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Finite data-rate feedback stabilization of switched and hybrid linear systems☆

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ABSTRACT

We study the problem of asymptotically stabilizing a switched linear control system using sampled and quantized measurements of its state. The switching is assumed to be slow enough in the sense of combined dwell time and average dwell time, each individual mode is assumed to be stabilizable, and the available data rate is assumed to be large enough but finite. Our encoding and control strategy is rooted in the one proposed in our earlier work on non-switched systems, and in particular the data-rate bound used here is the data-rate bound from that earlier work maximized over the individual modes. The main technical step that enables the extension to switched systems concerns propagating over-approximations of reachable sets through sampling intervals, during which the switching signal is not known; a novel algorithm is developed for this purpose. Our primary focus is on systems with time-dependent switching (switched systems) but the setting of state-dependent switching (hybrid systems) is also discussed.

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1. Introduction

The subject of this paper is control of switched systems based on limited information about their state. More specifically, by "limited information" here we mean that measurements being passed from the system to the controller are sampled and quantized using a finite alphabet, resulting in finite data-rate communication. Our aim is to bring together two research areas – switched systems and control with limited information – which have both enjoyed a lot of activity in the past two decades and made great impact on applications, but synergy between which has been lacking. The switched nature of the system enables one to capture many processes encountered in practice which switch between different modes of operation, while the limited nature of the communication link between the system and the controller allows one to incorporate scenarios where the controller is remotely located or the sensors are limited. Combining the two aspects is essential for the development of a unified framework for building and analyzing control systems capable of handling realistic dynamics and realistic control architectures.

Feedback control problems with limited information have been an active research area for some time now, as surveyed in Nair, Fagnani, Zampieri, and Evans (2007) (several specifically relevant works will be cited below). Information flow in a feedback control loop is an important consideration in many application-related scenarios. Even though in modern applications a lot of communication bandwidth is usually available, there are also multiple resources competing for this bandwidth (due to many control loops sharing a network cable or wireless medium, or microsystems with many sensors and actuators on a small chip). In many applications one is also faced with constraints on the sensors dictated by cost concerns or physical limitations, or constraints on information transmission dictated by security considerations. Besides multiple practical motivations, the questions of how much information is really needed to solve a given control problem, or what interesting control tasks can be performed with a given amount of information, are quite fundamental from the theoretical point of view. In the literature on control with limited information, Lyapunov analysis and data-rate/transmission-interval bounds are commonly employed tools (see, e.g., Liberzon, 2003b, Chapter 5; Nešić & Liberzon, 2009; Nešić & Teel, 2004).

Switched and hybrid systems are ubiquitous in realistic system models, because of their ability to capture the presence of two types of dynamical behavior within the system: continuous flow and discrete transitions. Amidst the large body of research on switched and hybrid systems, particularly relevant here is the work on stability analysis and stabilization of such systems, covered in the books (Liberzon, 2003b; van der Schaft & Schumacher,

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2000), the survey (Shorten, Wirth, Mason, Wulff, & King, 2007), and many references therein. Among the specific technical tools typically employed to study these problems, common and multiple Lyapunov functions and slow-switching conditions of the (average) dwell-time type are prominently featured.

Control problems with limited information do not seem to have received much attention so far in the context of switched systems. (Some work has been devoted to quantized control of Markov jump linear systems (Ling & Lin, 2010; Nair, Dey, & Evans, 2003; Xiao, Xie, & Fu, 2010; Zhang, Chen, & Dullerud, 2009), but these systems are quite different from the models we study. Besides, the information structure considered in these references – except for Xiao et al. (2010) which incorporates mode estimation – implies that the discrete mode is always known to the controller, which would remove most of the difficulties present in our problem formulation. On the other hand, control of hybrid systems with unknown discrete mode was also considered in Verma and Del Vecchio (2012) but there the continuous state was not quantized.) In view of the commonality of the technical tools employed for the analysis of switched systems and for control design with limited information, we contend that a marriage of these two research areas is quite natural (implicit or explicit evidence of this commonality can already be found in the above references as well as in several other sources, including Girard, Pola, and Tabuada (2010), Hespanha, Liberzon, and Teel (2008), and Vu and Liberzon (2012)). In particular, (average) dwell-time assumptions and multiple Lyapunov functions will play a crucial role in our analysis.

In order to understand how much information is needed – and how this information should be used – to stabilize a given system, we must understand how the uncertainty about the system's state evolves over time along its dynamics. In more precise terms, we need to be able to characterize propagation of reachable sets or their suitable over-approximations during the sampling interval (for a known control input). The reason is that at each sampling time, the quantizer singles out a bounded set which contains the continuous state and the controller determines the control signal to be applied to the system over the next sampling interval; no further information about the state is available during this interval, and at the next sampling time a bounded set containing the state must be computed to generate the next quantized measurement. The system can be stabilized if the factor by which the state estimation error is reduced at the sampling times is larger than the factor by which it grows between the sampling times. Thus propagation of reachable set bounds is a crucial ingredient in the available results on rate-constrained control of non-switched systems (such as Liberzon, 2003a which serves as the basis for the present work), and the bulk of the effort required to handle the switched system scenario is concentrated in implementing this step and analyzing its consequences.

If the switching signal were precisely known to the controller, then the problem of reachable set propagation would be just a sequence of corresponding problems for the individual modes, and as such would pose very little extra difficulty. (This would essentially correspond to the situation considered, in a discrete-time stochastic setting, in Nair et al., 2003.) On the other hand, if the switching signal were completely unknown, then the set of possible trajectories of the switched system would be too large to hope for a reasonable (not overly conservative) solution. To strike a balance between these two situations, we assume here that we have a partial knowledge of the switching signal; namely, we assume that the active mode of the switched system is known at each sampling time, and that the switching is subject to a fairly mild "slow-switching" assumption (described by a combination of a dwell time and an average dwell time). If in addition the allowed data rate is large enough, then we are able to design a provably correct communication and control strategy to stabilize the switched system.

We now outline in a bit more detail the sequence of steps that we follow. In Section 2 we define the switched linear system that we want to stabilize, explain what the information structure is, and state the basic assumptions and the main result. In Section 3 we describe the basic encoding and control strategy which assumes that appropriate bounds on reachable sets are available. Section 4 is devoted to generating such reachable set bounds. With these ingredients in place, Section 5 completes the analysis through revealing a cascade structure within the closed-loop system, constructing a (mode-dependent) Lyapunov function which decreases in the absence of switching if the data rate is large enough, and invoking the average dwell-time assumption to establish global asymptotic stability. Section 6 contains a short simulation example. In Section 7 we explain how our method can be adapted to hybrid systems by taking advantage of the knowledge of discrete dynamics. Some concluding remarks are given in Section 8.

This paper is based on two conference papers (Liberzon, 2011, 2013). The first of these was devoted to switched systems, while the second emphasized hybrid systems. The present paper addresses both system classes and provides proof details not included in the conference versions; these proof steps are important because that is where the interplay between the mode-dependent Lyapunov function and the average dwell-time is revealed. We also made significant structural and notational improvements in this version to make the presentation easier to follow.

2. Problem formulation

2.1. Switched system

The system to be controlled is the switched linear control system

$$\dot{x} = A_\sigma x + B_\sigma u, \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $\{(A_p, B_p) : p \in \mathcal{P}\}$ is a collection of matrix pairs defining the individual control systems ("modes") of the switched system, $\mathcal{P}$ is a finite index set, and $\sigma : [0, \infty) \to \mathcal{P}$ is a right-continuous, piecewise constant function called the *switching signal* which specifies the active mode at each time. The solution $x(\cdot)$ is absolutely continuous and satisfies the differential equation away from the discontinuities of σ (in particular, we assume for now that there are no state jumps, but state jumps can also be handled as explained in Section 7.3). The switching signal σ is fixed but not known to the controller a priori. The discontinuities of σ are called "switching times" or simply "switches" and we let $N_\sigma(t, s)$ stand for their number on a semi-open interval $(s, t]$:

$$N_\sigma(t, s) := \text{number of switches on } (s, t].$$

Our first basic assumption is that the switching is not too fast, in the following sense.

Assumption 1 (*Slow Switching*).

(1) There exists a number $\tau_d > 0$ (called a *dwell time*) such that any two switches are separated by at least τ_d, i.e., $N_\sigma(t, s) \le 1$ when $t - s \le \tau_d$;
(2) There exist numbers $\tau_a > \tau_d$ (called an *average dwell time*) and $N_0 \ge 1$ such that

$$N_\sigma(t, s) \le N_0 + \frac{t - s}{\tau_a} \quad \forall t > s \ge 0. \tag{2}$$

The concept of average dwell time was introduced in Hespanha and Morse (1999a) and has since then become standard; it includes dwell time as a special case (for $N_0 = 1$). Note that if the constraint $\tau_a > \tau_d$ were violated, the average dwell-time condition (item 2) would be implied by the dwell-time condition (item 1). Switching signals satisfying Assumption 1 were considered in Vu and Liberzon (2011), where they were called "hybrid dwell-time" signals.

Our second basic assumption is stabilizability of all individual modes.

Assumption 2 (*Stabilizability*)**.** For each $p \in \mathcal{P}$ the pair (A_p, B_p) is stabilizable, i.e., there exists a state feedback gain matrix K_p such that $A_p + B_pK_p$ is Hurwitz (all eigenvalues have negative real parts).

In the sequel, we assume that a family of such stabilizing gain matrices $K_p, p \in \mathcal{P}$ has been selected and fixed. We understand that (at least some of) the open-loop matrices $A_p, p \in \mathcal{P}$ are not Hurwitz. Note, however, that even if all the individual modes are stabilized by state feedback (or stable without feedback), stability of the switched system is not guaranteed in general (see, e.g., Liberzon, 2003b).

