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On Invertibility of Switched Linear Systems

L. Vu and D. Liberzon
Coordinated Science Laboratory
Department of Electrical and Computer Engineering
Univ. of Illinois at Urbana-Champaign
Urbana, IL 61801, U.S.A.
Email: {linhvu, liberzon}@control.csl.uiuc.edu

Abstract— **We address a new problem—the invertibility problem for continuous-time switched linear systems, which is the problem of recovering the switching signal and the input uniquely given an output and an initial state. In the context of hybrid systems, this corresponds to recovering the discrete state and the input from partial measurements of the continuous state. In solving the invertibility problem, we introduce the concept of singular pairs for two systems. We give a necessary and sufficient condition for a switched system to be invertible, which says that the subsystems should be invertible and there should be no singular pairs. When all the subsystems are invertible, we present an algorithm for finding switching signals and inputs that generate a given output in a finite interval when there is a finite number of such switching signals and inputs.**

I. INTRODUCTION

Switched systems—systems that comprise a family of dynamical subsystems together with a switching signal determining the active system—arise in many situations, both as a result of controller design, such as in switching supervisory control [5], and inherently by nature, such as when a physical plant has the capability of undergoing several operational modes (*e.g.*, an aircraft during different flying modes [4]).

In this paper, we address the *invertibility problem* for switched systems, which concerns with the following question: *What is the condition on the subsystems of a switched system so that, given an initial state x_0 and the corresponding output y generated with some switching signal σ and input u, we can recover the switching signal σ and the input u uniquely?* The aforementioned problem is in the same vein with the classic invertibility problem for non-switched linear systems, where one wishes to recover the input uniquely knowing the initial state and the output. The invertibility problem for non-switched linear systems has been studied extensively and completely solved, first by Brockett-Mesarovic [1], then with other algebraic criteria and the inversion constructions by Silverman [9], [10] and Sain-Massey [8], and also a geometric criterion by Morse-Wonham [6] (see also the discussions and the references in [7]). However, the invertibility problem for switched systems has not been investigated and it is the subject of this paper.

On the one hand, non-switched systems can be seen as switched systems with constant switching signals. In this regard, the invertibility problem for switched systems is an extension of the non-switched counterpart in the sense that we have to recover the switching signal in addition to the input, based on the output and the initial state. On the other hand, switched systems can be viewed as higher-level abstractions of hybrid systems. Recovering the switching signal for switched systems is equivalent to *mode identification* for hybrid systems. The mode identification problem for *hybrid systems with inputs* using known inputs and outputs has been studied, for example, in [3], [2], and for *switched systems without inputs* using the outputs in [11], [12]. Here, the difference is that we wish to do both *mode detection* and *input recovery* at the same time using the outputs of *switched systems with inputs*. Thus, the invertibility problem for switched systems can be seen as a nontrivial extension of the classic invertibility problem to switched systems as well as an extension of the mode identification problem for hybrid systems to the case with unknown inputs.

Our approach to the invertibility problem share some flavors with the approach used in the switching observability problem that was formulated and solved by Vidal *et. al.* [12], in which the objective is to recover the switching signal and the state uniquely from the output of a switched system *without inputs*. The basic idea is to do mode identification by utilizing relationship among the outputs and the states of the subsystems. For non-switched systems without inputs, this relationship is characterized via the observability matrix, which was used to solve the switching observability problem in [12]. For non-switched systems with inputs, the relationship among the output, the input, and the state is much more complicated and is realized using the structure algorithm [9], with the help of which our results for switched systems are subsequently developed.

II. PRELIMINARIES

Denote by C^n the set of n times differentiable functions; C^0 are continuous functions. Denote by $\mathcal{F}^{\mathrm{pc}}$ the set of *piecewise right-continuous functions*. Denote by $\mathcal{F}^n$ the subset of $\mathcal{F}^{\mathrm{pc}}$ whose elements are n times differentiable between two consecutive discontinuities. For $u : \mathcal{D} \to \mathbb{R}^n$, u_Q is the restriction of u onto a set $Q \subseteq \mathcal{D}$.

A *switched linear system* is written as

$$\Gamma_\sigma : \begin{cases} \dot{x} = A_\sigma x + B_\sigma u, \\ y = C_\sigma x + D_\sigma u, \end{cases} \tag{1}$$

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where $\sigma : [0,\infty) \to \mathcal{P}$ is a switching signal that indicates the active subsystem at every time, $\mathcal{P}$ is some index set, and A_p, B_p, C_p, D_p, $p \in \mathcal{P}$, are the matrices of the subsystems. A *switching signal* is a piecewise right-continuous function that has a finite number of discontinuities, which we call *switching times*, on every bounded time interval, and takes a constant value on every interval between two consecutive switching times. Denote by σ^p the switching signal such that $\sigma(t) = p \; \forall t \geq 0$. Denote by $\mathcal{S}$ the set of all admissible switching signals. Denote by $\Gamma_{p,x_0}(u)$ the trajectory of the subsystem with index p with input u starting at x_0 and by $\Gamma^{O}_{p,x_0}(u)$ the corresponding output.

Assume that all the subsystems live in the same state space $\mathbb{R}^n$ and there are no state jumps at switching times. We assume that the output dimensions of all the subsystems are the same since detecting switchings between two subsystems with different output dimensions is trivial. The input dimensions of the subsystems can be different. Due to space limitation, we neglect to define input sets of switched systems rigourously[1] with the understanding that inputs of the switched system (1) are concatenations of the corresponding inputs of the active subsystems.

