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Input-to-state stabilization with minimum number of quantization regions

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***Abstract*— We study control systems where the state measurements are quantized and time-sampled, and an unknown disturbance is being applied. We present a dynamic quantization scheme that switches between three modes of operation. We show that by using this scheme with a continuous static feedback controller we achieve a closed-loop system which has the Input-to-State Stability property (ISS). Our design does not use any characterization of the disturbance; as long as the disturbance is bounded the system will remain stable. We show that three quantization regions per dimension is sufficient to achieve the ISS property, and furthermore we show that the ISS property is achievable using a data rate that is arbitrarily close to the minimum required data rate when no disturbance is applied.**

I. INTRODUCTION

The effect of quantization on control systems has been a research subject for several works published since the late 80s, and especially in the last decade. By quantization we refer both to space quantization and to time sampling. By space quantization we mean that each measurement can only have a finite set of different values. By time sampling we mean that the measurement is only sampled once every certain time interval, and not continuously. Quantization can result from the technical properties of the sensors used to measure the state of the system. It can also result from limitations on the data rate that can be transmitted between the sensors and the controller.

One can neglect the quantization, and design a stabilizing controller assuming continuous state measurement is available. However, if the quantization regions are fixed over time, then even if the continuous controller is designed to make the system globally asymptotically stable (GAS), due to quantization the closed-loop system will at most be only locally practically stable. See [1] and [2] for more details.

If the quantization regions do change dynamically as the system evolves, then, as is shown in the several papers referenced below, it is possible to make the closed-loop system GAS. This paper is based on a minimum data rate approach which was presented for linear disturbance-free systems in [3],[4]. The approach was extended to nonlinear systems in [5] and [6], and disturbances were dealt with in [7], [8], [9] and [10]. The basic scheme of this approach, for disturbance-free systems, is as follows. Once the state is known to be in a bounded region, calculate the region where the state will be in the next sampling time, and position the quantizer to cover just this region. If there are enough quantization regions compared to the system growth rate, then the size of the quantization regions will be reduced every sampling time, and the state estimation error will converge to zero. To get the initial bounded region of where the state is, the size of the region covered by the quantizer needs to be enlarged each sampling time at a rate higher than the system growth rate.

The simple scheme described in the previous paragraph can achieve global asymptotic stability, but it may fail (for example the system might escape to infinity) if a disturbance is introduced. The GAS property is not applicable in the presence of a disturbance, and so we chose the Input-to-State Stability (ISS) property, first introduced in [11], as a natural extension of the GAS property to systems with a disturbance. In this paper we will show how to dynamically position the quantizer so that the closed-loop system will have the ISS property. We mentioned that several previous papers did deal with disturbances; however, some specifications of the disturbance were needed in the design of the quantization schemes, and it is possible that using their schemes the system will become unstable if the disturbance digresses from this specification. Our scheme regards the disturbance as completely unknown, and as long as the disturbance is finite, the state will remain bounded. Furthermore, once the disturbance vanishes the state will asymptotically converge to the origin.

The only paper we know of that showed how to achieve the ISS property with respect to a completely unknown disturbance in the presence of quantization is [12][1]. However, that paper used a different approach which assumes the quantizer can be changed dynamically in a more limited way than the approach we follow. That approach also appeared previously in [13] and [14] but for disturbance-free systems. While this other approach may be easier to implement in some scenarios because less flexibility is needed from the quantizer, the approach we follow uses fewer quantization regions. Using fewer quantization regions leads to a lower data rate, and in some other scenarios where using more

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[1] In [7] and [8], state boundedness in the presence of bounded disturbances is achieved by using the knowledge of a disturbance bound. In [9], mean square stability in the stochastic setting is obtained by utilizing statistical information about the disturbance. In [10] only stability in probability is proved, not ISS.

quantization regions makes the system more complex it may actually lead to a simpler implementation. We shall mention that although the two approaches solve the same problem (but under different conditions), the control design and analysis in this paper are significantly different from those in [12].

The paper is organized as follow. In §II we give a formal definition of the system, how the measurement is being quantized, and what is our goal (ISS). In §III we describe how the parameters of the quantization should be changed dynamically, and under what conditions our scheme will achieve ISS. Our main theorem is presented in that section. In §IV we give a more intuitive explanation of our scheme. In §V we give a brief discussion on how to choose the design parameters. In §VI we compare the minimum required data rate for disturbance-free systems as given in previous published results and the minimum required data rate for our scheme, and show that they can be made arbitrarily close. We extend our results to non-linear systems in §VII and give concluding remarks in §VIII. The formal proof of the validity of our scheme is omitted from this paper due to paper length limitations, but will be available on our website (see the appendix for the link) until a complete journal version will be published. The main steps of the proof, however, are presented in the appendix.

