0\}, \tag{13}$$

where t_0 is the initial time of y. If $\Sigma^{-1}(x_0, y)$ is empty, no subsystem is able to generate that y starting from x_0.

Begin of Function $\mathrm{H}_{x_0}^{-1}(y)$
  Let the domain of y be $[t_0, T)$.
  Let $\overline{\mathscr{P}} := \{p \in \mathscr{P} : y_{[t_0,t_0+\varepsilon)} \in \widehat{\mathscr{Y}}_p$ for some $\varepsilon > 0\}$.
  Let $t^* := \min\{t \in [t_0, T) : y_{[t,t+\varepsilon)} \notin \widehat{\mathscr{Y}}_p$
    for some $p \in \overline{\mathscr{P}}, \varepsilon > 0\}$ or $t^* = T$ otherwise.
  Let $\mathscr{P}^* := \Sigma^{-1}(x_0, y_{[t_0,t_0+\varepsilon)})$ for sufficiently small ε.
  If $\mathscr{P}^* \neq \emptyset$,
    Let $\mathscr{A} := \emptyset$.
    For each $p \in \mathscr{P}^*$,
      Let $u := \Gamma_{p,x_0}^{-1,\mathrm{O}}(y_{[t_0,t^*)})$,
      $\mathscr{T} := \{t \in (t_0, t^*) : (x(t), y_{[t,t^*)})$ is
        a singular pair of Γ_p, Γ_q for some $q \neq p\}$.
      If $\mathscr{T}$ is a finite set,
        For each $\tau \in \mathscr{T}$, let $\xi := \Gamma_p(u)(\tau)$.
        $\mathscr{A} := \mathscr{A} \cup \{(\sigma^p_{[t_0,\tau)}, u_{[t_0,\tau)}) \oplus H_\xi^{-1}(y_{[\tau,T)})\}$
        End For each
      Else If $\mathscr{T} = \emptyset$ and $t^* < T$, let $\xi = \Gamma_p(u)(t^*)$.
        $\mathscr{A} := \mathscr{A} \cup \{(\sigma^p_{[t_0,t^*)}, u) \oplus H_\xi^{-1}(y_{[t^*,T)})\}$
      Else If $\mathscr{T} = \emptyset$ and $t^* = T$,
        $\mathscr{A} := \mathscr{A} \cup \{(\sigma^p_{[t_0,T)}, u)\}$
      Else $\mathscr{A} := \emptyset$
      End If
    End For each
  Else $\mathscr{A} := \emptyset$
  End If
  Return $H_{x_0}^{-1}(y) := \mathscr{A}$
End of Function

The algorithm is based on the following relationship:

$$\mathrm{H}_{x_0}^{-1}(y_{[t_0,T)}) = \{(\sigma, u) \oplus \mathrm{H}^{-1}_{\mathrm{H}_{x_0}(\sigma,u)(t)}(y_{[t,T)}),\ (\sigma, u) \in \mathrm{H}_{x_0}^{-1}(y_{[t_0,t)})\}\ \forall t \in [t_0, T), \tag{14}$$

which follows from the semigroup property of trajectories of dynamical systems (which include switched systems). Now, if t in (14) is the first switching time after t_0, then we can find $\mathrm{H}_{x_0}^{-1}(y_{[t_0,t)})$ by singling out which subsystems are capable of generating $y_{[t_0,t)}$ using the index-matching map (13). The problem comes down to determining the first switching time t (and then the procedure is repeated for the function $y_{[t,T)}$).

In light of the discussion in the previous paragraph, it is noted that the switching inversion algorithm is a recursive procedure calling itself with different parameters within the main algorithm (e.g.$H_{x_0}^{-1}(y)$ uses the returns of $H_\xi^{-1}(y_{[\tau,T)})$). There are three stopping conditions: it terminates either when $\mathscr{P}^* = \emptyset$, in which case there is no subsystem that can generate y at time t_0 starting from x_0, or when $\mathscr{T}$ is not a finite set, in which case we cannot proceed because of infinitely many possible switching times, or when $\mathscr{T}$ is an empty set and $t^* = T$, in which case the switching signal is a constant switching signal.