Define the *concatenation map* $\oplus : \mathcal{F}^{\text{pc}} \times \mathcal{F}^{\text{pc}} \to \mathcal{F}^{\text{pc}}$ as

$$(f_1 \oplus f_2)(t) := \begin{cases} f_1(t), & \text{if } t \in [t_1, \tau_1), \\ f_2(t_2 + t - \tau_1), & \text{if } t \in [\tau_1, \tau_1 + \tau_2 - t_2), \end{cases}$$

where the domains of f_i are $[t_i, \tau_i)$, $i = 1,2$ (τ_i can be ∞). Note that $f_1 \oplus f_2 = f_1 \; \forall f_2$ if $\tau_1 = \infty$, and in general, $f_1 \oplus f_2 \neq f_2 \oplus f_1$. The concatenation of a function f and a set S is $f \oplus S := \{f \oplus g, \; g \in S\}$. By convention, $f \oplus \emptyset = \emptyset \; \forall f$. The concatenation of two sets S and T is $S \oplus T := \{f \oplus g, \; f \in S, g \in T\}$; by convention, $S \oplus \emptyset = \emptyset$, $\emptyset \oplus S = \emptyset \; \forall S$.

III. Invertibility of LTI systems and the structure algorithm

Consider a non-switched linear system

$$\Gamma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, and $y \in \mathbb{R}^\ell$. The invertibility problem[2] for the system (2) concerns with finding conditions on A, B, C, D so that for a given initial state x_0, the input-output map $H_{x_0} : \mathcal{U} \to \mathcal{Y}$ is one-to-one where $\mathcal{U}$ is the set of C^0 functions and $\mathcal{Y}$ is the set of corresponding outputs.

For a number $a \leq \ell$, define $\bar{I}_a := [I_{a\times a} \; 0_{a\times(\ell-a)}]$, $\widetilde{I}_a := [0_{a\times(\ell-a)} \; I_{(\ell-a)\times(\ell-a)}]$ and $M_a := \begin{bmatrix} \bar{I}_a \\ \widetilde{I}_{\ell-a}(d/dt) \end{bmatrix}$. Denote by $\mathcal{M}^{\ell\times\ell}$ the set of non-singular $\ell \times \ell$ matrices.

The structure algorithm *Let $q_0 = rank(D)$. $\exists S_0 \in \mathcal{M}^{\ell\times\ell}$ such that $\widetilde{I}_{q_0} S_0 D = 0$. Let $D_0 := S_0 D$ and $\overline{D}_0 := \bar{I}_{q_0} D_0$. Let $y_0 = S_0 y$ and $C_0 = S_0 C$. Then $y_0 = C_0 x + D_0 u$. Note that $rank(\overline{D}_0) = q_0$ and $\widetilde{I}_{q_0} D_0 = 0$. Suppose that at step k, we have $y_k = C_k x + D_k u$, where D_k is such that $rank(\bar{I}_{q_k} D_0) = q_k$ and $\widetilde{I}_{q_k} D_k = 0$. Let $\overline{C}_k := \bar{I}_{q_k} C_k$ and $\widetilde{C}_k := \widetilde{I}_{q_k} C_k$. If $q_k < m$, then $M_{q_k} y_k = \begin{bmatrix} \overline{C}_k \\ \widetilde{C}_k A \end{bmatrix} x + \begin{bmatrix} \overline{D}_k \\ \widetilde{C}_k B \end{bmatrix} u$. Let $q_{k+1} = rank\left(\begin{bmatrix} \overline{D}_k \\ \widetilde{C}_k B \end{bmatrix}\right)$. Then $\exists S_{k+1} \in \mathcal{M}^{\ell\times\ell}$ such that $\widetilde{I}_{q_{k+1}} S_{k+1} \begin{bmatrix} \overline{D}_k \\ \widetilde{C}_k B \end{bmatrix} = 0$. Let $C_{k+1} := S_{k+1} \begin{bmatrix} \overline{C}_k \\ \widetilde{C}_k A \end{bmatrix}$, $D_{k+1} := S_{k+1} \begin{bmatrix} \overline{D}_k \\ \widetilde{C}_k B \end{bmatrix}$, and $\overline{D}_{k+1} := \bar{I}_{q_{k+1}} D_{k+1}$. Let $y_{k+1} := S_{k+1} M_{q_k} y_k$. Then $y_{k+1} = C_{k+1} x + D_{k+1} u$ and we can repeat the procedure.*

It was shown in [10] that $\exists$ a smallest integer $\alpha \leq n$ such that $q_k = q_\alpha \; \forall k \geq \alpha$. The system is invertible iff [3] $q_\alpha = m$.

Let $N_k := \prod_{i=0}^{k} S_{k-i} M_{q_{k-i-1}}$, $(M_{-1} := I)$. Then $y_k = N_k y$. Let $\overline{N}_k := \bar{I}_{q_k} N_k$, $\widetilde{N}_k := \widetilde{I}_{q_k} N_k$, $\tilde{y}_k = \widetilde{N}_k y$, and $\bar{y}_k = \overline{N}_k y$. We have $\begin{bmatrix} \widetilde{N}_0 \\ \vdots \\ \widetilde{N}_k \end{bmatrix} y = \begin{pmatrix} \tilde{y}_0 \\ \vdots \\ \tilde{y}_k \end{pmatrix} = \begin{bmatrix} \widetilde{C}_0 \\ \vdots \\ \widetilde{C}_k \end{bmatrix} x =: L_k x \; \forall k$ by virtue of $\tilde{y}_k = \widetilde{C}_k x$. Silverman and Payne have shown in [10] that $\exists$ a smallest number β, $\alpha \leq \beta \leq n$, such that $\text{rank}(L_k) = \text{rank}(L_\beta) \; \forall k \geq \beta$. Also, $\exists$ a number δ, $\beta \leq \delta \leq n$ such that $\widetilde{C}_\delta = \sum_{i=0}^{\delta-1} P_i \left(\prod_{j=i+1}^{\delta} \widetilde{R}_j\right) \widetilde{C}_i$ for some matrices $\widetilde{R}_j$ related to the structure algorithm and some constant matrices P_i (see [10, p.205] for detail). The number δ is not easily determined as α and β but the significance of δ is that it can be used to characterize the set of all outputs of a linear system as in the range theorem [10, Theorem 4.3], which we paraphrase in Lemma 1 below. Let $\widetilde{R}_{a,b} := \prod_{j=a}^{b} \widetilde{R}_j$ and define the differential operators $\mathbf{M}_1 := \left(\frac{d^\delta}{dt^\delta} - \sum_{i=0}^{\delta-1} P_i \frac{d^i}{dt^i}\right) \widetilde{R}_{0,\alpha}$, $\mathbf{M}_2 := \sum_{j=0}^{\delta} \widetilde{R}_{j+1,\alpha} K_j \frac{d^{\delta-1}}{dt^{\delta-j}} - \sum_{j=0}^{\delta-1} P_j \sum_{k=0}^{j} \widetilde{R}_{k+1,\alpha} K_k \frac{d^{j-k}}{dt^{j-k}}$ where K_i are matrices related to the structure algorithm. Let $\mathbf{N} := [\widetilde{N}_0^T \ldots \widetilde{N}_{\beta-1}^T]^T$ and $L := L_{\beta-1}$. Denote by $\widehat{\mathcal{Y}}$ the set of functions $f : \mathcal{D} \to \mathbb{R}^\ell \; \forall \mathcal{D} \subseteq [0,\infty)$ such that $N_\delta f \in C^0$, $(\mathbf{M}_1 - \mathbf{M}_2 \overline{N}_\alpha) f \equiv 0$. The notation $|_{t^+}$ means "evaluating the limit as $s \downarrow t$".