II. Definitions

In this paper we will use the ∞-norm unless otherwise specified. For vectors, $|\boldsymbol{x}| \doteq |\boldsymbol{x}|_\infty \doteq \max_i |x_i|$. For signals, $\|\boldsymbol{w}\|_{[0,t]} \doteq \sup_{\tau\in[0,t]} |\boldsymbol{w}(\tau)|_\infty$ and $\|\boldsymbol{w}\| \doteq \|\boldsymbol{w}\|_{[0,\infty)}$. For matrices, $\|M\| \doteq \max_{\boldsymbol{x}} \frac{|M\boldsymbol{x}|}{|\boldsymbol{x}|} \equiv \max_i \left(\sum_j |M_{ij}|\right)$.

The system we want to stabilize is described by the usual linear equation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) + D\boldsymbol{w}(t) \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the control input, and $\boldsymbol{w}(t) \in \mathbb{R}^l$ is an unknown bounded, measurable disturbance. We assume that (A,B) is a stabilizable pair, so there exists a K such that using $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ the system will be driven to zero when no disturbance is applied.

We use the concept of Input-to-State Stability (ISS), which was first introduced in [11], to describe the goal of our design. A system is said to be ISS if the norm of the system can be bounded as follows:

$$|\boldsymbol{x}(t)| \le \beta\left(|\boldsymbol{x}_0|, t\right) + \gamma_w\left(\|\boldsymbol{w}\|_{[0,t]}\right), \qquad \forall t \ge 0 \tag{2}$$

where γ_w is a class $\mathcal{K}_\infty$ function[2] and β is a class $\mathcal{KL}$ function[3].

Fix an odd integer $N \ge 3$. The quantizer we are assumed to be given is one that assigns N different labels for each

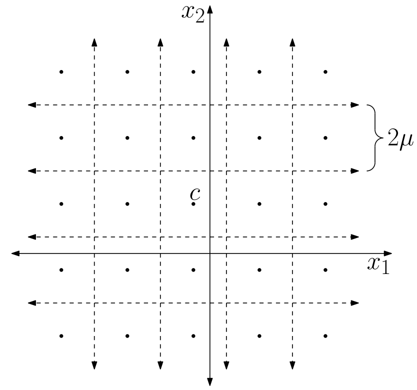

Fig. 1. Illustration of the quantizer for the two-dimensional state space, $N = 5$. The dashed lines define the boundaries of the quantization regions. The black dots define where the quantizer estimates the state to be, given the index of the quantization region that currently contains the state. Note that the quantizer output gives a displacement from $\boldsymbol{c}$.

dimension of the state. The center point $\boldsymbol{c} \in \mathbb{R}^n$ of the quantizer can be dynamically defined by the controller, together with a zoom factor $\mu \in \mathbb{R}_{\ge 0}$. The quantizer Q, which takes values in a finite subset of $\mathbb{R}^n$, is defined mathematically below and is also illustrated in Figure 1.

$$Q_i\left(\boldsymbol{x}; \boldsymbol{c}, \mu\right) \doteq \begin{cases} (-N+1)\mu & x_i - c_i \le (-N+2)\mu \\ (-N+3)\mu & (-N+2)\mu < x_i - c_i \le (-N+4)\mu \\ \vdots & \vdots \\ 0 & -\mu < x_i - c_i \le \mu \\ \vdots & \vdots \\ (N-3)\mu & (N-4)\mu < x_i - c_i \le (N-2)\mu \\ (N-1)\mu & (N-2)\mu < x_i - c_i \end{cases} \tag{3}$$

Note that if $|\boldsymbol{x} - \boldsymbol{c}| \le N\mu$ then the measurement error, $|\boldsymbol{x} - Q\left(\boldsymbol{x}; \boldsymbol{c}, \mu\right)|$, is bounded by μ. We refer to the (bounded) regions for which $\forall i \; Q_i\left(\boldsymbol{x}; \boldsymbol{c}, \mu_k\right) \ne \pm(N-1)\mu_k$ as the inner regions, and to all the other (unbounded) regions as the outer regions.

The design of the quantizer will be under the constraint that a measurement (sampling) of the state using the device implementing the described quantizer can only be taken every T_s seconds. Using this sampling time interval we define the open-loop maximum growth rate:

$$\Lambda \doteq \left\|\exp\left(AT_s\right)\right\|.$$

We are now ready to describe the design of the controller using this quantizer.