If the algorithm returns an empty set, it means that there is no σ and u that can generate y, or there is an infinite number of possible switching times (it is possible to further distinguish between these two cases by using an extra variable in the algo-

[1] The symbol $2^{\mathscr{P}}$ denotes the set of all subsets of a set $\mathscr{P}$.

rithm that is assigned different values for the different cases). Note the utilization of the concatenation notation here: if at any instant of time, the return of the procedure is an empty set, then that branch of the search will be empty because $f \oplus \emptyset = \emptyset$.

Remark 10. When $y \in \mathscr{F}^n$ and the input and output dimensions are the same, t^* in the algorithm can be simplified to be the first discontinuity of y in $[t_0, T)$ (or $t^* = T$ if y is continuous). That is because in this case, C^n functions are always in $\widehat{\mathscr{Y}}_p$ $\forall p \in \mathscr{P}$ (see Remark 3) and thus, $\overline{\mathscr{P}} = \mathscr{P}$, and $y_{[t,t+\varepsilon)} \notin \widehat{\mathscr{Y}}_p$ only if y is discontinuous at t.

5. Examples

Example 1. Consider the switched system generated by the following two subsystems:

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \\ y = [0 \quad 1]\, x, \end{cases} \quad \Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u, \\ y = [0 \quad 2]\, x. \end{cases}$$

Using the structure algorithm, we can check that the two systems are invertible. The operators $\mathbf{N}_1, \mathbf{N}_2$ are matrix operators and $\mathbf{N}_1 = W_1 = [1]$, $L_1 = [0\ \ 1]$, $\mathbf{N}_2 = W_2 = [1]$, $L_2 = [0\ \ 2]$. In this example, the input and output dimensions are the same. The rank condition (9) is satisfied. By Theorem 2, we conclude that the switched system generated by $\{\Gamma_1, \Gamma_2\}$ is invertible on $\overline{\mathscr{Y}} := \{y \in \mathscr{F}^{\mathrm{pc}} : W_{1,2}\, y|_{t^+} \neq 0\ \forall t\} = \{y \in \mathscr{F}^{\mathrm{pc}} : y(t) \neq 0\ \forall t\}$.

Example 2. We return to the example in the introduction. Using the structure algorithm, we can check that the two systems are invertible. We have $\mathbf{N}_1 = W_1 = [1]$, $L_1 = [0\ \ 1]$, $\mathbf{N}_2 = W_2 = [1]$, $L_2 = [1\ \ 2]$. The rank condition is violated, and hence, the switched system generated by Γ_1, Γ_2 is not invertible.

We now illustrate how the inversion algorithm works. Using (13) with x_0 and $y(0) = -1$, we obtain $\mathscr{P}^* := \Sigma^{-1}(x_0, y_{[0,t^*)}) = \{2\}$ (see Remark 10). The structure algorithm for Γ_2 on $[0, t^*)$ yields the inverse

$$\Gamma_2^{-1} : \begin{cases} \dot{z} = \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \dot{y}, \\ u(t) = \dot{y} - [-1 \quad 4]\, z, \end{cases} \qquad t \in [0, t^*)$$

with $z(0) = x_0$, which then gives

$$\begin{aligned} z(t) &= (-e^t, -e^t + e^{2t})^{\mathrm{T}} =: x(t), \\ u(t) &= 0, \end{aligned} \qquad t \in [0, t^*). \tag{15}$$

We find $\mathscr{T} = \{t \leqslant t^* : (x(t), y_{[t,t^*)})$ is a singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving $W_1 y(t) = L_1 x(t)$ (since we already have $W_2 y(t) = L_2 x(t)\ \forall t \in [0, t^*)$). This leads to the equation $2e^{2t} - 3e^t = x_2(t) = -e^t + e^{2t}$, $t \in [0, t^*)$. The foregoing equation has a solution $t = \ln 2 =: t_1 < t^*$, and hence, $\mathscr{T} = \{t_1\}$, which is a finite set. We have $\xi = x(t_1) = (-2, 2)^{\mathrm{T}}$ and we repeat the procedure for the initial state ξ and the output $y_{[t_1,T)}$. Now $\mathscr{P}^* = \Sigma^{-1}(\xi, y_{[t_1,t^*)}) = \{1, 2\}$.