Lemma 1: *For a linear system Γ, using the structure algorithm on the system matrices, construct a set $\widehat{\mathcal{Y}}$ of functions and a differential operator $\mathbf{N} : \widehat{\mathcal{Y}} \to C^0$ and a matrix L. There exists $u \in C^0$ such that $y = \Gamma_{x_0}(u)$ iff $y \in \widehat{\mathcal{Y}}$ and $\mathbf{N}y|_{t_0^+} = Lx_0$ where t_0 is the initial time.*

Remark 1: A special case is $m = p = q_\alpha$ (input and output dimensions are the same and the system is invertible). Then $\mathbf{M}_1 = \mathbf{M}_2 = 0$ and thus, C^δ functions are always in $\widehat{\mathcal{Y}}$. ▷

Roughly speaking, the set $\widehat{\mathcal{Y}}$ characterizes C^0 functions that can be generated by the system from all initial states (in some sense, $\mathbf{M}_1$, $\mathbf{M}_2$ capture the coupling among output components). The condition $\mathbf{N}y|_{t_0^+} = Lx_0$ guarantees that the particular y can be generated starting from x_0 at time t_0. We use $\mathbf{N}y|_{t_0^+}$ to reflect that y does not need to be defined for $t < t_0$ (which is useful later for switched systems where inputs and outputs can be piecewise right-continuous).

The differential operator $\mathbf{N}$ in Lemma 1 is to deal with

[1] which would require the concept of "hybrid functions" whose segments are functions not necessarily of the same dimension, and also, the concept of compatible input functions and switching signals.

[2] more precisely, the left invertibility property; from here onward, when we say invertibility we mean left invertibility.

[3] iff is the abbreviation for "if and only if".

y that may not be differentiable but $\mathbf{N}y$ exists. From the construction of $\mathbf{N}$, $\mathbf{N}$ has the following form:

$$\mathbf{N}y := N_0 y + \frac{d}{dt}(N_1 y + \ldots + \frac{d}{dt}(N_{\beta-2} y + \frac{d}{dt} N_{\beta-1} y)), \quad (3)$$

where N_i are $\ell \times \ell$ matrices resulting from the structure algorithm (more details are in the journal version of the paper). We then have the following lemma.

Lemma 2: Consider the linear system (2). Let β be the number and $\mathbf{N}$ be the differential operator as described in the structure algorithm. For any $\kappa \geq \beta$, for every y such that Ny exists and is continuous, we have $\mathbf{N}y = N_0 y + \frac{d}{dt}(N_1 + \cdots + \frac{d}{dt}(N_{\kappa-2} y + \frac{d}{dt} N_{\kappa-1} y))$, where $N_i, 0 \leq i \leq \beta - 1$, are as in (3) and $N_i = 0, i \geq \beta$.

IV. Invertibility of switched systems

THE INVERTIBILITY PROBLEM. *Consider the map* $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$ *for the switched system* (1)*. Find a set* $\mathcal{Y}$ *and a condition on the subsystems, independent of* x_0*, such that the map* H_{x_0} *is one-to-one.*

Recall that $\mathcal{S}$ is the set of all possible switching signals. We do not specify what the set $\mathcal{U}$ is but instead, we specify the set $\mathcal{Y}$ and then $\mathcal{U}$ will be the corresponding set of inputs that together with $\mathcal{S}$ generate $\mathcal{Y}$. The domains of $\mathcal{Y}$, $\mathcal{S}$, and $\mathcal{U}$ are the same and can be $[0,\infty)$ or any interval.

We say that $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$ is *invertible at* y if $\mathrm{H}_{x_0}(\sigma_1, u_1) = \mathrm{H}_{x_0}(\sigma_2, u_2) = y \Rightarrow \sigma_1 = \sigma_2, u_1 = u_2$ (similarly for non-switched systems, $\mathrm{H}_{x_0} : \mathcal{U} \to \mathcal{Y}$ is invertible at y if $\mathrm{H}_{x_0}(u_1) = \mathrm{H}_{x_0}(u_2) = y \Rightarrow u_1 = u_2$). H_{x_0} is *invertible on* $\mathcal{Y}$ if it is invertible at y $\forall y \in \mathcal{Y}$. The switched system is *invertible on* $\mathcal{Y}$ if H_{x_0} is invertible on $\mathcal{Y}$ $\forall x_0$.

There is a major difference between the maps H_{x_0} for non-switched systems and for switched systems. The former is a linear map. The latter is a nonlinear map on $\mathcal{S} \times \mathcal{U}$, of which $\mathcal{S}$ is not a linear space. The map H_{x_0} for non-switched LTI systems has the nice property that if H_{x_0} is invertible at y for one pair (x_0, y), then the map H_{x_0} is invertible on the set of all possible outputs generated by continuous inputs and for all x_0. In contrast, for switched systems, uniqueness of (σ, u) for one pair (x_0, y) does not imply uniqueness for other pairs. A switched system may be invertible on one output set $\mathcal{Y}_1$ but not invertible on another set $\mathcal{Y}_2$ (which is not the case for non-switched systems). This situation prompts a more delicate definition of the output set $\mathcal{Y}$ for switched systems (*cf.* output sets are irrelevant for invertibility of LTI systems).