III. Quantized Controller Design

The controller will operate in either of three modes: *zoom-out*, *zoom-in/measurement update* or *zoom-in/escape detection*, where the initial mode will be *zoom-out*. The controller will also use $\hat{\boldsymbol{x}} \in \mathbb{R}^n$, $\mu \in \mathbb{R}$ and $\rho \in \mathbb{N}$ as auxiliary variables. The variable $\hat{\boldsymbol{x}}$ will be changed continuously between sampling times, and "abruptly" at the sampling times. If

[2]A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and also unbounded

[3]A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \ge 0$.

kT_s is a sampling time, then $\hat{\boldsymbol{x}}^-(kT_s) \doteq \lim_{\tau \nearrow kT_s} \hat{\boldsymbol{x}}(\boldsymbol{\tau})$ is the value of $\hat{\boldsymbol{x}}$ just before the abrupt change. We initialize $\hat{\boldsymbol{x}}(t_0) = \mathbf{0}$. The variable μ will be changed only at sampling times, and we will use the notation $\mu_k \doteq \mu(kT_s)$. Its initial value, μ_0, can be any positive value, and will be regarded as a design parameter. The controller will also have four other design parameters: $\alpha > 0$, $s > 0$, $\Omega_{\text{out}} > \Lambda$, $P \in \mathbb{N}$. The effect of the choice of the parameters on the performance of the controller will be discussed in §V. Define

$$\Omega_{\text{in,m}} \doteq \frac{\Lambda}{N-\alpha} \qquad \Omega_{\text{in,e}} \doteq \frac{\Lambda+\alpha}{N-2}. \tag{4}$$

The control law will be as follows:

1) Between sampling times:

$$\boldsymbol{u}(t) = K\hat{\boldsymbol{x}}(t)$$

$$\dot{\hat{\boldsymbol{x}}}(t) = A\hat{\boldsymbol{x}}(t) + B\boldsymbol{u}(t)$$

2) At sampling times ($t = kT_s$):

a) In *zoom-out* mode:
If $\forall i \quad Q_i\left(\boldsymbol{x}(t); \hat{\boldsymbol{x}}^-(t), \mu_k\right) \neq \pm(N-1)\mu_k$
Then update $\rho = P$,

$$\hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{x}}^-(t) + Q\left(\boldsymbol{x}(t); \hat{\boldsymbol{x}}^-(t), \mu_k\right), \tag{5}$$

$$\mu_{k+1} = \Omega_{\text{in,m}}\mu_k \tag{6}$$

and switch to *zoom-in/measurement update* mode
Else update

$$\mu_{k+1} = \Omega_{\text{out}}\mu_k, \qquad \hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{x}}^-(t)$$

b) In *zoom-in/measurement update* mode:
Update $\hat{\boldsymbol{x}}$ using (5) and also update $\rho = \rho - 1$.
If $\rho = 0$
Then update

$$\mu_{k+1} = \Omega_{\text{in,e}}\mu_k \tag{7}$$

and switch to *zoom-in/escape detection* mode
Else update μ using (6).

c) In *zoom-in/escape detection* mode:
If $\forall i \quad Q_i\left(\boldsymbol{x}(t); \hat{\boldsymbol{x}}^-(t), \mu_k\right) \neq \pm(N-1)\mu_k$
Then update $\hat{\boldsymbol{x}}$ using (5), μ using (6), update $\rho = P$, and switch to *zoom-in/measurement update* mode
Else update

$$\mu_{k+1} = \frac{s}{N-2}, \qquad \hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{x}}^-(t) \tag{8}$$

and switch to *zoom-out* mode.

We are now ready to state our main result.

Theorem 1: Consider the system (1) with the quantizer described in §II and assume a state feedback gain, K, is given such that $(A+BK)$ is a Hurwitz matrix. Applying the controller described above, with any choice of design parameters such that

$$0 < \Omega_{\text{in,m}}^P \Omega_{\text{in,e}} \equiv \frac{\Lambda^P(\Lambda+\alpha)}{(N-\alpha)^P(N-2)} < 1 \tag{9}$$

holds, will make the closed-loop system Input-to-State Stable with respect to the disturbances, i.e. there exist a class $\mathcal{KL}$ function β_{cl} and a class $\mathcal{K}_\infty$ function γ_{cl} such that the following holds:

$$|\boldsymbol{x}(t)| \leq \beta_{cl}\left(|\boldsymbol{x}_0|, t\right) + \gamma_{cl}\left(\|\boldsymbol{w}\|_{[0,t]}\right). \tag{10}$$

Note that a necessary and sufficient condition for existence of design parameters for which (9) holds is that $\Lambda^{P+1} < N^P(N-2)$, and since $\lim_{T_s \to 0}\Lambda = 1$ this condition can always be achieved if the sampling times are frequent enough or N is large enough.