2.2. Information structure

The task of the controller is to generate a control input $u(\cdot)$ based on limited information about the state $x(\cdot)$ and about the switching signal $\sigma(\cdot)$. The information to be communicated to the controller is subject to the following two constraints.

Sampling: State measurements are taken at times $t_k := k\tau_s, k = 0, 1, 2, \ldots$, where $\tau_s > 0$ is a fixed *sampling period*.

Quantization: Each state measurement $x(t_k)$ is encoded by an integer from 0 to N^n, where N is an odd positive integer, and sent to the controller. In addition, the value of $\sigma(t_k) \in \mathcal{P}$ is also sent to the controller.

As a consequence, data is transmitted to the controller at the rate of $(\log_2(N^n + 1) + \log_2 |\mathcal{P}|)/\tau_s$ bits per time unit, where $|\mathcal{P}|$ is the number of elements in $\mathcal{P}$. We assume the data transmission to be noise-free and delay-free. We take the sampling period τ_s to be no larger than the dwell time from Assumption 1 (item 1):

$$\tau_s \le \tau_d. \tag{3}$$

This guarantees that at most one switch occurs within each sampling interval, i.e., σ does not take any values other than $\sigma(t_k)$ and $\sigma(t_{k+1})$ on $[t_k, t_{k+1}]$.[2] Since the average dwell time τ_a in Assumption 1 (item 2) is larger than τ_d, we know that switches actually occur less often than once every sampling period. The reason for taking the integer N to be odd is to ensure that the control strategy described later preserves the equilibrium at the origin.

Throughout the paper, we work with the ∞-norm $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ on $\mathbb{R}^n$ and the corresponding induced matrix norm $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |A_{ij}|$ on $\mathbb{R}^{n\times n}$, both of which we denote simply by $\|\cdot\|$. To formulate our final basic assumption, we define

$$\Lambda_p := \|e^{A_p\tau_s}\|, \quad p \in \mathcal{P}. \tag{4}$$

Assumption 3 (*Data Rate*)**.** $\Lambda_p < N$ for all $p \in \mathcal{P}$.

We can view the above inequality as a data-rate bound because it requires N to be sufficiently large relative to τ_s, thereby imposing (indirectly) a lower bound on the available data rate. A very similar data-rate bound but for the case of a single mode appears in Liberzon (2003a), where it is shown to be sufficient for stabilizing a non-switched linear system. That bound is slightly conservative compared to known bounds that characterize the minimal data rate necessary for stabilization (see, e.g., Hespanha, Ortega, & Vasudevan, 2002; Tatikonda & Mitter, 2004). However, the control scheme of Liberzon (2003a) can be refined by tailoring it better to the structure of the system matrix A, and then the data rate that it requires will approach the minimal data rate (see also the discussion in Sharon & Liberzon, 2012, Section V). Therefore, it is fair to say that Assumption 3 does not introduce a significant conservatism beyond requiring that the data rate be sufficient to stabilize each individual mode of the switched system (1).

2.3. Main objective

The control objective is to asymptotically stabilize the system defined in Section 2.1 while respecting the information constraints described in Section 2.2. More concretely, we want to provide a constructive proof of the following result.

Theorem 1 (*Main Result*)**.** *Consider the switched linear system (1) and let Assumptions 1–3 and the inequality (3) hold. If the average dwell time τ_a is large enough, then there exists an encoding and control strategy that yields the following two properties:*

Exponential convergence: *There exist a number $\lambda > 0$ and a function $g : [0, \infty) \to (0, \infty)$ such that for every initial condition x_0 and every time $t \ge 0$ we have*

$$\|x(t)\| \le e^{-\lambda t} g(\|x_0\|). \tag{5}$$

Lyapunov stability: *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \ge 0. \tag{6}$$

A precise lower bound on the average dwell time τ_a will be derived in the course of the proof (see the formula (38) in Section 5.3 as well as Remark 2 there). The exponential decay rate λ will also be explicitly characterized (see the formula (43) in Section 5.4). As for the function g in the exponential convergence property, from the proof it will be clear that $g(r)$ does not go to 0 as $r \to 0$ and that, in general, g grows faster than any linear function at infinity (see the formula (44) in Section 5.4 and the discussion at the end of Section 4.3). For this reason, Lyapunov stability needs to be established separately, and the two properties (exponential convergence and Lyapunov stability) combined still do not give the standard global exponential stability, but rather just global asymptotic stability with an exponential convergence rate.

The control strategy that we will develop to prove Theorem 1 is a dynamic one: it involves an additional state denoted by $\hat{x}$. Theorem 1 only discusses the behavior of the state x, which is the main quantity of interest, but it can be deduced from the proof that the controller state $\hat{x}$ satisfies analogous bounds. We will also see that $\hat{x}$ is potentially discontinuous at the sampling times t_k (which are not synchronized with the switching times of the original system); in other words, our controller is a hybrid one.

3. Basic encoding and control strategy

In this section we outline our encoding and control strategy, assuming for now that the state x satisfies known bounds at the sampling times. The problem of generating such state bounds is solved in the next section.

First, suppose that at some sampling time t_{k_0} we have

$$\|x(t_{k_0})\| \le E_{k_0}$$

[2] This assumption is made for simplicity. It could be relaxed to allow multiple switches, up to a fixed number, per sampling interval. To extend our approach to this setting, we would need to consider all possible sequences of "hidden" switches consistent with the received data. This would make our formulas more complicated but would not cause conceptual difficulties.

where $E_{k_0} > 0$ is a number known to the controller. (In Section 4.3 we will show how such a bound can be generated for an arbitrary initial state x_0, by using a "zooming-out" procedure.) At the first such sampling time our controller is initialized. The encoder works by partitioning the hypercube $\{x \in \mathbb{R}^n : \|x\| \le E_{k_0}\}$ into N^n equal hypercubic boxes, N per each dimension, and numbering them from 1 to N^n in some specific way. It then records the number of the box that contains[3] $x(t_{k_0})$ and sends it to the controller, along with the value of $\sigma(t_{k_0})$. We assume that the controller knows the box numbering system used by the encoder, so it can decode the box number. It lets $c_{k_0} \in \mathbb{R}^n$ be the center of the box containing $x(t_{k_0})$. We then have

$$\|x(t_{k_0}) - c_{k_0}\| \le \frac{E_{k_0}}{N}.$$

For $t \in [t_{k_0}, t_{k_0+1})$, the control is set to

$$u(t) = K_{\sigma(t_{k_0})}\hat{x}(t)$$

where $\hat{x}$ is defined to be the solution of

$$\dot{\hat{x}} = (A_{\sigma(t_{k_0})} + B_{\sigma(t_{k_0})}K_{\sigma(t_{k_0})})\hat{x} = A_{\sigma(t_{k_0})}\hat{x} + B_{\sigma(t_{k_0})}u$$

with the boundary condition

$$\hat{x}(t_{k_0}) = c_{k_0}.$$

At a general sampling time t_k, $k \ge k_0 + 1$, suppose that a point $x_k^* \in \mathbb{R}^n$ and a number $E_k > 0$ are known such that

$$\|x(t_k) - x_k^*\| \le E_k. \tag{7}$$

Of course the encoder has precise knowledge of x; the quantities x_k^* and E_k have to be obtainable on the decoder/controller side, based on the knowledge of the system matrices (but not the switching signal) and previously received measurements. We explain later how such x_k^* and E_k can be generated. The encoder also computes x_k^* and E_k in the same way, to ensure that the encoder and the decoder are synchronized. The encoding is then done as follows. Partition the hypercube $\{x \in \mathbb{R}^n : \|x - x_k^*\| \le E_k\}$ into N^n equal hypercubic boxes, N per each dimension. Send the number of the box to the controller, along with the value of $\sigma(t_k)$. On the decoder/controller side, let c_k be the center of the box containing $x(t_k)$. This gives

$$\|x(t_k) - c_k\| \le \frac{E_k}{N} \tag{8}$$

and also

$$\|c_k - x_k^*\| \le \frac{N-1}{N}E_k. \tag{9}$$

Note that the formula (9) is also valid for $k = k_0$ if we set $x_{k_0}^* := 0$, a convention that we follow in the sequel. For $t \in [t_k, t_{k+1})$ define the control, along the same lines as before, by

$$u(t) = K_{\sigma(t_k)}\hat{x}(t)$$

where $\hat{x}$ is the solution of

$$\dot{\hat{x}} = (A_{\sigma(t_k)} + B_{\sigma(t_k)}K_{\sigma(t_k)})\hat{x} = A_{\sigma(t_k)}\hat{x} + B_{\sigma(t_k)}u \tag{10}$$

with the boundary condition

$$\hat{x}(t_k) = c_k. \tag{11}$$

The above procedure is to be repeated for each subsequent value of k. Note that $\hat{x}$ is, in general, discontinuous (only right-continuous) at the sampling times, and we will use the notation $\hat{x}(t_k^-) := \lim_{t \nearrow t_k} \hat{x}(t)$. In the earlier work (Liberzon, 2003a), x_k^* was obtained directly from $\hat{x}$ via $x_k^* := \hat{x}(t_k^-)$. On sampling intervals containing a switch this construction is no longer suitable (cf. Remark 1), and the task of defining x_k^* as well as E_k becomes more challenging.

4. Generating state bounds: over-approximations of reachable sets

Proceeding inductively, we start with known x_k^* and E_k satisfying (7), where $k \ge k_0$, and show how to find x_{k+1}^* and E_{k+1} such that

$$\|x(t_{k+1}) - x_{k+1}^*\| \le E_{k+1}. \tag{12}$$

Generation of E_{k_0} is addressed at the end of the section.