One special case is $x_0 = 0$, $y \equiv 0$. It is obvious that with $u \equiv 0$ and any switching signal, we always have $\mathrm{H}_0(\sigma, 0) = 0$ $\forall \sigma$ regardless of the subsystem dynamics, and therefore, the map H_0 is not one-to-one if the function $0 \in \mathcal{Y}$. This illustrates the issue of why we cannot take the output set $\mathcal{Y}$ to be all the possible outputs. We call those pairs (x_0, y) for which H_{x_0} is not invertible at y *singular pairs*. Fortunately, for other pairs $(x_0, y) \neq (0,0)$, invertibility of H_{x_0} at y depends on the subsystems' dynamics and properties of y.

Definition 1: Let $x_0 \in \mathbb{R}^n$ *and* $y \in C^0$*. The pair* (x_0, y) *is a* singular pair *of the two subsystems* Γ_p, Γ_q *if there exist* u_1, u_2 *such that* $\Gamma^{\mathrm{O}}_{p,x_0}(u_1) = \Gamma^{\mathrm{O}}_{q,x_0}(u_2) = y$*.*

For the subsystem indexed by p, denote by $\mathbf{N}_p$, L_p, and $\widehat{\mathcal{Y}}_p$ the corresponding objects of interest as in Lemma 1. It follows from Definition 1 and Lemma 1 that (x_0, y) is a singular pair iff $y \in \widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q$ and

$$\mathbf{N}_{p,q} y\big|_{t_0^+} = L_{p,q} x_0, \quad (4)$$

where $\mathbf{N}_{p,q} := \begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix}$, $L_{p,q} := \begin{bmatrix} L_p \\ L_q \end{bmatrix}$ and t_0 is the initial time of y. For a given (x_0, y), Eq. (4) can be directly verified since $\widehat{\mathcal{Y}}_p, \widehat{\mathcal{Y}}_q, \mathbf{N}_p, \mathbf{N}_q, L_p$, and L_q are known. Observe that $0 \in \mathrm{Im} L_{p,q}$ and we can always have (4) with $x_0 = 0$ and y such that $\mathbf{N}_{p,q} y\big|_{t_0^+} = 0$. In particular, if $y_{[t_0, t_0+\varepsilon)} \equiv 0$ and $x_0 = 0$, then (4) holds regardless of $\mathbf{N}_p, \mathbf{N}_q$, L_p, L_q; this is the only case of strong singular pairs. Apart from this case, in general, $\mathbf{N}_{p,q} y\big|_{t_0^+} = 0$ depends on $\mathbf{N}_p, \mathbf{N}_q$ and y, and it is possible to find conditions on $\mathbf{N}_p, \mathbf{N}_q, L_p, L_q$ and y so that there is no x_0 satisfying (4) if $\mathbf{N}_{p,q} y\big|_{t_0^+} \neq 0$.

A. A solution of the invertibility problem

Let $\mathcal{Y}^{\mathrm{all}}$ be the set of outputs of the system (1) generated by all possible piecewise continuous inputs and switching signals from all possible initial states (the set $\mathcal{Y}^{\mathrm{all}}$ can be seen as all the possible concatenations of all elements of $\widehat{\mathcal{Y}}_p$ $\forall p \in \mathcal{P}$). Let $\overline{\mathcal{Y}} \subset \mathcal{Y}^{\mathrm{all}}$ be the largest set of functions in $\mathcal{Y}^{\mathrm{all}}$ on the time domain $[0,\infty)$ such that if $y \in \overline{\mathcal{Y}}$ and $y_{[t_0, t_0+\varepsilon)} \in \widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q$ for some $p \neq q, p, q \in \mathcal{P}$, $\varepsilon > 0$, then $\mathbf{N}_{p,q} y\big|_{t_0^+} \neq 0$. Literally speaking, we avoid functions whose segments can form singular pairs with $x_0 = 0$. Excluding such functions from our output set, we can impose conditions on the subsystems to eliminate the possibility of singular pairs $\forall x_0 \in \mathbb{R}^n, y \in \overline{\mathcal{Y}}$. Note that the singular pair concept in Definition 1 is defined for $y \in C^0$. For switched systems, we check for singular pairs for the continuous output segments in between consecutive discontinuities at the output.

Theorem 1: Consider the switched system (1) *and the output set* $\overline{\mathcal{Y}}$*. The switched system is invertible on* $\overline{\mathcal{Y}}$ *iff all the subsystems are invertible and the subsystem dynamics are such that* $\forall x_0 \in \mathbb{R}^n, y \in \overline{\mathcal{Y}} \cap C^0$*, the pairs* (x_0, y) *are not singular pairs of* Γ_p, Γ_q $\forall p \neq q, p, q \in \mathcal{P}$*.*

Proof: (Sketched).
Sufficiency: Suppose $\mathrm{H}_{x_0}(\sigma_1, u_1) = \mathrm{H}_{x_0}(\sigma_2, u_2) = y$. Let $t_1 > 0$ be the first time such that u_1 or u_2 is discontinuous at t_1. From the nonsingular pair condition, it must be that $\sigma_1(t) = \sigma_2(t) = p$ $\forall t \in [0, t_1)$ for some $p \in \mathcal{P}$. Then $u_{1_{[0,t_1)}} = u_{2_{[0,t_1)}} = u_{[0,t_1)}$ is uniquely recovered on $[0, t_1)$ by invertibility of Γ_p. By continuity, $x(t_1) = x(t_1^-) =: x_1$. If $t_1 = \infty$, then $\sigma_1(t) = \sigma_2(t)$, $u_1(t) = u_2(t)$ $\forall t \in [0,\infty)$. Otherwise, repeat the argument with x_1 and $y_{[t_1,\infty)}$.
Necessity: Invertibility of all the subsystems can be seen by picking a constant switching signal. If there is a singular pair (x_0, y) for some $p \neq q$, then the switched system is not invertible by definition. ■

In the case the subsystem input and output dimensions are equal, we can have a rank condition for invertibility of switched systems. For an index p, let $W_p :=$

$[N_{p,0}\, N_{p,1}\, \dots\, N_{p,n-1}]$ where $N_{p,0}, \dots, N_{p,n-1}$ are the matrices as in Lemma 2 for the subsystem with index p.

