Quasi-ISS Reduced-Order Observers and Quantized Output Feedback

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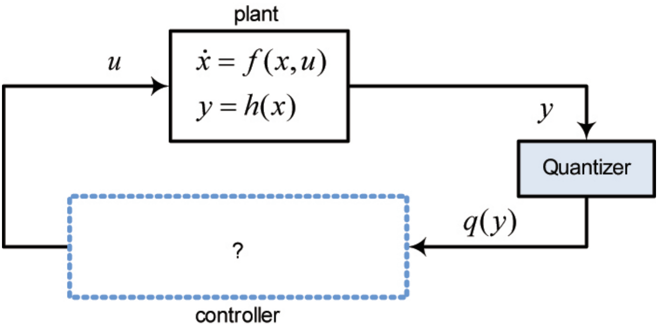
Quantized Output Feedback



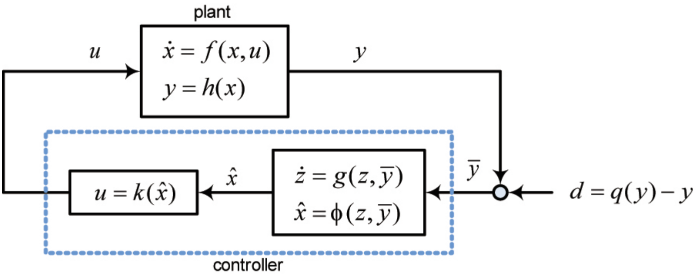
Motivation:

- limited communication between sensor and controller
- large quantization error with low-cost communication

Objective:

- design output feedback stabilizers robust to quantization errors

ISS Approach of [Liberzon, 08]



IF

ISS observer:

$$|\hat{x}(t) - x(t)| \leq \max\{\beta_o(|\hat{x}(0) - x(0)|, t), \gamma_o(\|d\|_{[0,t]})\}$$

ISS controller:

$$|x(t)| \leq \max\{\beta_c(|x(0)|, t), \gamma_c(\|\hat{x} - x\|_{[0,t]})\}$$

THEN

$$\limsup_{t\to\infty} |x(t)| \leq \gamma(\Delta)$$

where $\Delta \geq |d(t)|$

robust to quantization error!

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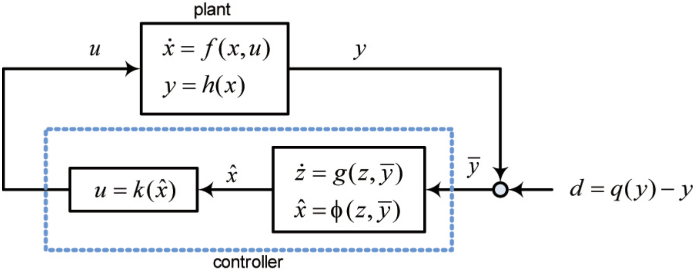

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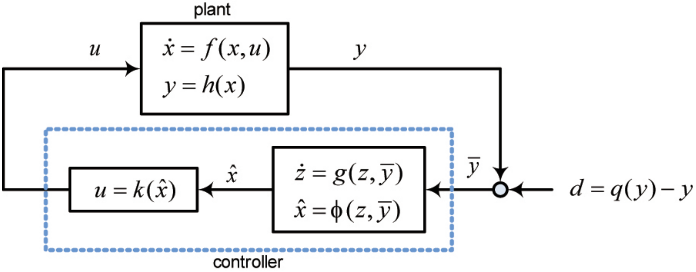

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Quasi-ISS Observer

ISS observers hardly exist for nonlinear systems

plant:

$$\dot{x} = -x + x^2 u, \qquad y = x$$

observer:

$$\dot{\hat{x}} = -\hat{x} + \bar{y}^2 u \qquad \text{with} \quad \bar{y} = y + d$$

With $\tilde{x} := \hat{x} - x$, $\dot{\tilde{x}} = -\tilde{x} + 2xud + ud^2$
$\limsup_{t\to\infty} |\tilde{x}(t)|$ depends on size of x and u as well as d

Definition: Quasi-ISS Observer

$\exists \beta_o \in \mathcal{KL}$, $\gamma_{o,K} \in \mathcal{K}_\infty$ depending on K, s.t.