4.1. Sampling interval with no switch

We first consider the simpler case when $\sigma(t_k) = \sigma(t_{k+1}) = p \in \mathcal{P}$. By (3) we know that no switch has occurred on $(t_k, t_{k+1}]$, since two switches would have been impossible. So, we know that on the whole interval $[t_k, t_{k+1}]$ mode p is active. We can then proceed as in Liberzon (2003a). It is clear from (1) and (10) that the error $e := x - \hat{x}$ satisfies $\dot{e} = A_p e$ on $[t_k, t_{k+1})$, and we know from (11) and (8) that $\|e(t_k)\| \le E_k/N$, hence

$$\|e(t_{k+1}^-)\| \le \Lambda_p \frac{E_k}{N} =: E_{k+1} \tag{13}$$

where Λ_p was defined in (4). It remains to let

$$x_{k+1}^* := \hat{x}(t_{k+1}^-) = e^{(A_p+B_pK_p)\tau_s}c_k \tag{14}$$

and recall that x is continuous to see that (12) indeed holds.

4.2. Sampling interval with a switch

Suppose now that $\sigma(t_k) = p$ and $\sigma(t_{k+1}) = q \ne p$. Then again by (3) the controller knows that exactly one switch, from mode p to mode q, has occurred somewhere on the interval $(t_k, t_{k+1}]$, but it does not know exactly where. This case is more challenging.

Let the (unknown) time of the switch from p to q be $t_k + \bar{t}$, where $\bar{t} \in (0, \tau_s]$.

4.2.1. Analysis before the switch

On $[t_k, t_k + \bar{t})$ mode p is active, and we can derive as before that

$$\|x(t_k + \bar{t}) - \hat{x}(t_k + \bar{t})\| \le \|e^{A_p\bar{t}}\|\frac{E_k}{N}.$$

But $\hat{x}(t_k + \bar{t})$ is unknown, so we need to describe a set that contains it. Choose an arbitrary $t' \in [0, \tau_s]$ (which may vary with k). By (10) and (11) we have[4]

$$\hat{x}(t_k + t') = e^{(A_p+B_pK_p)t'}c_k \tag{15}$$

and

$$\hat{x}(t_k + \bar{t}) = e^{(A_p+B_pK_p)(\bar{t}-t')}\hat{x}(t_k + t')$$

hence

$$\begin{aligned}&\|\hat{x}(t_k + \bar{t}) - \hat{x}(t_k + t')\| \\ &\quad \le \|e^{(A_p+B_pK_p)(\bar{t}-t')} - I\|\|\hat{x}(t_k + t')\| \\ &\quad \le \|e^{(A_p+B_pK_p)(\bar{t}-t')} - I\|\|e^{(A_p+B_pK_p)t'}\|\|c_k\|.\end{aligned}$$

We also have from (9) that

$$\|c_k\| \le \|x_k^*\| + \frac{N-1}{N}E_k. \tag{16}$$

By the triangle inequality, we obtain

$$\begin{aligned}\|x(t_k + \bar{t}) - \hat{x}(t_k + t')\| &\le \|e^{(A_p+B_pK_p)(\bar{t}-t')} - I\|\|e^{(A_p+B_pK_p)t'}\| \\ &\quad \times \left(\|x_k^*\| + \frac{N-1}{N}E_k\right) + \|e^{A_p\bar{t}}\|\frac{E_k}{N} =: D_{k+1}(\bar{t}).\end{aligned}$$

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[3] In case $x(t_{k_0})$ lies on the boundary of several boxes, either one of these boxes can be chosen.

[4] In case either $t_k + t'$ or $t_k + \bar{t}$ equals t_{k+1}, the value of $\hat{x}$ at that time should be replaced by the left limit $\hat{x}(t_k + t'^-)$ or $\hat{x}(t_k + \bar{t}^-)$, respectively.

4.2.2. Analysis after the switch

On the interval $[t_k + \bar{t}, t_{k+1})$, the closed-loop dynamics are

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A_q & B_q K_p \\ 0 & A_p + B_p K_p \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}. \tag{17}$$

Due to the mismatch between the modes governing the evolution of x and $\hat{x}$ (these modes are q and p, respectively), the error $e = x - \hat{x}$ no longer satisfies a closed-form differential equation, and the behavior of x and $\hat{x}$ needs to be considered jointly. Letting

$$z := \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \qquad \bar{A}_{pq} := \begin{pmatrix} A_q & B_q K_p \\ 0 & A_p + B_p K_p \end{pmatrix} \tag{18}$$

we can write (17) in the more compact form

$$\dot{z} = \bar{A}_{pq} z. \tag{19}$$

The previous analysis shows that

$$\left\| z(t_k + \bar{t}) - \begin{pmatrix} \hat{x}(t_k + t') \\ \hat{x}(t_k + t') \end{pmatrix} \right\| \le D_{k+1}(\bar{t})$$

(noting the property $\|(a^T, b^T)^T\| = \max\{\|a\|, \|b\|\}$ of the ∞-norm). Consider the auxiliary system copy (on $\mathbb{R}^{2n}$)

$$\dot{\bar{z}} = \bar{A}_{pq} \bar{z}, \qquad \bar{z}(0) = \begin{pmatrix} \hat{x}(t_k + t') \\ \hat{x}(t_k + t') \end{pmatrix}.$$

We have

$$\|z(t_{k+1}^-) - \bar{z}(\tau_s - \bar{t})\| \le \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| D_{k+1}(\bar{t}).$$

We now need to generate a bound for the unknown $\bar{z}(\tau_s - \bar{t})$. Similarly to what we did before, pick a $t'' \in [0, \tau_s]$. Then $\bar{z}(t'') = e^{\bar{A}_{pq} t''} \bar{z}(0)$ and $\bar{z}(\tau_s - \bar{t}) = e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} \bar{z}(t'')$, hence

$$\begin{aligned} \|\bar{z}(\tau_s - \bar{t}) - \bar{z}(t'')\| &\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \|\bar{z}(t'')\| \\ &\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \|e^{\bar{A}_{pq} t''}\| \|\bar{z}(0)\| \\ &= \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \|e^{\bar{A}_{pq} t''}\| \|\hat{x}(t_k + t')\| \\ &\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \|e^{\bar{A}_{pq} t''}\| \|e^{(A_p + B_p K_p) t'}\| \left(\|x_k^*\| + \frac{N-1}{N} E_k \right) \end{aligned}$$

where we used (15) and (16) in the last step. By the triangle inequality,

$$\begin{aligned} \|z(t_{k+1}^-) - \bar{z}(t'')\| &\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \\ &\quad \times \|e^{\bar{A}_{pq} t''}\| \|e^{(A_p + B_p K_p) t'}\| \left(\|x_k^*\| + \frac{N-1}{N} E_k \right) \\ &\quad + \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| D_{k+1}(\bar{t}) =: E_{k+1}(\bar{t}). \end{aligned}$$

To eliminate the dependence on the unknown $\bar{t}$, we take the maximum over $\bar{t}$ (with t' and t'' fixed as above):

$$\begin{aligned} E_{k+1} := \max_{0 \le \bar{t} \le \tau_s} E_{k+1}(\bar{t}) = \max_{0 \le \bar{t} \le \tau_s} \Bigg\{ &\|e^{\bar{A}_{pq}(\tau_s - \bar{t} - t'')} - I\| \\ &\times \|e^{\bar{A}_{pq} t''}\| \|e^{(A_p + B_p K_p) t'}\| \left(\|x_k^*\| + \frac{N-1}{N} E_k \right) \\ &+ \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \Bigg(\|e^{(A_p + B_p K_p)(\bar{t} - t')} - I\| \|e^{(A_p + B_p K_p) t'}\| \\ &\times \left(\|x_k^*\| + \frac{N-1}{N} E_k \right) + \|e^{A_p \bar{t}}\| \frac{E_k}{N} \Bigg) \Bigg\}. \end{aligned}$$

We can use the inequalities

$$\|M - I\| \le \|M\| + 1, \qquad \|e^{As}\| \le e^{\|A\| |s|} \tag{20}$$

to obtain a more conservative upper bound which is more useful for computations:

$$\begin{aligned} E_{k+1} \le &\left(e^{\|\bar{A}_{pq}\| \max\{t'', \tau_s - t''\}} + 1 \right) \|e^{\bar{A}_{pq} t''}\| \|e^{(A_p + B_p K_p) t'}\| \\ &\times \left(\|x_k^*\| + \frac{N-1}{N} E_k \right) + e^{\|\bar{A}_{pq}\| \tau_s} \\ &\times \Bigg(\left(e^{\|A_p + B_p K_p\| \max\{t', \tau_s - t'\}} + 1 \right) \|e^{(A_p + B_p K_p) t'}\| \\ &\times \left(\|x_k^*\| + \frac{N-1}{N} E_k \right) + e^{\|A_p\| \tau_s} \frac{E_k}{N} \Bigg). \end{aligned} \tag{21}$$

Note that this formula simplifies considerably if we set $t' = t'' = 0$, but the original expression for E_{k+1} is not necessarily minimized with this choice of t' and t''. Finally, x_{k+1}^* is defined by projecting $\bar{z}(t'')$ onto the x-component:

$$\begin{aligned} x_{k+1}^* &:= \begin{pmatrix} I_{n \times n} & 0_{n \times n} \end{pmatrix} \bar{z}(t'') \\ &= \begin{pmatrix} I_{n \times n} & 0_{n \times n} \end{pmatrix} e^{\bar{A}_{pq} t''} \begin{pmatrix} \hat{x}(t_k + t') \\ \hat{x}(t_k + t') \end{pmatrix} \\ &= \begin{pmatrix} I_{n \times n} & 0_{n \times n} \end{pmatrix} e^{\bar{A}_{pq} t''} \begin{pmatrix} I_{n \times n} \\ I_{n \times n} \end{pmatrix} e^{(A_p + B_p K_p) t'} c_k. \end{aligned} \tag{22}$$

Remark 1. Another possibility would be to still define x_{k+1}^* via (14), which is simpler than the above expression and does not involve choosing t' and t''. However, the corresponding bound E_{k+1} may then be much larger, especially if the switch happens close to the beginning of the sampling interval (because after the switch, $\hat{x}$ is not a good approximation of x).