Lemma 3: Consider the switched system (1) *and the output set* $\overline{\mathcal{Y}}$. *Consider the following two statements:*

S1. The subsystem dynamics are such that for all $x_0 \in \mathbb{R}^n$ *and* $y \in \overline{\mathcal{Y}}$, *the pairs* $(x_0, y_{[t_0,t_0+\varepsilon)})$ *are not singular pairs of* Γ_p, Γ_q *for all* $p \neq q, p, q \in \mathcal{P}$ *and for all* $t_0 \geq 0, \varepsilon > 0$, *such that* $y_{[t_0,t_0+\varepsilon)} \in C^0$;

S2. The subsystem dynamics are such that

$$\text{rank}\begin{bmatrix} W_p & L_p \\ W_q & L_q \end{bmatrix} = \text{rank}\begin{bmatrix} W_p \\ W_q \end{bmatrix} + \text{rank}\begin{bmatrix} L_p \\ L_q \end{bmatrix} \tag{5}$$

for all $p \neq q, p, q \in \mathcal{P}$ *such that* $\widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q \neq \{0\}$.

Then S2 $\Rightarrow$ *S1. If the subsystems are invertible and the input and output dimensions are the same, then S1* $\Rightarrow$ *S2.*

Proof of Lemma 3 is omitted due to space limitation. From Theorem (1) and Lemma 3, we arrive at the following result.

Theorem 2: Consider the switched system (1) *and the output set* $\overline{\mathcal{Y}}$. *The switched system is invertible on* $\overline{\mathcal{Y}}$ *if all the subsystems are invertible and the rank condition* (5) *holds. If the input and output dimensions of all the subsystems are the same, then invertibility of all the subsystems together with the rank condition is also necessary for invertibility of the switched system.*

When a switched system is invertible (as stated in Theorem 1), a *switched inverse system* can be constructed as follows. Define the *index inversion function* $\overline{\Sigma}^{-1} : \mathbb{R}^n \times \overline{\mathcal{Y}} \to \mathcal{P}$ as

$$\overline{\Sigma}^{-1} : (x_0, y) \mapsto p : y \in \widehat{\mathcal{Y}}_p,\, \mathbf{N}_p y|_{t_0^+} = L_p x_0, \tag{6}$$

where t_0 is the initial time of y. The function $\overline{\Sigma}^{-1}$ is well-defined since p is unique by the fact that there is no singular pair, and so, the index at every time is uniquely determined from the output y and state x. In the invertibility problem, it is assumed that y is an output so the existence of p in (6) is guaranteed. A switched inverse system is:

$$\Gamma_\sigma^{-1} : \begin{cases} \sigma(t) &= \overline{\Sigma}^{-1}(z(t), y_{[t,\infty)}), \\ \dot{z} &= (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)} z + (B\overline{D}_\alpha^{-1} N_\alpha)_{\sigma(t)}\, y, \\ u &= -(\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)}\, z + (\overline{D}_\alpha^{-1} N_\alpha)_{\sigma(t)}\, y \end{cases} \tag{7}$$

with the initial condition $z(0) = x_0$. The notation $(\cdot)_{\sigma(t)}$ denotes the object in the parentheses calculated for the subsystem with index $\sigma(t)$. The initial active subsystem $\sigma(0) = \overline{\Sigma}^{-1}(y_{[0,\infty)}, x_0)$ at $t = 0$, from which time onwards, the active system indexes and the input as well as the state are determined uniquely and simultaneously via (7).

Remark 2: Our result can also be readily extended to include the more general case of different output dimensions (the only difference is the use of "hybrid functions" for outputs instead of functions). The result in this section can also be extended to include the case where the state spaces of the subsystems are different and where there are state jumps at switching times by appropriately modifying the concept of singular pairs (taking into account different state space dimensions and jump maps). For these cases, the statements of Theorem 1 remain largely unchanged. Due to space limitation, we do not provide details here and point to later journal version of this paper. ▷

V. OUTPUT GENERATION

In the previous section, we considered the question of whether one can recover (σ, u) uniquely $\forall x_0 \in \mathbb{R}^n, y \in \mathcal{Y}$. In this section, we address a different but closely related problem which concerns with finding (σ, u) (there maybe more than one) such that $\mathrm{H}_{x_0}(\sigma, u) = y$ for a given y and x_0. For the invertibility problem, we find conditions on the subsystems and the set $\mathcal{Y}$ so that H_{x_0} is injective $\forall x_0$. Here, we are given one particular (x_0, y) and wish to find the *preimage* $\mathrm{H}_{x_0}^{-1}$ of the map H_{x_0}:

$$\mathrm{H}_{x_0}^{-1}(y) := \{(\sigma, u) : \mathrm{H}_{x_0}(\sigma, u) = y\}. \tag{8}$$

By convention, $\mathrm{H}_{x_0}^{-1}(y) = \emptyset$ if y is not in the image set of H_{x_0}. In general, $\mathrm{H}_{x_0}^{-1}(y)$ is a set for a given y (when $\mathrm{H}_{x_0}^{-1}(y)$ is a *singleton*, the map H_{x_0} is invertible at y).

Lemma 4: Suppose that the a non-switched linear system Γ is not invertible. Consider an arbitrary interval $[a, b]$. For every $u \in \mathcal{F}_{[a,b]}^{\text{pc}}$ and $x_a \in \mathbb{R}^n$, $\exists$ infinitely many different $v \in \mathcal{F}_{[a,b]}^{\text{pc}}$ such that $\Gamma_{x_a}^{\text{O}}(v) = \Gamma_{x_a}^{\text{O}}(u)$ and $\Gamma_{x_a}(v)\big|_b = \Gamma_{x_a}(u)\big|_b$.