$$|\hat{x}(t) - x(t)| \leq \max\{\beta_o(|\hat{x}(0) - x(0)|, t), \gamma_{o,K}(\|d\|_{[0,t]})\}$$

whenever $\|u\|_{[0,t]} \leq K$ and $\|x\|_{[0,t]} \leq K$

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Construction of Quasi-ISS Observer:

plant: $$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u)\end{bmatrix}, \qquad y = x_1 \in \mathbb{R}^p, \quad x_2 \in \mathbb{R}^{n-p}$$

with $\xi := x_2 + L(x_1)$,

$$\dot{\xi} = f_2(x_1, \xi - L(x_1), u) + \frac{\partial L}{\partial x_1}(x_1) f_1(x_1, \xi - L(x_1), u) =: F(x_1, \xi, u)$$

Assumption: L & V

$\exists\, L : \mathbb{R}^p \to \mathbb{R}^{n-p}$ and $V : \mathbb{R}^{n-p} \to \mathbb{R}$ such that

$$\text{(a)}\ \alpha_1(|e|) \le V(e) \le \alpha_2(|e|), \ \left|\frac{\partial V}{\partial e}(e)\right| \le \alpha_4(|e|),$$

$$\text{(b)}\ \frac{\partial V}{\partial e}(e)\Big(\Big[f_2(x_1, e + x_2, u) + \frac{\partial L}{\partial x_1}(x_1) f_1(x_1, e + x_2, u)\Big] - \Big[f_2(x_1, x_2, u) + \frac{\partial L}{\partial x_1}(x_1) f_1(x_1, x_2, u)\Big]\Big) \le -\alpha_3(|e|),$$

$$\text{(c)}\ \alpha(s)\alpha_4(s) \le \alpha_3(s)$$

where α and α_i, $i = 1, \ldots, 4$, are class-$\mathcal{K}_\infty$.

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plant: $\begin{bmatrix} \dot{x}_1 \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} f_1(x_1, \xi - L(x_1), u) \\ F(x_1, \xi, u) \end{bmatrix}, \qquad y = x_1$

Quasi-ISS (Reduced-order) Observer:

$$\dot{z} = f_2(\bar{y}, z - L(\bar{y}), u) + \frac{\partial L}{\partial x_1}(\bar{y}) f_1(\bar{y}, z - L(\bar{y}), u) = F(\bar{y}, z, u)$$

$$\hat{x}_1 = \bar{y}$$

$$\hat{x}_2 = z - L(\bar{y}) \qquad \bar{y} = y + d = x_1 + d$$

Existence of c and ρ: from [Freeman & Kokotovic, 93/96]

$$|F(x_1 + d, \xi, u) - F(x_1, \xi, u)| \leq c(x_1, \xi, u)\rho(|d|), \qquad \rho \in \mathcal{K}$$

Robustness of $e := z - \xi$:

$$\dot{V}(e) = \frac{\partial V}{\partial e}\{F(\bar{y}, z, u) - F(\bar{y}, \xi, u)\} + \frac{\partial V}{\partial e}\{F(\bar{y}, \xi, u) - F(y, \xi, u)\}$$
$$\leq -\alpha_3(|e|) + \alpha_4(|e|)c(x_1, \xi, u)\rho(|d|)$$

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Robustness of $e := z - \xi$ (cont.):

$$\dot{V}(e) \le -\alpha_3(|e|) + \alpha_4(|e|)c(x_1, \xi, u)\rho(|d|)$$

$$< 0 \quad \text{if } \alpha_3(|e|) \ge \alpha(|e|)\alpha_4(|e|) > \alpha_4(|e|)c(x_1, x_2 + L(x_1), u)\rho(|d|)$$

$\Rightarrow$ if $\|x\|_{[0,t]} \le K$ and $\|u\|_{[0,t]} \le K$

$$|e(t)| \le \max\{\bar{\beta}(|e(0)|, t), \bar{\gamma}_K(\|d\|_{[0,t]})\}$$

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$$|\tilde{x}| \le |d| + |e| + \theta_K(|d|), \qquad |e| \le |\tilde{x}_2| + \theta_K(|d|)$$

$\theta_K \in \mathcal{K}$, parameterized by K

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Robust Redesign of Existing Nonlinear Observers

- how to find such $L(x_1)$ and $V(e)$?
- what is the class of systems that admit Quasi-ISS observers?
- I already have a nonlinear observer. Can I make it robust to measurement disturbance?