4.3. Generating an initial state bound E_{k_0}

Initially, set the control to $u \equiv 0$. At time 0, choose an arbitrary $E_0 > 0$ and partition the hypercube $\{x \in \mathbb{R}^n : \|x\| \le E_0\}$ into N^n equal hypercubic boxes, N per each dimension. If x_0 belongs to one of these boxes, then send the number of the box to the controller. Otherwise send 0 (the "overflow" symbol). Choose an increasing sequence $E_1, E_2, \ldots$ that grows fast enough to dominate the rate of growth of the open-loop dynamics. For example, we can pick a small $\varepsilon > 0$ and let

$$E_k := e^{(2+\varepsilon) \max_{p \in \mathcal{P}} \|A_p\| t_k} E_0, \quad k = 1, 2, \ldots. \tag{23}$$

There are other options but for concreteness we assume that the specific "zooming-out" sequence (23) is implemented. Repeat the above encoding procedure at each step. (As long as the quantization symbol is 0, there is no need to send the value of σ to the controller.) Then we claim that there will be a time t_{k_0} such that, for the corresponding value E_{k_0}, the symbol received by the controller will not be 0. At this time, the encoding strategy described in Section 3 can be initialized.

To see why the above claim is true, consider a sampling interval $[t_k, t_{k+1}]$ on which $u \equiv 0$. If $\sigma \equiv p \in \mathcal{P}$ on this interval, then the dynamics are $\dot{x} = A_p x$ from which it follows that $\|x(t)\| \le \bar{\Lambda}_p \|x(t_k)\|$ on this interval, where

$$\bar{\Lambda}_p := \max_{0 \le s \le \tau_s} \|e^{A_p s}\|. \tag{24}$$

If, on the other hand, the sampling interval contains a switch from mode p to another mode q, then the dynamics become $\dot{x} = A_q x$ after the switch and a (conservative) bound is now $\|x(t)\| \le \bar{\Lambda}_q \bar{\Lambda}_p \|x(t_k)\|$ for $t \in [t_k, t_{k+1}]$. Iterating, we obtain that $\|x(t)\| \le \max_{p \in \mathcal{P}} \bar{\Lambda}_p^{2k} \|x_0\|$ on $[0, t_k]$ as long as $u \equiv 0$ there. Since $\bar{\Lambda}_p \le e^{\|A_p\| \tau_s}$, the values of E_k in (23) grow faster than the largest values that $\|x(t)\|$ can attain on the intervals $[t_{k-1}, t_k]$ under zero control. It follows that k_0 is indeed well defined and there exist functions $\eta : [0,\infty) \to \mathbb{Z}_{\geq 0}$ and $\gamma : [0,\infty) \to (0,\infty)$ such that

$$k_0 \leq \eta(\|x_0\|), \qquad E_{k_0} \leq \gamma(\|x_0\|) \tag{25}$$

and

$$\|x(t)\| \leq \gamma(\|x_0\|) \quad \forall\, t \in [0, t_{k_0}]. \tag{26}$$

Both functions depend on the initial choice of E_0. Note that we can pick them so that $\eta(r) = 0$ and $\gamma(r) = E_0$ for all $r \leq E_0$. For large values of its argument, $\gamma(\cdot)$ is in general super-linear. In fact, we can calculate that $\gamma(r)$ is of the order of r^2/E_0, and $\eta(r)$ is of the order of $(\max_{p\in\mathcal{P}} \|A_p\|\tau_s)^{-1}\log(r/E_0)$, for large values of r.

5. Stability analysis

In this section we prove that the encoding and control strategy developed in Sections 3 and 4 fulfills the properties listed in Theorem 1.

5.1. Sampling interval with no switch

Consider an interval $[t_k, t_{k+1}]$, $k \geq k_0$ on which $\sigma \equiv p \in \mathcal{P}$, as in Section 4.1. Rewrite (14) as

$$\begin{aligned} x^*_{k+1} &= e^{(A_p+B_pK_p)\tau_s}c_k = e^{(A_p+B_pK_p)\tau_s}(x^*_k + \Delta_k) \\ &= S_p x^*_k + S_p\Delta_k \end{aligned} \tag{27}$$

where

$$\Delta_k := c_k - x^*_k, \qquad S_p := e^{(A_p+B_pK_p)\tau_s}. \tag{28}$$

We know from (9) that

$$\|\Delta_k\| \leq \frac{N-1}{N}E_k \tag{29}$$

and we know that S_p is Schur stable because $A_p + B_pK_p$ is Hurwitz. Also, (13) and Assumption 3 give us

$$E_{k+1} = \frac{\Lambda_p}{N}E_k < E_k. \tag{30}$$

We see that, as long as there are no switches, E_k decays exponentially and x^*_k evolves according to an exponentially stable discrete-time linear system whose input Δ_k is bounded in terms of E_k. It is then well known that the overall "cascade" system describing the joint evolution of x^*_k and E_k is exponentially stable. We now formalize this fact by constructing a Lyapunov function in the form of a weighted sum of a quadratic form in x^*_k and E_k^2, along standard lines. This Lyapunov function will depend on p, the currently active mode. Let $P_p = P_p^T > 0$ and $Q_p = Q_p^T > 0$ be such that

$$S_p^T P_p S_p - P_p = -Q_p < 0. \tag{31}$$

We let $\underline{\lambda}(\cdot)$ and $\overline{\lambda}(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. Define

$$\begin{aligned} \alpha_{1,p} &:= \frac{1}{2}\underline{\lambda}(Q_p), \\ \beta_{1,p} &:= \left(\frac{2n^2\|S_p^TP_pS_p\|^2}{\underline{\lambda}(Q_p)} + n\|S_p^TP_pS_p\|\right)\left(\frac{N-1}{N}\right)^2. \end{aligned} \tag{32}$$

Let ρ_p be a positive constant large enough to satisfy

$$\frac{\beta_{1,p}}{\rho_p} + \frac{\Lambda_p^2}{N^2} < 1$$

(such a ρ_p exists because the second fraction is less than 1 by Assumption 3). We now define

$$V_p(x^*_k, E_k) := (x^*_k)^T P_p x^*_k + \rho_p E_k^2. \tag{33}$$

Lemma 1. *The function V_p satisfies*

$$V_p(x^*_{k+1}, E_{k+1}) \leq \nu V_p(x^*_k, E_k)$$

where

$$\begin{aligned} \nu &:= \max_{p\in\mathcal{P}} \nu_p, \\ \nu_p &:= \max\left\{1 - \frac{\alpha_{1,p}}{n\overline{\lambda}(P_p)}, \frac{\beta_{1,p}}{\rho_p} + \frac{\Lambda_p^2}{N^2}\right\} < 1. \end{aligned} \tag{34}$$

Proof. This is a slightly lengthy but straightforward calculation; see Appendix A.1. □

5.2. Sampling interval with a switch

Next, consider an interval $[t_k, t_{k+1}]$, $k \geq k_0$ which contains a switch from mode p to mode q, as in Section 4.2. We know from (22) that $x^*_{k+1} = H_{pq}c_k = H_{pq}(x^*_k + \Delta_k)$ where H_{pq} is a matrix defined by

$$H_{pq} := \begin{pmatrix} I_{n\times n} & 0_{n\times n}\end{pmatrix} e^{\bar{A}_{pq}t''}\begin{pmatrix} I_{n\times n} \\ I_{n\times n}\end{pmatrix} e^{(A_p+B_pK_p)t'}$$

(note that $H_{pq} = I$ if $t' = t'' = 0$) and Δ_k is defined in (28) and satisfies (29). This gives

$$\|x^*_{k+1}\| \leq h_{pq}\left(\|x^*_k\| + \frac{N-1}{N}E_k\right)$$

where $h_{pq} := \|H_{pq}\|$. We also know from (21) that

$$E_{k+1} \leq \alpha_{2,pq}\|x^*_k\| + \beta_{2,pq}E_k$$

where

$$\begin{aligned} \alpha_{2,pq} := &\left(e^{\|\bar{A}_{pq}\|\max\{t'',\tau_s-t''\}} + 1\right)\|e^{\bar{A}_{pq}t''}\|\|e^{(A_p+B_pK_p)t'}\| \\ &+ e^{\|\bar{A}_{pq}\|\tau_s}\left(e^{\|A_p+B_pK_p\|\max\{t',\tau_s-t'\}} + 1\right)\|e^{(A_p+B_pK_p)t'}\| \end{aligned} \tag{35}$$

(this simplifies to $\alpha_{2,pq} = \left(e^{\|\bar{A}_{pq}\|\tau_s} + 1\right) + e^{\|\bar{A}_{pq}\|\tau_s}\left(e^{\|A_p+B_pK_p\|\tau_s} + 1\right)$ if $t' = t'' = 0$) and

$$\beta_{2,pq} := \alpha_{2,pq}\frac{N-1}{N} + e^{\|\bar{A}_{pq}\|\tau_s}e^{\|A_p\|\tau_s}\frac{1}{N}. \tag{36}$$

Extending the construction of the mode-dependent Lyapunov function (33) to all modes $p \in \mathcal{P}$, we have the following result.