Proof: Proof is omitted due to space limitation. ■

A corollary of Lemma 4 is that for our output generation problem, if one active subsystem is not invertible, then $\mathrm{H}_{x_0}^{-1}(y)$ will have infinite number of elements if $\mathrm{H}_{x_0}^{-1}(y) \neq \emptyset$. This motivates us to introduce the following assumption.

Assumption 1: The subsystems Γ_p *are invertible* $\forall p \in \mathcal{P}$.

We have no other assumption on the subsystem dynamics and the switched system may not be invertible as the subsystems may not satisfy the invertibility condition in the previous section. Since we look for an algorithm to find $\mathrm{H}_{x_0}^{-1}(y)$, we only consider y of finite intervals (and hence, there is a finite number of switches) to avoid infinite loop reasoning when there are infinitely many switchings. Even though the subsystems are invertible and the number discontinuities is finite, finding $\mathrm{H}_{x_0}^{-1}(y)$ is nontrivial since output discontinuities do not always imply switches. We can have a switch and y is still smooth at that switching time, and likewise, we can have no switching even if y loses continuity (because for a subsystem, the input maybe discontinuous and the output may depend directly on the input through D).

We now present a switching inversion algorithm[4] for switched systems that takes $x_0 \in \mathbb{R}^n$, $y \in \mathcal{F}_{\mathcal{D}}^{\text{pc}}$ as parameters and returns $\mathrm{H}_{x_0}^{-1}(y)$ as in (8) where $\mathcal{D}$ is a finite interval, when $\mathrm{H}_{x_0}^{-1}(y)$ is a finite set. Define the *index-matching map*[5] $\Sigma^{-1} : \mathbb{R}^n \times \mathcal{F}^{\text{pc}} \to 2^{\mathcal{P}}$ that returns the indexes of the subsystems capable of generating y starting from x_0:

$$\Sigma^{-1}(x_0, y) := \{p : y \in \widehat{\mathcal{Y}}_p,\, \mathbf{N}_p y|_{t_0^+} = L_p x_0\}, \tag{9}$$

where t_0 is the initial time of y. Note that the map Σ^{-1} in (9) is defined for every pair (x_0, y) and returns a set, whereas the index inversion function $\overline{\Sigma}^{-1}$ in (6) is defined for non-singular pairs and returns an element of $\mathcal{P}$.

[4]In the algorithm, "$\leftarrow$" reads "assigned as", and "$:=$" reads "defined as".
[5]The symbol $2^{\mathcal{P}}$ denotes the set of all subsets of a $\mathcal{P}$.

Begin of Function $\mathrm{H}_{x_0}^{-1}(y)$
Let $[t_0,T) :=$ the domain of y;
$\overline{P} := \{p \in \mathcal{P} : y_{[t_0,t_0+\varepsilon)} \in \widehat{\mathcal{Y}}_p$ for some $\varepsilon > 0\}$;
$t^* := \min\{t \in [t_0,T) : y_{[t,t+\varepsilon)} \notin \widehat{\mathcal{Y}}_p$ for some $p \in \overline{\mathcal{P}}, \varepsilon > 0\}$
  or $t^* = T$ otherwise.
$\mathcal{P}^* := \Sigma^{-1}(x_0, y_{[t_0,t^*)})$.
If $\mathcal{P}^* \neq \emptyset$, Let $\mathcal{A} := \emptyset$.
  For each $p \in \mathcal{P}^*$,
    Let $u := \Gamma_{p,x_0}^{-1,\mathrm{O}}(y_{[t_0,t^*)})$,
    $\mathcal{T} := \{t \in (t_0,t^*) : (x(t), y_{[t,t^*)})$ is
      a singular pair of Γ_p, Γ_q for some $q \neq p\}$.
    If $\mathcal{T}$ is a finite set,
      For each $\tau \in \mathcal{T}$, let $\xi := \Gamma_p(u)(\tau)$.
      $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma^p_{[t_0,\tau)}, u_{[t_0,\tau)}) \oplus H_\xi^{-1}(y_{[\tau,T)})\}$
      End For each
    Else If $\mathcal{T} = \emptyset$ and $t^* < T$, let $\xi = \Gamma_p(u)(t^*)$.
      $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma^p_{[t_0,t^*)}, u) \oplus H_\xi^{-1}(y_{[t^*,T)})\}$
    Else If $\mathcal{T} = \emptyset$ and $t^* = T$,
      $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma^p_{[t_0,T)}, u)\}$
    Else $\mathcal{A} := \emptyset$
    End If
  End For each
Else $\mathcal{A} := \emptyset$
End If
Return $H_{x_0}^{-1}(y) := \mathcal{A}$
End of Function

The switching inversion algorithm is a recursive procedure calling itself with different parameters within the main loop. There are three stopping conditions: it terminates either when $\mathcal{P}^* = \emptyset$, in which case there is no subsystem that can generate y starting from x_0 at time t_0, or when $\mathcal{T}$ is not a finite set, in which case we cannot proceed due to infinitely many possible switching times, or when $\mathcal{T} = \emptyset$ and $t^* = T$, in which case the switching signal is a constant signal.

If the return is a nonempty set, the set must be finite and contains pairs of switching signals and inputs that generate the given y starting from x_0. If the return is an empty set, it means that there is no switching signal and input that generate y, or there is an infinite number of possible switching times (it is possible to further distinguish between these two cases by using an extra variable in the algorithm that assigned different values for different cases). Notice the utilization of the concatenation notation: if at any instant of time, the return of the procedure is an empty set, then that branch of the search will be empty because $f \oplus \emptyset = \emptyset$.

VI. Examples

Example 1: Consider the two subsystems:

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x, \end{cases}, \quad \Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 2 \end{bmatrix} x. \end{cases}$$

Using the structure algorithm, we can check that Γ_1, Γ_2 are invertible. We have $\mathbf{N}_1 = W_1 = \mathbf{N}_2 = W_2 = [1]$, $L_1 = [0\ 1]$, $L_2 = [0\ 2]$. In this example, the input and output dimensions are the same. The rank condition (5) is satisfied. By Theorem 2, we conclude that the switched system generated by $\{\Gamma_1, \Gamma_2\}$ is invertible on $\overline{\mathcal{Y}}^1 := \{y \in \mathcal{F}^{\mathrm{pc}} : \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} y|_{t^+} \neq 0\ \forall t\} = \{y \in \mathcal{F}^{\mathrm{pc}} : y(t) \neq 0\ \forall t\}$.