IV. Explanation and Intuitive Justification of the Proposed Controller Design

In this section we will describe in words the algorithm presented in the previous section, and will try to give the intuition as to why this specific design achieves the desired properties.

The controller uses the state estimate, $\hat{\boldsymbol{x}}$, and the feedback gain for the unquantized system, K, to calculate the control input. If we define $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ to be the estimation error, then the system can be written as:

$$\dot{\boldsymbol{x}}(t) = (A+BK)\boldsymbol{x}(t) - BK\boldsymbol{e}(t) + D\boldsymbol{w}(t). \tag{11}$$

Because $(A+BK)$ is Hurwitz, the system (11) is ISS with respect to both the estimation error, $\boldsymbol{e}(t)$, and the disturbance, $\boldsymbol{w}(t)$. See [15, §4.9] for more details.

Between sampling times, the state estimation error is propagated as:

$$\begin{aligned}\dot{\boldsymbol{e}} &= \dot{\boldsymbol{x}} - \dot{\hat{\boldsymbol{x}}} = A\boldsymbol{x} + BK\hat{\boldsymbol{x}} + D\boldsymbol{w} - (A\hat{\boldsymbol{x}} + BK\hat{\boldsymbol{x}}) \\ &= A\boldsymbol{e} + D\boldsymbol{w}.\end{aligned} \tag{12}$$

The important observation from (12) is that the dynamics of the estimation error between the sampling times are independent of the state. The proposed controller design preserves this independency at the sampling times – knowledge of the estimation error but not of the state just before a sampling time is sufficient to determine the estimation error immediately after the sampling time. Thus, if we show that using the proposed controller makes the relation between the disturbance and the estimation error ISS, then we can use the cascade theorem [11, Proposition 7.2] to show that the closed-loop system is ISS.

Note that the quantization regions can be divided into inner and outer regions. The inner regions are between $(-N+2)\mu$ and $(N-2)\mu$ in all dimensions; each inner region is a bounded box whose size is 2μ. The outer regions are unbounded. The basic logic of the above controller is to zoom out until the state falls inside an inner region (and so the measurement error can be bounded). Once the state does fall inside an inner region, the controller switches to a sequence of measurement updates and escape detections. In the *measurement update* mode all the quantization regions are used to improve the estimation. However, this is done under the assumption that the estimation error is less than $N\mu$. By covering the space in which the state is expected to be with only the inner regions, the *escape detection* mode is used to verify that this assumption still holds, and also to improve the estimation when $N > 3$. If it is found that the assumption no longer holds (due to the disturbance), the controller switches back to *zoom-out*.

The distinction between the two *zoom-in* modes is essential to achieve the minimum number of quantization regions (or minimum data rate). To see this, take for example the extreme, but perhaps the most interesting, case where $N = 3$. In this case the estimation improvement is done solely in the *measurement update* mode, and not in the *escape detection* mode. However the *escape detection* mode is still essential to make the system robust to the disturbance. Between sampling times, without a disturbance, the estimation error will always grow at most by the factor of Λ; however, to make the system ISS, the controller must assume that the error will grow even more due to the disturbance. The size of a disturbance that will drive the estimation error to one of the outer regions in the *escape detection* mode is proportional to the current zoom factor μ where the proportionality constant is determined by α. Therefore, the controller will switch to the *zoom-out* mode only when the estimation error is already small enough compared to the disturbance. This is essential for achieving the ISS property.

By taking $\Omega_{\text{out}} > \Lambda$ we are guaranteed that for any bounded disturbance, we will eventually catch the state during the *zoom-out*. The condition (9) is needed in order to have μ converge to zero in the *zoom-in* sequence. This condition is equivalent to saying that the amount of information we receive during each sequence of P *zoom-in/measurement updates* and one *zoom-in/escape detection*, which is determined by $N^P(N-2)$, is large enough compared to the error's growth rate, which is determined by Λ^{P+1}, and the "slack" we give for the disturbance, given by α.

Figure 2 shows a simulation that visually illustrates the behavior of the controller.