Lemma 2. *The functions V_p and V_q satisfy*

$$V_q(x^*_{k+1}, E_{k+1}) \leq \mu V_p(x^*_k, E_k)$$

where

$$\begin{aligned} \mu &:= \max_{\mathcal{P}\ni p\neq q\in\mathcal{P}} \mu_{pq}, \\ \mu_{pq} &:= \max\left\{\frac{2n\overline{\lambda}(P_q)h_{pq}^2 + 2\rho_q\alpha_{2,pq}^2}{\underline{\lambda}(P_p)},\right. \\ &\qquad \left.\frac{2n\overline{\lambda}(P_q)h_{pq}^2}{\rho_p}\left(\frac{N-1}{N}\right)^2 + \frac{2\rho_q\beta_{2,pq}^2}{\rho_p}\right\}. \end{aligned} \tag{37}$$

Proof. This is another direct calculation; see Appendix A.2. □

5.3. Combined bound for sampling times

We now invoke item 2 of Assumption 1 (the average dwell-time property) and derive a lower bound on the average dwell time τ_a that guarantees convergence.

Lemma 3. *Let ν and μ come from Lemmas 1 and 2, respectively. If*

$$\tau_a > \left(1 + \frac{\log \mu}{\log \frac{1}{\nu}}\right)\tau_s \tag{38}$$

then there exists a $\theta \in (0, 1)$ such that

$$V_{\sigma(t_k)}(x_k^*, E_k) \le \left(\frac{\mu}{\nu}\right)^{N_0} \theta^{k-k_0} (\max_{p\in\mathcal{P}} \rho_p) E_{k_0}^2 \quad \forall\, k \ge k_0.$$

Proof. This is again a direct calculation; see Appendix A.3. □

Lemma 3 immediately leads to the bounds

$$\|x_k^*\| \le \sqrt{\frac{V_{\sigma(t_k)}(x_k^*, E_k)}{\min_{p\in\mathcal{P}} \underline{\lambda}(P_p)}} \le \left(\frac{\mu}{\nu}\right)^{N_0/2} \theta^{(k-k_0)/2} \sqrt{\frac{\max_{p\in\mathcal{P}} \rho_p}{\min_{p\in\mathcal{P}} \underline{\lambda}(P_p)}} E_{k_0} \tag{39}$$

and

$$E_k \le \left(\frac{\mu}{\nu}\right)^{N_0/2} \theta^{(k-k_0)/2} \sqrt{\frac{\max_{p\in\mathcal{P}} \rho_p}{\min_{p\in\mathcal{P}} \rho_p}} E_{k_0} \tag{40}$$

for all $k \ge k_0$. Recalling that, by (7), $\|x(t_k)\| \le \|x_k^*\| + E_k$ for all k, we obtain an exponential decay bound for $\|x(t_k)\|$ given by the sum of the right-hand sides of (39) and (40). We will not need this combined bound, though; only the individual bounds (39) and (40) will end up being used in what follows.

Remark 2. Lemma 3 provides considerable insight into the interplay between the speed of switching, data rate, and stability, although the relationships between the various quantities are not simple. First, the condition (38) makes it clear that, as expected, larger average dwell time (slower switching) is favorable for stability of the closed-loop switched system. The effect of increasing N, which means increasing the data rate, can be traced through the formulas derived earlier in this section. If N is increased then the second term inside the maximum that defines ν_p in (34) decreases (assuming that ρ_p and all other constants stay the same), which may thus lead to a decrease in ν. At the same time, it is easy to see from (35)–(37) that as N is increased μ stays bounded (because $\beta_{2,pq}$ approaches $\alpha_{2,pq}$ which does not depend on N). It is also interesting to notice that, since $\alpha_{1,p}$ is defined by (32), the *first* term inside the maximum defining ν_p in (34) decreases as the ratio $\underline{\lambda}(Q_p)/\overline{\lambda}(P_p)$ increases. In view of (31), increasing this ratio corresponds to moving the eigenvalues of the matrix S_p defined in (28) closer to the origin. Hence, we may decrease ν_p by choosing K_p that improves Schur stability of this matrix (and/or by increasing N as we already explained), although this of course affects μ_p as well. If we are able to decrease ν without increasing μ, then we obtain a decrease of the right-hand side of (38), which means that the lower bound on the average dwell time τ_a required in Lemma 3 becomes less restrictive. Finally, we note that the right-hand side of (38) depends on τ_s not only directly but also through the constants μ and ν, and this dependence goes all the way back to the matrix S_p.

5.4. Intersample bound and exponential convergence

We are now ready to establish the first claim of Theorem 1 (exponential convergence). To do this, we modify relevant calculations from Section 4 to derive bounds that are simpler (in particular, we work with $t' = t'' = 0$) and more conservative, but apply to the whole sampling intervals and not just to the sampling times. Consider an interval $[t_k, t_{k+1}]$ with a possible switch at a time $t_k + \bar{t}$ in its interior. For all $t \in [t_k, \bar{t})$ some mode $p \in \mathcal{P}$ is active, and we have

$$\|x(t) - \hat{x}(t)\| \le \bar{\Lambda}_p \frac{E_k}{N}$$

where $\bar{\Lambda}_p$ was defined in (24). Since $\hat{x}(t) = e^{(A_p+B_pK_p)(t-t_k)}c_k$, we have

$$\begin{aligned}\|\hat{x}(t) - c_k\| &\le \max_{0\le s\le\tau_s} \|e^{(A_p+B_pK_p)s} - I\|\|c_k\| \\ &\le \max_{0\le s\le\tau_s} \|e^{(A_p+B_pK_p)s} - I\| \left(\|x_k^*\| + \frac{N-1}{N}E_k\right)\end{aligned}$$

and so, by the triangle inequality,

$$\begin{aligned}\|x(t) - c_k\| &\le \max_{0\le s\le\tau_s} \|e^{(A_p+B_pK_p)s} - I\| \\ &\quad\times \left(\|x_k^*\| + \frac{N-1}{N}E_k\right) + \bar{\Lambda}_p \frac{E_k}{N} =: \bar{D}_{k+1}.\end{aligned} \tag{41}$$

After the switch (if there is indeed a switch) to another mode q, the closed-loop dynamics are given by (17)–(19). The previous formulas show that

$$\left\| z(t_k + \bar{t}) - \begin{pmatrix} c_k \\ c_k \end{pmatrix} \right\| \le \bar{D}_{k+1}.$$

Consider the auxiliary system copy (in $\mathbb{R}^{2n}$)

$$\dot{\bar{z}} = \bar{A}_{pq}\bar{z}, \qquad \bar{z}(0) = \begin{pmatrix} c_k \\ c_k \end{pmatrix}.$$

For all $t \in [t_k + \bar{t}, t_{k+1})$ we have

$$\|z(t) - \bar{z}(t - t_k - \bar{t})\| \le \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s}\|\bar{D}_{k+1}.$$

Next, $\bar{z}(t - t_k - \bar{t}) = e^{\bar{A}_{pq}(t-t_k-\bar{t})}\bar{z}(0)$, hence

$$\begin{aligned}\|\bar{z}(t - t_k - \bar{t}) - \bar{z}(0)\| &\le \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s} - I\|\|\bar{z}(0)\| \\ &\le \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s} - I\|\|c_k\| \\ &\le \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s} - I\| \left(\|x_k^*\| + \frac{N-1}{N}E_k\right).\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}\|z(t) - \bar{z}(0)\| &\le \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s} - I\| \\ &\quad\times \left(\|x_k^*\| + \frac{N-1}{N}E_k\right) + \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s}\|\bar{D}_{k+1} =: \bar{E}_{k+1}.\end{aligned}$$

Projecting onto the x-component, we deduce that

$$\|x(t) - c_k\| \le \bar{E}_{k+1}.$$

This subsumes the earlier bound (41) valid for the times before the switch, i.e., we can use this bound on the whole interval. We obtain

$$\begin{aligned}\|x(t)\| &\le \|c_k\| + \|x(t) - c_k\| \le \|x_k^*\| + \frac{N-1}{N}E_k \\ &\quad + \bar{E}_{k+1} = \alpha_{3,pq}\|x_k^*\| + \beta_{3,pq}E_k\end{aligned}$$

where

$$\begin{aligned}\alpha_{3,pq} &:= 1 + \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s} - I\| \\ &\quad + \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s}\| \max_{0\le s\le\tau_s} \|e^{(A_p+B_pK_p)s} - I\|\end{aligned}$$

and

$$\beta_{3,pq} := \alpha_{3,pq}\frac{N-1}{N} + \max_{0\le s\le\tau_s} \|e^{\bar{A}_{pq}s}\| \max_{0\le s\le\tau_s} \|e^{A_ps}\|\frac{1}{N}.$$

(As before, we can use the inequalities (20) to derive more conservative but more computationally friendly upper bounds.) Invoking the earlier bounds (39) and (40), we conclude that for all $t \in [t_k, t_{k+1})$, $k \geq k_0$ we have

$$\|x(t)\| \leq \bar{c}\theta^{(k-k_0)/2} E_{k_0} \tag{42}$$

where

$$\bar{c} := \left(\frac{\mu}{\nu}\right)^{N_0/2} \left(\max_{p,q\in\mathcal{P}} \alpha_{3,pq} \sqrt{\frac{\max_{p\in\mathcal{P}} \rho_p}{\min_{p\in\mathcal{P}} \underline{\lambda}(P_p)}} + \max_{p,q\in\mathcal{P}} \beta_{3,pq} \sqrt{\frac{\max_{p\in\mathcal{P}} \rho_p}{\min_{p\in\mathcal{P}} \rho_p}} \right).$$