Lemma: Quadratic Error Lyapunov Function

IF the system admits a full-order observer

$$\dot{\hat{x}} = g(y, \hat{x}, u) \qquad \hat{x} \in \mathbb{R}^n$$

whose convergence is verified with a quadratic ELF:

$$W(\tilde{x}) = \frac{1}{2}\tilde{x}^T P \tilde{x}, \quad \tilde{x} := \hat{x} - x, \quad P = P^T > 0, \quad \dot{W} \le -k_1|\tilde{x}|^2$$

THEN $\exists$ such $L(x_1)$ and $V(e)$ of the Assumption.

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Existence of $L(x_1)$ and $V(e)$: motivated by [Shim & Praly, 03]

$$W = \frac{1}{2}[\tilde{x}_1^T, \tilde{x}_2^T] \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

$$\dot{W} = [\tilde{x}_1^T, \tilde{x}_2^T] \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} g_1(y,\hat{x},u) - f_1(x,u) \\ g_2(y,\hat{x},u) - f_2(x,u) \end{bmatrix} \leq -k_1|\tilde{x}|^2$$

since $g(y,\hat{x},u) = f(\hat{x},u) = f(x_1,\hat{x}_2,u)$ when $\tilde{x}_1 = 0$

$$\dot{W}_{\tilde{x}_1=0} = \tilde{x}_2^T P_3[(f_2(x_1,\hat{x}_2,u) - f_2(x_1,x_2,u)) \\ + P_3^{-1}P_2(f_1(x_1,\hat{x}_2,u) - f_1(x_1,x_2,u))] \leq -k_1|\tilde{x}_2|^2$$

$\Rightarrow$ Take $L(x_1) = P_3^{-1}P_2x_1$ and $V(e) = \frac{1}{2}e^TP_3e$.

This Lemma applies, e.g., to

- Linearized error dynamics by (Krener & Isidori, 83)
- Nonlinear observer by (Tsinias, 89)
- High-gain observers by (Gauthier et al., 92), (Khalil, 99)
- Circle criterion observer by (Arcak & Kokotovic, 01)

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Closed-loop Robustness

Assumption: ISS Controller

$\exists$ ISS controller $u = k(x)$ such that

$$\dot{x} = f(x, k(\hat{x})) = f(x, k(x + \tilde{x}))$$
$$\Rightarrow \quad |x(t)| \le \max\{\beta_c(|x(0)|, t), \gamma_c(\|\tilde{x}\|_{[0,t]})\}$$

Consult with [Freeman & Kokotovic '93, '96, Freeman '97, Fah '99, Jiang et al. '99, Sanfelice & Teel '05, Ebenbauer, Raff & Allgower '07, '08] for design.

Quasi-ISS from measurement disturbance: by the cascade ISS argument

$$\left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \le \max\left\{\beta\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, t\right), \gamma_K(\|d\|_{[0,t]})\right\}$$

if $\|u\|_{[0,t]} \le K$, $\|x\|_{[0,t]} \le K$

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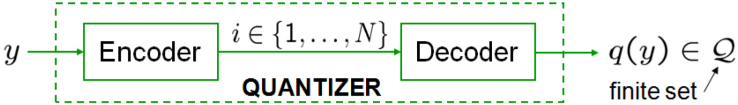
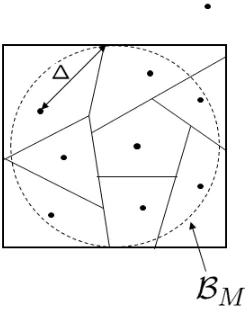
Quantizer



Output space is divided into quantization regions

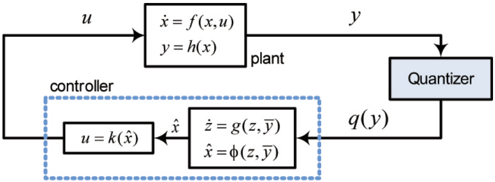
Assume $\exists M, \Delta > 0$ such that

1. $|y| \leq M \Rightarrow |q(y) - y| \leq \Delta$
2. $|y| > M \Rightarrow |q(y)| > M - \Delta$ (saturate)