Case 1: $p = 1$. Using the structure algorithm, we obtain the inverse system of Γ_1,

$$\Gamma_1^{-1} : \begin{cases} \dot{z} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{y}, \\ u = \dot{y} - [0 \quad -1]\, z, \end{cases}$$

with the initial state $z(t_1) = \xi$, which yields $z(t) = ((-13 + 6\ln 2)e^t + 6e^{2t} - 6te^t, 2e^{2t} - 3e^t)^{\mathrm{T}} =: x(t)$ and $u(t) = 6e^{2t} - 6e^t$, $t \geqslant t_1$. We find $\mathscr{T} = \{t_1 < t \leqslant t^* : (x(t), y_{[t,t^*)})$ is a singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving $W_2 y(t) = 2e^{2t} - 3e^t = L_2 x(t) = (-19 + 6\ln 2)e^t + 10e^{2t} - 6te^t$ for $t_1 < t \leqslant t^*$. It is not difficult to check that the foregoing equation does not have a solution. We let $\xi = x(t^*) = (15 + 18\ln(\frac{2}{3}), 9)$ and repeat the procedure with ξ and $y_{[t^*,T)}$, which yields $\sigma = \sigma^2_{[t^*,T)}$ and $u_{[t^*,T)} = 0$.

Case 2: $p = 2$. This case means that t_1 is not a switching time. Then $u(t) = 0$ up to time t^* by the structure algorithm, and hence, $x(t) = (-e^t, -e^t + e^{2t})^{\mathrm{T}}$, $\tau \leqslant t \leqslant t^*$ in view of (15). We then repeat the procedure with $\xi = x(t^*) = (-3, 6)$ and $y_{[t^*,T)}$. We have $y(t^*) = 33 + 18\ln(\frac{2}{3})$, and since $L_1\xi \notin W_1 y(t^*)$ and $L_2\xi \notin W_2 y(t^*)$, we have $\Sigma^{-1}(\xi, y_{[t^*,T)}) = \emptyset$.

Thus, the inversion algorithm returns $\{(\sigma, u)\}$, which were given in Section 1.1.

As we can see, there is a switching at $t_1 < t^*$ whilst the output is smooth at t_1. Using the switching inversion algorithm, we can detect the switching at t_1 and recover the switching signal and subsequently the input.

6. Conclusions

We formulated the invertibility problem for switched systems, which seeks a condition on the subsystems that guarantees unique recovery of the switching signal and the input from given output and initial state. We introduced the concept of singular pairs and presented a necessary and sufficient condition for invertibility of continuous-time switched linear systems, which says that the subsystems should be invertible and there should be no singular pairs with respect to the output set. For switched systems, not necessarily invertible but with invertible subsystems, we gave an algorithm for finding switching signals and inputs that generate a given output from a given initial state.

The invertibility problem for discrete-time switched systems remains an open problem. Other research topic is a geometric approach for continuous-time switched systems, which will complement the matrix-oriented approach presented here. Another direction is to extend the results to switched nonlinear systems, for which the extension of the structure algorithm for non-switched nonlinear systems (Singh, 1981) or the geometric approach (Benedetto et al., 1989) may be useful.

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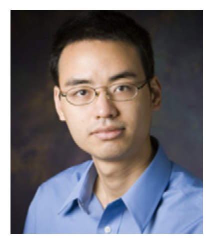

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