We can now establish a continuous-time exponential decay bound: rewriting (42) as

$$\begin{aligned} \|x(t)\| &\leq \bar{c}\theta^{\frac{1}{2}\left(\lfloor \frac{t}{\tau_s} \rfloor - k_0\right)} E_{k_0} \leq \bar{c}\theta^{\frac{1}{2}\left(\frac{t}{\tau_s} - 1 - k_0\right)} E_{k_0} \\ &= \bar{c}\left(\frac{1}{\sqrt{\theta}}\right)^{1+k_0} \theta^{\frac{t}{2\tau_s}} E_{k_0} \end{aligned}$$

and recalling the initial bounds (25) and (26), we finally arrive at the desired exponential convergence property (5) with

$$\lambda := \frac{1}{2\tau_s} \log \frac{1}{\theta} \tag{43}$$

and

$$g(r) := \bar{c}\left(\frac{1}{\sqrt{\theta}}\right)^{1+\eta(r)} \gamma(r). \tag{44}$$

5.5. Lyapunov stability

The proof of the second claim of Theorem 1 (Lyapunov stability) proceeds along the lines of Liberzon (2003a) and Liberzon and Hespanha (2005). Suppose that for the first few time steps $k = 0, 1, \ldots, k_1 - 1$ we have $c_k = 0$, i.e., $x(t_k)$ is in the central quantization box. (Recall that N is odd.) Then in particular $k_0 = 0$. The following is true for these values of k. First, the formulas for propagation of x_k^* and E_k from Section 4 can be refined. If there is no switch on $(t_k, t_{k+1}]$, then (14) implies that $x_{k+1}^* = 0$, and (30) holds for some p. If there is a switch from p to q, then x_{k+1}^* is still 0 by (22), and

$$E_{k+1} = \max_{0\leq s\leq\tau_s} \|e^{\bar{A}_{pq}s}\| \frac{\bar{\Lambda}_p}{N} E_k = \Omega_{pq} \frac{\Lambda_p}{N} E_k$$

where

$$\Omega_{pq} := \max_{0\leq s\leq\tau_s} \|e^{\bar{A}_{pq}s}\| \frac{\bar{\Lambda}_p}{\Lambda_p}$$

and $\bar{\Lambda}_p$ was defined in (24). The intersample bounds can also be refined, because $\hat{x} \equiv 0$; the same analysis as in Section 4.3 applies, giving us

$$\|x(t)\| \leq \max_{p\in\mathcal{P}} \bar{\Lambda}_p^{2k_1} \|x_0\| \quad \forall\, t \in [0, t_{k_1}]. \tag{45}$$

Adopting (48), we have

$$\begin{aligned} E_k &\leq \Omega_{pq}^{N_\sigma(0,t_k)} \left(\frac{\Lambda_p}{N}\right)^k E_0 \leq \Omega_{pq}^{N_0 + \frac{k}{m}} \left(\frac{\Lambda_p}{N}\right)^k E_0 \\ &= \Omega_{pq}^{N_0} \left(\Omega_{pq}^{1/m} \frac{\Lambda_p}{N}\right)^k E_0 \end{aligned} \tag{46}$$

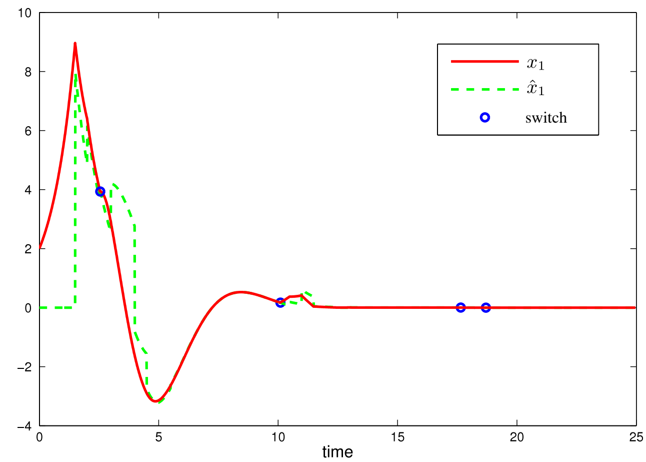

Fig. 1. Simulation example.

for $k = 0, 1, \ldots, k_1 - 1$, and this decays exponentially to 0 if m is large enough so that

$$\Omega_{pq}^{1/m} \frac{\Lambda_p}{N} < 1.$$

We note that the earlier formula (42) is valid for every $k_1 \geq k_0$ in place of k_0:

$$\|x(t)\| \leq \bar{c}\theta^{(k-k_1)/2} E_{k_1} \quad \forall\, t \in [t_k, t_{k+1}),\ k \geq k_1. \tag{47}$$

Now Lyapunov stability can be proved in three steps. First, given an arbitrary $\varepsilon > 0$, we see from (47) that $E_{k_1} < \varepsilon/\bar{c}$ guarantees that $\|x(t)\| < \varepsilon$ for all $t \geq t_{k_1}$. Second, taking E_0 as fixed, use the decay bound (46) for E_k to calculate a value of k_1 for which $E_{k_1} < \varepsilon/\bar{c}$. Finally, pick a δ small enough so that $\max_{p\in\mathcal{P}} \bar{\Lambda}_p^{2k_1}\delta < \varepsilon$ and

$$\max_{p\in\mathcal{P}} \bar{\Lambda}_p^{2k_1-2}\delta < \left(\frac{\min_{p\in\mathcal{P}} \Lambda_p}{N}\right)^{k_1-1} \frac{E_0}{N}.$$

This ensures that, by (45), $\|x(t)\|$ stays below ε for $0 \leq t \leq t_{k_1}$ and that x falls within the central quantization box at the sampling times t_k, $0 \leq k \leq k_1 - 1$, making the above analysis valid. With δ so chosen, we have the implication (6). This completes the proof of the main result.

6. Simulation example

We simulated the above control strategy with the following data: $\mathcal{P} = \{1, 2\}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $K_1 = \begin{pmatrix} -2 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} 0 & -1 \end{pmatrix}$, $x_0 = (2, 2)^T$, $E_0 = 0.5$, $\tau_s = 0.5$, $N = 5$ (Assumption 3 is satisfied), $\tau_d = 1.05$, $\tau_a = 7.55$, and $N_0 = 5$. We chose both t' and t'' to be 0 (a few other values we tried did not lead to better results). Fig. 1 plots a typical behavior of the first component x_1 of the continuous state (in solid red) and the corresponding component $\hat{x}_1$ of the state estimate (in dashed green) versus time; switches are marked by blue circles. Observe the initial "zooming-out" phase and the nonsmooth behavior of x when $\hat{x}$ experiences a jump (causing a jump in the control u). The above value of the average dwell time τ_a was picked empirically to be just large enough to provide consistent convergence in simulations. For this example, the theoretical lower bound on the average dwell time τ_a from the formula (38) is about 85.5 which is, not surprisingly, quite conservative.

Guided by the observations made in Remark 2, we can adjust parameter values and see how this affects the theoretical average dwell-time bound given by (38). Changing the first control gain in K_1 (let us label it as $K_{1,1}$) from -2 to a smaller value moves one eigenvalue of $A_1 + B_1K_1$ to the left while the other eigenvalue remains at -1, and this does not lead to a decrease of the average dwell-time bound. In fact, the opposite is true: for $K_{1,1} < -2$ the average dwell-time bound goes up (due to an increase in μ), while for $K_{1,1} = -1.5$ it goes down to about 81.7. On the other hand, decreasing slightly the second control gain in K_2 (let us label it as $K_{2,2}$) from -1, thereby moving both eigenvalues of $A_2 + B_2K_2$ to the left, does yield a decrease in the average dwell-time bound. With $K_{2,2} = K_{1,1} = -1.5$, the right-hand side of (38) is just below 75 (although for $K_{2,2} < -1.5$ it starts going up again). Increasing N for this system is not beneficial (at least as far as our theory goes) because it increases μ. Here N can be as low as 2 to satisfy Assumption 3, and the optimal value appears to be $N = 3$ which gives (with both control gains still at -1.5) the average dwell-time bound of about 74.3. Clearly, we are not able to approach the value $\tau_a = 7.55$ observed to work in simulations.

7. State-dependent switching

Here we consider systems with state-dependent switching (hybrid systems). In a hybrid system, the abstract notion of the switching signal σ that we used to define the switched system (1) is replaced (or, we may say, realized) by a discrete dynamics model which generates the sequence of modes, based typically on the evolution of the continuous state. Many specific modeling formalisms for hybrid systems exist in the literature, but a common paradigm which we also have in mind here is that each mode corresponds to a region in the continuous state space (sometimes called the *invariant* for that mode) where the corresponding continuous dynamics are active, and transitions (or switchings) between different modes take place when the continuous state x crosses boundaries (called *switching surfaces*, or *guards*) between these regions. At the times of these discrete transitions, the value of x in general can also jump to a new value according to some *reset map*. (For more details on such hybrid system models see, e.g., van der Schaft & Schumacher, 2000.)

Thus, compared to the switched system model (1), the two main new aspects that must be incorporated are *switching surfaces* and *state jumps*. We will address both these aspects in what follows. However, since we saw that propagating (over-approximations of) reachable sets is a key ingredient of our control strategy, we first discuss some relevant prior work on reachable set computation for hybrid systems in order to put our present developments in a proper context.

7.1. Comparison with existing reachable set algorithms

Without aiming for completeness, we give here an overview of some representative results. We classify them roughly according to the type of dynamics in the considered hybrid system model and the shapes of the sets used for reachable set approximation.