Example 2: Consider the two subsystems:

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x, \end{cases}, \quad \Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u, \\ y = \begin{bmatrix} 1 & 2 \end{bmatrix} x. \end{cases}$$

Using the structure algorithm, we can check that Γ_1, Γ_2 are invertible and $\mathbf{N}_1 = W_1 = \mathbf{N}_2 = W_2 = [1]$, $L_1 = [0\ 1]$, $L_2 = [1\ 2]$. Since the rank condition (5) is violated, the switched systems generated by Γ_1, Γ_2 does not satisfy Theorem 2. Consider an output

$$y(t) = \begin{cases} 2e^{2t} - 3e^t, & \text{if } t \in [0,t^*), \\ c_1 e^t + c_2 e^{2t}, & \text{if } t \in [t^*,T), \end{cases}$$

where $t^* = \ln 3$, $T = \frac{6}{5}$, $c_1 = 15 + 18\ln(\frac{2}{3})$, $c_2 = -\frac{4}{3} - 4\ln(\frac{2}{3})$ and the initial state $x_0 = (-1,0)^T$.

We illustrate how the switching inversion algorithm works. In this case, the input and output dimensions are the same so smooth functions are always in $\widehat{\mathcal{Y}}_1, \widehat{\mathcal{Y}}_2$. It follows that $\overline{\mathcal{P}} = \mathcal{P}$ and t^* in the algorithm is the same as t^* in the definition of y since $y_{[t,t+\varepsilon)} \notin \widehat{\mathcal{Y}}_p$ only if y is discontinuous at t. Now, $\mathcal{P}^* := \Sigma^{-1}(x_0, y_{[0,t^*)}) = \{2\}$ by using (9) with x_0, $y(0) = -1$. An inverse of Γ_2 (by the structure algorithm) is

$$\Gamma_2^{-1} : \begin{cases} \dot{z} = \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \dot{y}, \\ u(t) = \dot{y} - \begin{bmatrix} -1 & 4 \end{bmatrix} x, \end{cases} \quad t \in [0,t^*)$$

with $z(0) = x_0$, which yields

$$z(t) = \begin{pmatrix} -e^t \\ -e^t + e^{2t} \end{pmatrix} =: \bar{x}_2(t),\ u(t) = 0,\ t \in [0,t^*). \qquad (10)$$

We find $\mathcal{T} = \{t \le t^* : (x(t), y_{[t,t^*)})$ is a singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving $W_1 y(t) = L_1 \bar{x}_2(t), t \in [0,t^*) \Leftrightarrow 2e^{2t} - 3e^t = x_2(t) = -e^t + e^{2t}$, $t \in [0,t^*)$, which has a solution $t = \ln 2 =: t_1$. Thus, $\mathcal{T} = \{t_1\}$, which is a finite set. We repeat the procedure for $\xi = x(t_1) = (-2,2)^T$ and $y_{[t_1,T)}$. Now, $\mathcal{P}^* = \Sigma^{-1}(\xi, y_{[t_1,t^*)}) = \{1,2\}$.

Case 1: $p = 1$. An inverse system of Γ_1 is

$$\Gamma_1^{-1} : \begin{cases} \dot{z} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{y}, \\ u = \dot{y} - \begin{bmatrix} 0 & -1 \end{bmatrix} z, \end{cases}$$

with the initial state $z(t_1) = \xi$, which yields

$$\begin{aligned} z(t) &= \begin{pmatrix} (-13 + 6\ln 2)e^t + 6e^{2t} - 6te^t \\ 2e^{2t} - 3e^t \end{pmatrix} =: \bar{x}_1(t), \\ u(t) &= 6e^{2t} - 6e^t, \end{aligned} \quad t \ge t_1.$$

We find $\mathcal{T} = \{t \in (t_1,t^*] : (x(t), y_{[t,t^*)})$ is a singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving $W_2 y(t) = 2e^{2t} - 3e^t = L_2 \bar{x}_1(t) = (-19 + 6\ln 2)e^t + 10e^{2t} - 6te^t$, $t_1 < t \le t^*$. It can be checked that the foregoing equation does not have a solution. Repeating the procedure with $\xi = \bar{x}_1(t^*) = (15 + 18\ln(\frac{2}{3}), 9)$ and $y_{[t^*,T)}$, we get the solution $\sigma = \sigma^2_{[t^*,T)}$, $u_{[t^*,T)} = 0$.

Case 2: $p = 2$. This case means that t_1 is not a switching time. Then $u(t) = 0$ up to time t^* by the structure algorithm, and hence, $x(t) = \begin{pmatrix} -e^t \\ -e^t + e^{2t} \end{pmatrix}$, $\tau \le t \le t^*$, in view of (10). We repeat the procedure with $\xi = x(t^*) = (-3, 6)$ and $y_{[t^*,T)}$. We have $y(t^*) = 33 + 18\ln(2/3)$. Since $L_1\xi \notin W_1 y(t^*)$ and $L_2\xi \notin W_2 y(t^*)$, we get $\Sigma^{-1}(\xi, y_{[t^*,T)}) = \emptyset$.

The switching inversion algorithm returns $\{(\sigma, u)\}$, where

$$\sigma(t) = \sigma^2_{[0,t_1)} \oplus \sigma^1_{[t_1,t^*)} \oplus \sigma^2_{[t^*,T)},$$

$$u(t) = \begin{cases} 0, & \text{if } 0 \le t < t_1, \\ 6e^{2t} - 6e^t, & \text{if } t_1 \le t < t^*, \\ 0, & \text{if } t^* \le t \le T. \end{cases}$$

We see that there is a switching at t_1 whilst the output is smooth at t_1. Without the concept of singular pairs, one might falsely conclude that there is no switching signal and input after trying all obvious combinations of the switching signals (*i.e.* $\sigma = \sigma^i_{[0,t^*)} \oplus \sigma^j_{[t^*,T)}$, $i, j \in \{1, 2\}$). Here, we can recover the switching signals and the inputs (uniquely in this case). This clearly demonstrates the usefulness of the singular pair concept.