M is the range, Δ is the quantization error bound

Quantized Output Feedback



$$\left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \leq$$

$$\max\left\{\beta\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, t\right), \gamma_K(\|d\|_{[0,t]})\right\}$$

where $d = q(y) - y$

if $\|u\|_{[0,t]} \leq K$, $\|x\|_{[0,t]} \leq K$

Let K, Δ and M such that

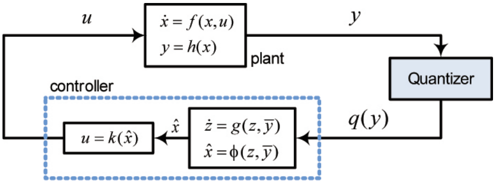
$$K \geq \bar{K}(\Delta, M), \quad \gamma_K(\Delta) < M$$ (they always exist with small Δ)

Closed-loop Robustness to Quantization:

$$\beta\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, 0\right) < M \quad \Rightarrow \quad \begin{cases} \|u\|_{[0,t]} \leq K, \quad \|x\|_{[0,t]} \leq K \\ \limsup_{t\to\infty} \left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \leq \gamma_K(\Delta) \end{cases}$$

Bounds of K, Δ, M can be improved by "zooming" strategy [Liberzon,03].

Quantized Output Feedback



$$\left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \leq$$
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$$K \geq \bar{K}(\Delta, M), \quad \gamma_K(\Delta) < M \quad \text{(they always exist with small } \Delta\text{)}$$

Closed-loop Robustness to Quantization:

$$\beta\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, 0\right) < M \quad \Rightarrow \quad \begin{cases} \|u\|_{[0,t]} \leq K, \quad \|x\|_{[0,t]} \leq K \\ \limsup_{t\to\infty} \left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \leq \gamma_K(\Delta) \end{cases}$$

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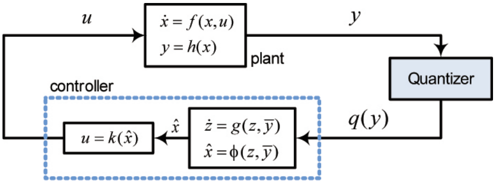
Quantized Output Feedback



$$\left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \le$$

$$\max\left\{\beta\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, t\right), \gamma_K(\|d\|_{[0,t]})\right\}$$

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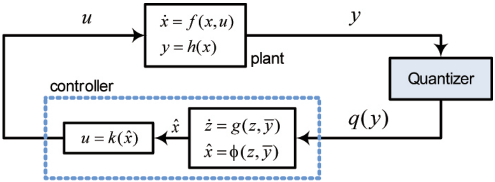
Quantized Output Feedback



$$\left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \le \max\left\{\beta\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, t\right), \gamma_K(\|d\|_{[0,t]})\right\}$$

where $d = q(y) - y$

if $\|u\|_{[0,t]} \le K,\ \|x\|_{[0,t]} \le K$

Let K, Δ and M such that

$$K \ge \bar{K}(\Delta, M), \quad \gamma_K(\Delta) < M$$ (they always exist with small Δ)

Closed-loop Robustness to Quantization:

$$\beta\left(\left|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right|, 0\right) < M \quad \Rightarrow \quad \begin{cases} \|u\|_{[0,t]} \le K, \quad \|x\|_{[0,t]} \le K \\ \limsup_{t\to\infty} \left|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right| \le \gamma_K(\Delta) \end{cases}$$

Bounds of K, Δ, M can be improved by "zooming" strategy [Liberzon,03].

Example

plant:

$$\dot{x}_1 = x_1 + 2x_2 + 4x_2^3 + 2u$$
$$\dot{x}_2 = x_2^3 + u, \qquad y = x_1$$

quasi-ISS observer:
with $L(x_1) = -\frac{1}{4}x_1$, $V(e) = \frac{1}{2}e^2$

$$\dot{z} = -\frac{1}{4}\bar{y} - \frac{1}{2}(z + \frac{1}{4}\bar{y}) + \frac{1}{2}u$$
$$\hat{x}_1 = \bar{y}, \qquad \hat{x}_2 = z + \frac{1}{4}\bar{y}$$

ISS controller:
from [Ebenbauer et al., 07]

$$u = k(x) = -x_1 - x_2 - x_2^3$$

$\Rightarrow$ the plant is stabilized by quantized output feedback

$$\dot{z} = -\frac{1}{4}q(y) - \frac{1}{2}(z + \frac{1}{4}q(y)) + \frac{1}{2}u$$
$$u = -q(y) - (z + \frac{1}{4}q(y)) - (z + \frac{1}{4}q(y))^3$$