Early work by Puri, Borkar, and Varaiya on differential inclusions (Puri, Borkar, & Varaiya, 1996) approximates a general nonlinear differential inclusion by a piecewise constant one, and computes over-approximations of reachable sets which are unions of polyhedra. Henzinger, Ho, and Wong-Toi (1998) and Preußig, Stursberg, and Kowalewski (1999) approximate hybrid systems by *rectangular automata* (hybrid systems whose regions and flow in each region are defined by constant lower and upper bounds on state and velocity components, respectively) and base reachable set computation on the tool `HyTech`; Frehse later developed a refined tool, `PHAVer` (Frehse, 2005), for a similar purpose. Also related to this is reachability analysis using "face lifting" (Dang & Maler, 1998). Asarin, Bournez, Dang, and Maler (2000) and Asarin, Dang, and Maler (2002) work with linear dynamics and rectangular polyhedra and develop the tool called `d/dt`. They reduce the conservatism due to the so-called "wrapping effect" by combining propagation of exact reachable sets at sampling instants with convex over-approximation during intersample intervals. Similar ideas appeared in the earlier work of Greenstreet and Mitchell (1998) who also handle nonlinear models and non-convex polyhedra by using two-dimensional projections. Mitchell and Tomlin (2000) and Tomlin, Mitchell, Bayen, and Oishi (2003) work with general nonlinear dynamics and compute reachable sets as sublevel sets of value functions for differential games, which are solutions of Hamilton–Jacobi PDEs. Kurzhanski and Varaiya (2005) work with affine open-loop dynamics and use ellipsoids for reachable set approximation (based on ellipsoidal methods for continuous systems developed in their prior work). They handle discrete transitions by taking the union of reachable sets over possible switching times and covering it with one bounding ellipsoid. Chutinan and Krogh (2003) compute optimal polyhedral approximations of continuous flow pipes for general nonlinear dynamics, using the tool `CheckMate`. Stursberg and Krogh (2003) work with nonlinear dynamics and "oriented rectangular hulls" relying on principal component analysis. Girard (2005) and Girard and Le Guernic (2008) use a procedure similar to that of Asarin et al. mentioned earlier, but work with *zonotopes* (affine transforms of hypercubes) which allow more efficient computation for linear dynamics. More recently, this approach was refined with the help of support-function representations (Le Guernic & Girard, 2009) and the accompanying tool `SpaceEx` (Frehse et al., 2011) was developed. Other examples of very recent work in this area are the result of Kim, Mitra, and Kumar (2011) on computation of ε-reach sets and the timed relational abstraction scheme of Zutshi, Sankaranarayanan, and Tiwari (2012) for computation of reachable set over-approximations.

There are several similarities between our method and the previous ones just mentioned. Like `d/dt` and related techniques, we also reduce the conservatism due to the "wrapping effect" by making a distinction between sampling and intersample approximations (the bounds derived in Sections 4 and 5.3 are valid at sampling times only and they are sharper than the intersample bounds derived in Section 5.4). Also, similarly to Kurzhanski and Varaiya, we handle discrete transitions by taking the union of reachable sets over possible switching times and covering it with one bounding set, except we work with hypercubes rather than ellipsoids. On the other hand, in spite of the multitude of available methods, these methods were designed for reachability verification and are not directly tailored to control problems of the kind considered here. There are at least two important reasons why we prefer to build on the method from Section 4 rather than just adopt one of the above methods for dealing with hybrid systems:

(i) The methods just mentioned are computational (on-line) in nature; by this we mean that approximations of reachable sets are generated in real time as the system evolves. By contrast, the method from Section 4 is analytical (off-line). Indeed, the size of the reachable set bound E_k at each time step, as well as the center point x_k^*, are obtained iteratively from the formulas given in Section 4. In other words, knowing the system data (the matrices A_p and B_p as well as the control gains K_p), we can pre-compute these bounds; there is no need to synchronize their computation with the evolution of the system. Consequently, the corresponding lower bound on the data rate required for stabilization can be obtained *a priori*, which makes more sense in the context of applications where communication strategies are designed separately from on-line control tasks. (On the negative side, this makes the bounds on reachable sets that our method provides more conservative.)

(ii) Our method is tailored specifically to linear dynamics and to sets in the shapes of hypercubes. Our choice of hypercubes as bounding sets is very natural from the point of view of quantizer design with rectilinear quantization boxes, such as those arising from simple sensors. (However, in other application contexts it may be possible to work with different set shapes. For example, zonotopes – which are affine transforms of hypercubes – would correspond to pre-processing the continuous state by an affine transformation before passing it to a digital encoder; this generalization appears to be quite promising for more efficient computations.)

7.2. Switching surfaces

With regards to hybrid systems where mode switching occurs on switching surfaces, the first observation is that our Theorem 1 already covers such systems, because our reachable set over-approximation is computed by taking the union over all possible switching times $\bar{t}$ (see Section 4). Indeed, a switched system admits more solutions than a hybrid system (for which it serves as a high-level abstraction), and so our stabilization result conservatively captures the hybrid system solutions. The main issue is to verify that a given hybrid system fulfills the slow-switching assumption (Assumption 1), i.e., that all solutions satisfy the dwell-time and average dwell-time properties specified there. This can be difficult, but is possible in some cases. Notable examples are hybrid systems whose switching surfaces are concentric circles with respect to some norm, or lines through the origin in the plane. (Average dwell time is not directly helpful but in these cases we can compute dwell time, assuming linear dynamics in each mode.) Some more interesting examples where time-dependent properties (of dwell-time type) are established a posteriori for control systems with state-dependent switching can be found in Hespanha and Morse (1999b) and Liberzon and Trenn (2010). Thus, translating a hybrid system to a switched system and applying our previous result off-the-shelf via verifying the slow-switching condition can actually be a reasonable route to follow. In fact, since our strategy guarantees containment of the reachable set at each sampling time within a bounding hypercube, we can just run it and verify empirically whether or not the switching is slow enough for convergence. This is what we actually did in the simulation example given in Section 6; in some sense it moves us closer to the on-line computational methods cited above.

A better approach, however, is to improve our reachable set bounds by explicitly incorporating the information available in a hybrid system about where in the continuous state space the switching can occur. Recall that our information structure makes the current mode available to the controller at each sampling time t_k. So, for example, if we know as in Section 4.1 that no switch has occurred on an interval $(t_k, t_{k+1}]$ and $\sigma(t) = p$ there, then the hypercube $\{x \in \mathbb{R}^n : \|x - x^*_{k+1}\| \leq E_{k+1}\}$, which contains the reachable set at time $t = t_{k+1}$, can be reduced by intersecting it with the invariant for mode p. In other words, if a guard passes through this hypercube then we keep only the portion lying on that side of the guard on which mode p is active; the point x^*_{k+1} can also be redefined at this step. The resulting reduction in the size of the bounding set can be quite significant, especially if the set $\{x : \|x - x^*_k\| \leq E_k\}$ at time $t = t_k$ was close to some of the switching surfaces. (Note, however, that if the reachable set over-approximation at time t_{k+1} must be a hypercube, then some or all of this size reduction might become undone when passing to a bounding hypercube.) Or, consider the situation of Section 4.2 where a sampling interval $(t_k, t_{k+1}]$ contains a switch from $\sigma = p$ to $\sigma = q$ at an unknown time $\bar{t}$. The bounding set before the switch, $\{x : \|x - \hat{x}(t_k + t')\| \leq D_{k+1}(\bar{t})\}$ (see Section 4.2(a)) can be reduced in the same way as above by intersecting it with the invariant for mode p. (Since $\bar{t}$ is unknown, we should either treat it as a parameter for this computation or take the maximum over $\bar{t}$ first; (20) is helpful for doing the latter.) Then, when this possibly reduced intermediate bounding set is used to calculate the bounding set after the switch, which we previously defined as $\{x : \|x - x^*_{k+1}\| \leq E_{k+1}\}$ (see Section 4.2(b)), we may reduce it again, this time intersecting it with the invariant for mode q. Overall, this can lead to a significant reduction in the size of the reachable set over-approximation compared to the method of Section 4 which does not assume any relation between the continuous state x and the switching signal (but again, working with hypercubes would not allow us to take full advantage of this size reduction).

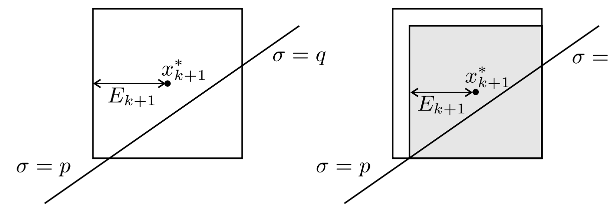

Fig. 2. Reachable set over-approximation: before reduction (left) and after reduction (right).

We illustrate the last point with a simple example. Consider again the scenario when $\sigma(t_k) = p$ and $\sigma(t_{k+1}) = q$. Suppose that switching from $\sigma = p$ to $\sigma = q$ occurs on a hyperplane, represented by the slanted line in Fig. 2, with mode p being active on the left of this hyperplane and mode q being active on the right of the hyperplane. Suppose that our original algorithm gives the bounding hypercube shown on the left side of the figure. Here, x^*_{k+1} is on the "wrong" side of the switching hyperplane, since we know that $x(t_{k+1})$ lies on the other side, i.e., it belongs to the part of the hypercube that is below the hyperplane. Accordingly, we can find a smaller hypercube that is still guaranteed to contain $x(t_{k+1})$; this reduced bounding set is shaded in gray on the right side of the figure. Of course, the relative size of the two hypercubes depends on the position of the hyperplane relative to the larger hypercube.

Additionally, the knowledge of switching surfaces can be used to obtain some information about the unknown switching time $\bar{t}$: for example, if at time t_k we are far from any switching surface, then using the system dynamics we can calculate a lower bound on the time that must pass before a switch can occur.