VII. The invertibility problem for discrete-time switched systems

In this section, we outline the invertibility problem for discrete-time switched systems, which is currently under further investigation by the authors. The continuous-time case and discrete-time case are different as we will explain below. Consider a *discrete-time switched system*

$$\Gamma_\sigma : \begin{cases} x[k+1] = A_{\sigma[k]}x[k] + B_{\sigma[k]}u[k], \\ y[k] = C_{\sigma[k]}x[k] + D_{\sigma[k]}u[k]. \end{cases} \quad (11)$$

We assume that the the individual subsystems live in the same state space $\mathbb{R}^n$. The discrete-time switching signal σ is a function $\sigma : \{0, 1, 2, \dots\} \to \mathcal{P}$; switching times are k such that $\sigma(k) \ne \sigma(k-1)$, $k \ge 1$. Unlike the case of continuous-time switching signals, we do not have further restrictions on σ because a discrete-time signal already implies that there can only be finitely many switches in any finite interval. We assume that there is no jump at switching times. Denote by $f_{[a:b]}$ the restriction of a discrete-time function f on an interval $[a : b]$. While invertibility for the continuous-time case is defined as injectivity of the switching signal $\times$ input-output map H_{x_0}, invertibility for the discrete-time case is defined with a delay as follows:

PROBLEM *For the discrete-time switched system* (11)*, find a condition on the matrices* A_p, B_p, C_p, D_p, $p \in \mathcal{P}$, *a set* $\mathcal{S}$ *of switching signals, a set* $\mathcal{U}$ *of inputs, and a set* $\mathcal{Y}$ *of outputs such that* $\mathrm{H}_{x_0}(\sigma_{1[0:k]}, u_{1[0:k]}) = \mathrm{H}_{x_0}(\sigma_{2[0:k]}, u_{2[0:k]}) = y_{[0:k]} \Rightarrow \sigma_{1[0:k-d]} = \sigma_{2[0:k-d]}, u_{1[0:k-d]} = u_{2[0:k-d]}\ \forall x_0 \in \mathbb{R}^n, y \in \mathcal{Y}, \sigma \in \mathcal{S}, u \in \mathcal{U}, k \ge d$ *for some number* $d \ge 0$.

Compared to the continuous-time case, the invertibility problem for discrete-time switched systems differs from the former in that the latter requires specifying the set $\mathcal{S}$ and the delay d. For continuous-time non-switched systems, an output in an infinitesimally small interval is completely determined by the input in the same interval. Carrying over to switched systems, this property enables us to identify switching times and the active subsystem index by comparing the subsystem dynamics using the output in an infinitesimally small interval. For discrete-time switched systems, note that if every subsystem is invertible with 0 delay, then we can have a result similar to the continuous-time case using the current output sample. However, in general, for a discrete-time non-switched system, an output in an interval is determined by the input in a shorter interval. This delay behavior of the individual subsystems makes it difficult to determine switching times and the active subsystem index of a switched system. Suppose that for the subsystems, we need α sample delay to recover the input uniquely. For the switched system, there will be output segments with length α (around switching times) such that in that segments, there will be mixing of more than one dynamics. Since the switching signal is not known, it is not clear how to use that output segment to recover the input (and the switching signal).

VIII. Conclusion

We have formulated a new problem, namely, the invertibility problem for switched systems. We introduced the concept of singular pairs and presented a necessary and sufficient condition for invertibility of continuous-time switched linear systems. For continuous-time switched linear systems, not necessarily invertible, with invertible subsystems, we provided an algorithm that finds switching signals and inputs that generate a given output with given initial state. Future research direction is to investigate invertibility of discrete-time switched linear systems and switched nonlinear systems.

References

[1] R. W. Brockett and M. D. Mesarovic. The reproducibility of multivariable control systems. *J. Math. Anal. Appl.*, 11:548–563, 1965.

[2] G. Ferrari-Trecate, M. Muselli, D. Liberati, and M. Morari. A clustering technique for the identification of piecewise affine systems. *Automatica*, 39(2):205–217, 2003.

[3] K. Huang, A. Wagner, and Y. Ma. Identification of hybrid linear time-invariant systems via subspace embedding and segmentation. In *Proc. 43st IEEE Conf. on Decision and Control*, volume 3, pages 3227–3234, 2004.

[4] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3):349–370, 1999.

[5] A. S. Morse. Supervisory control of families of linear set-point controllers-part 1: exact matching. *IEEE Trans. Automat. Control*, 41(10):1413–1431, 1996.

[6] A. S. Morse and W. M. Wonham. Status of noninteracting control. *IEEE Trans. Automat. Control*, 16(6):568–581, 1971.

[7] P. J. Moylan. Stable inversion of linear systems. *IEEE Trans. Automat. Control*, 22(1):74–78, 1977.

[8] M. K. Sain and J. L. Massey. Invertibility of linear time-invariant dynamical systems. *IEEE Trans. Automat. Control*, 14:141–149, 1969.

[9] L. M. Silverman. Inversion of multivariable linear systems. *IEEE Trans. Automat. Control*, 14(3):270–276, 1969.

[10] L. M. Silverman and H. J. Payne. Input-output structure of linear systems with application to the decoupling problem. *SIAM J. Control*, 9(2):199–233, 1971.

[11] R. Vidal, A. Chiuso, and S. Soatto. Observability and identification of jump linear systems. *Proc. 41st IEEE Conf. on Decision and Control*, 41:3614–3619, 2002.

[12] R. Vidal, A. Chiuso, S. Soatto, and S. Sastry. Observability of linear hybrid systems. In *Hybrid Systems: Computation and Control*, volume 2623, pages 526–539. Springer-Verlag, 2003.