7.3. State jumps

The reachable set propagation method of Section 4 assumes that there are no state jumps at the switching times, i.e., the reset map is the identity. However, it is not very difficult to augment it to nontrivial reset maps. Specifically, if we have a reset map $R_{pq} : \mathbb{R}^n \to \mathbb{R}^n$ which defines the new state $R_{pq}(x)$ to which x jumps at the time of mode transition from p to q, all we need is a knowledge of some affine Lipschitz bound of the form $\|R_{pq}(x_1) - R_{pq}(x_2)\| \leq a\|x_1 - x_2\| + b$. Then, we can apply the transformations $c \mapsto R_{pq}(c)$ and $D \mapsto aD + b$ to the reachable set over-approximations of the form $\{x : \|x - c\| \leq D\}$ obtained at each time that corresponds to a switch (these times are $t_k + \bar{t}$ on sampling intervals containing a switch, see Section 4.2). We can continue working with hypercubes because after incorporating resets in this way we still obtain hypercubes. We see that accounting for state jumps does not lead to substantial complications in our reachable set algorithm. (The same claim is true for most of the other existing reachable set algorithms from the literature: many of them assume the identity reset map but can be generalized with not much difficulty.) The stability analysis can proceed similarly, with the constants a, b affecting the evolution of the Lyapunov function and leading to a modified average dwell time bound.

8. Conclusions

We presented a result on sampled-data quantized state feedback stabilization of switched linear systems, which relies on a slow-switching condition in the sense of combined dwell time and average dwell time and on a novel method for propagating over-approximations of reachable sets for switched systems. We also explained how this result can be applied in the setting of hybrid systems, where it can actually be improved by utilizing the knowledge of discrete dynamics. Future work will focus on refining the reachable set bounds (by trying to find the optimal choice of the intermediate points t' and t'', by working with bounding sets other than hypercubes, by fleshing out the ideas of Section 7 for specific classes of hybrid systems, and possibly by combining our method with other known reachable set algorithms for hybrid systems); relaxing the main assumptions (to allow, in particular, multiple switches per sampling interval and less frequent transmissions of the mode value σ); and addressing more general systems (by incorporating external disturbances, modeling uncertainty, nonlinear dynamics, and output feedback).

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Appendix. Proofs of the technical lemmas

A.1. Proof of Lemma 1

The ∞-norm $\|\cdot\|$ and the Euclidean norm $|\cdot|$ are related by $\|x\| \le |x| \le \sqrt{n}\|x\|$, hence

$$\underline{\lambda}(Q_p)\|x\|^2 \le x^T Q_p x \le n\overline{\lambda}(Q_p)\|x\|^2.$$

Proceeding similarly to Jiang and Wang (2001, Example 3.4), we have from (27) that

$$\begin{aligned}
&(x^*_{k+1})^T P_p x^*_{k+1} - (x^*_k)^T P_p x^*_k \\
&\quad = (S_p x^*_k + S_p \Delta_k)^T P_p (S_p x^*_k + S_p \Delta_k) - (x^*_k)^T P_p x^*_k \\
&\quad = (x^*_k)^T S_p^T P_p S_p x^*_k - (x^*_k)^T P_p x^*_k + 2(x^*_k)^T S_p^T P_p S_p \Delta_k + \Delta_k^T S_p^T P_p S_p \Delta_k \\
&\quad = -(x^*_k)^T Q_p x^*_k + 2(x^*_k)^T S_p^T P_p S_p \Delta_k + \Delta_k^T S_p^T P_p S_p \Delta_k \\
&\quad \le -\underline{\lambda}(Q_p)\|x^*_k\|^2 + 2n\|x^*_k\|\|S_p^T P_p S_p\|\|\Delta_k\| + n\|S_p^T P_p S_p\|\|\Delta_k\|^2 \\
&\quad = -\frac{1}{2}\underline{\lambda}(Q_p)\|x^*_k\|^2 - \frac{1}{2}\Bigg(\sqrt{\underline{\lambda}(Q_p)}\|x^*_k\| \\
&\qquad - \frac{2n\|S_p^T P_p S_p\|\|\Delta_k\|}{\sqrt{\underline{\lambda}(Q_p)}}\Bigg)^2 + \Bigg(\frac{2n^2\|S_p^T P_p S_p\|^2}{\underline{\lambda}(Q_p)} \\
&\qquad + n\|S_p^T P_p S_p\|\Bigg)\|\Delta_k\|^2 \le -\frac{1}{2}\underline{\lambda}(Q_p)\|x^*_k\|^2 \\
&\qquad + \left(\frac{2n^2\|S_p^T P_p S_p\|^2}{\underline{\lambda}(Q_p)} + n\|S_p^T P_p S_p\|\right)\|\Delta_k\|^2.
\end{aligned}$$

Using (29) we obtain

$$(x^*_{k+1})^T P_p x^*_{k+1} - (x^*_k)^T P_p x^*_k \le -\alpha_{1,p}\|x^*_k\|^2 + \beta_{1,p}E_k^2.$$

By definition of V_p we now have

$$\begin{aligned}
V_p(x^*_{k+1}, E_{k+1}) &= (x^*_{k+1})^T P_p x^*_{k+1} + \rho_p E_{k+1}^2 \\
&\le (x^*_k)^T P_p x^*_k - \alpha_{1,p}\|x^*_k\|^2 + \beta_{1,p}E_k^2 + \rho_p \frac{\Lambda_p^2}{N^2}E_k^2 \\
&\le (x^*_k)^T P_p x^*_k - \frac{\alpha_{1,p}}{n\overline{\lambda}(P_p)}(x^*_k)^T P_p x^*_k + \left(\frac{\beta_{1,p}}{\rho_p} + \frac{\Lambda_p^2}{N^2}\right)\rho_p E_k^2 \\
&\le \nu_p V_p(x^*_k, E_k) \le \nu V_p(x^*_k, E_k)
\end{aligned}$$

as claimed.

A.2. Proof of Lemma 2

We have

$$\begin{aligned}
&V_q(x^*_{k+1}, E_{k+1}) = (x^*_{k+1})^T P_q x^*_{k+1} + \rho_q E_{k+1}^2 \\
&\quad \le n\overline{\lambda}(P_q)\|x^*_{k+1}\|^2 + \rho_q E_{k+1}^2 \le n\overline{\lambda}(P_q)h_{pq}^2\Bigg(\|x^*_k\| \\
&\qquad + \frac{N-1}{N}E_k\Bigg)^2 + \rho_q(\alpha_{2,pq}\|x^*_k\| + \beta_{2,pq}E_k)^2 \\
&\quad \le 2n\overline{\lambda}(P_q)h_{pq}^2\|x^*_k\|^2 + 2n\overline{\lambda}(P_q)h_{pq}^2\left(\frac{N-1}{N}\right)^2 E_k^2 \\
&\qquad + 2\rho_q\alpha_{2,pq}^2\|x^*_k\|^2 + 2\rho_q\beta_{2,pq}^2E_k^2 = \big(2n\overline{\lambda}(P_q)h_{pq}^2 \\
&\qquad + 2\rho_q\alpha_{2,pq}^2\big)\|x^*_k\|^2 + \Bigg(2n\overline{\lambda}(P_q)h_{pq}^2\left(\frac{N-1}{N}\right)^2 \\
&\qquad + 2\rho_q\beta_{2,pq}^2\Bigg)E_k^2 \le \frac{2n\overline{\lambda}(P_q)h_{pq}^2 + 2\rho_q\alpha_{2,pq}^2}{\underline{\lambda}(P_p)}(x^*_k)^T P_p x^*_k \\
&\qquad + \left(\frac{2n\overline{\lambda}(P_q)h_{pq}^2}{\rho_p}\left(\frac{N-1}{N}\right)^2 + \frac{2\rho_q\beta_{2,pq}^2}{\rho_p}\right)\rho_p E_k^2 \\
&\quad \le \mu_{pq}V_p(x^*_k, E_k) \le \mu V_p(x^*_k, E_k)
\end{aligned}$$

as claimed.

A.3. Proof of Lemma 3

The average dwell time property (2) implies that

$$N_\sigma(t_{k_0}, t_k) \le N_0 + \frac{k-k_0}{m}$$

for every m such that

$$\tau_a \ge m\tau_s. \tag{48}$$

We know from (3) that $N_\sigma(t_{k_0}, t_k)$ equals the number of intervals of the form $(t_\ell, t_{\ell+1}]$, $k_0 \le \ell \le k-1$ which contain a switch (among the total number $k-k_0$ of such intervals). Combining the conclusions of Lemmas 1 and 2, we have the following bound for all $k \ge k_0$:

$$\begin{aligned}
V_{\sigma(t_k)}(x^*_k, E_k) &\le \mu^{N_0+\frac{k-k_0}{m}}\nu^{k-k_0-N_0-\frac{k-k_0}{m}}V_{\sigma(t_{k_0})}(x^*_{k_0}, E_{k_0}) \\
&= \left(\frac{\mu}{\nu}\right)^{N_0}\left(\mu^{1/m}\nu^{(m-1)/m}\right)^{k-k_0}\rho_{\sigma(t_{k_0})}E_{k_0}^2
\end{aligned}$$

(since $x^*_{k_0} = 0$). We need to ensure that $\theta := \mu^{1/m}\nu^{(m-1)/m} < 1$, which is equivalent to

$$m > 1 + \frac{\log\mu}{\log\frac{1}{\nu}}$$

and the claim follows.

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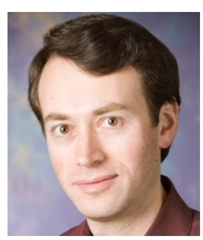

Daniel Liberzon was born in the former Soviet Union in 1973. He did his undergraduate studies in the Department of Mechanics and Mathematics at Moscow State University from 1989 to 1993.

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