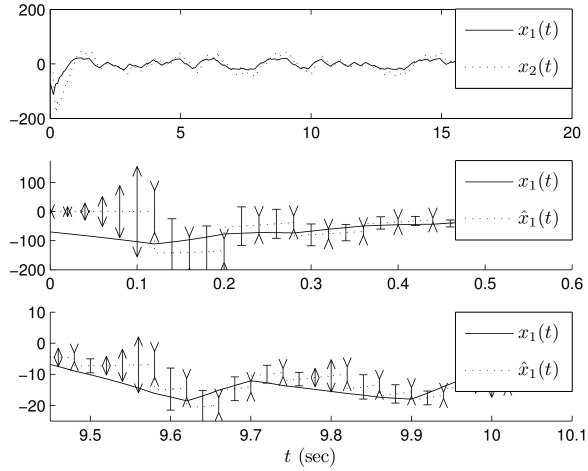

Fig. 2. A simulation of using the described controller on a 2-dimensional, open-loop unstable, system, $N = 3$. The top chart shows the state of the system over 20 seconds of simulation. The middle chart focuses on the first second of the simulation to show the initial transient. The bottom chart focuses on the 10th second to show the steady state behavior. Only one dimension of the state and its estimate are shown in the middle and bottom charts. The vertical lines indicate the (single) inner region of the quantizer. The lines with arrows pointing outward correspond to *zoom-out*, the lines with arrows pointing inward correspond to *zoom-in/measurement update*, and the lines with horizontal boundaries correspond to *zoom-in/escape detection*. Simulation parameters: $A = [4, -3; 2, -1]$, $B = [-1; 4]$, $K = [4.0294, -1.7827]$, $T_s = 0.02$, $\alpha = 0.5$, $\Omega_{out} = \sqrt{3}$, $s = 3$, $\mu_0 = 10$, $\boldsymbol{x}_0 = [-70; -10]$, $\|\boldsymbol{w}\| = 50$.

V. Sensitivity to Design Parameters

Several design parameters are used by the controller. Any choice for the design parameters will render the closed-loop system ISS as long as (9) holds. However, different choices will result in a different ISS gain and overshoot, and may also affect performance measures which are not expressed by the ISS definition, such as energy gain and data rate. By ISS gain we refer to the γ function in the ISS definition, and by overshoot we refer to the β function in the ISS definition. A gain or overshoot will be smaller (or bigger) if it is smaller (or bigger) for any chosen bound on the disturbance, any initial condition and for all $t \geq 0$.

The parameter α expresses the sensitivity of the system to the disturbance and it is bounded from above via (9). The ISS gain and the overshoot will decrease as α is increased. However, increasing α will slow down the convergence of the system in the *zoom-in* sequence. By taking more quantization regions we may use a larger α but that will also require higher data rate. The parameter P will have similar effects: higher P will result in faster convergence and smaller data rate, but the ISS gain and overshoot will be increased.

The parameters μ_0, s, and Ω_{out} have more complicated effect on the ISS gain and the overshoot, and their optimal values, when the goal is to minimize the ISS gain or the overshoot, depend on the characteristics of the disturbance and the initial condition. The choice of μ_0 will depend on the expected magnitude of the initial condition, and it will affect the overshoot only. The choice of s will depend on the expected magnitude of the disturbance and it will affect the ISS gain only. Last, the choice of Ω_{out} will depend on the expected deviation of the initial condition and the disturbance from their expected values. None of these three design parameters will affect the data rate.

VI. Approaching the Minimal Data Rate

As mentioned in the introduction, several papers ([3],[7],[8],[9],[10]) present (the same) lower bound on the data rate necessary to stabilize a given system. In all these cases, including ours, the lower bound on the necessary data rate is independent of the disturbance characteristics, and is derived only from the open-loop characteristics. However, while the minimum rate required is independent of the disturbance, the control design in all previous cases does depend on some of the characteristics of the disturbance[4].

[4]An exception is [10] which does suggest a method that does not depend on any characteristics of the disturbance. However, the method does not achieve the ISS property

Our paper is unique since we show how to achieve the ISS property while approaching the minimum rate, and do so without using any knowledge on the characteristics of the disturbance.

The bound on the minimum data rate, in terms of the bit-rate that needs to be transmitted (R), is:

$$R > \frac{\sum_{|\eta_j|\geq 1} \log_2 |\eta_j|}{T_s} \tag{13}$$

where $\eta_j's$ are the eigenvalues of the discrete open-loop matrix $\Phi \doteq \exp(AT_s)$. The data rate in our scheme is given by $\log_2 (N^n)/T_s$, where N is bounded from below via (9). The value Λ in (9) measures the ∞-norm growth of the open-loop disturbance-free system. The choice of the ∞-norm came from the specific rectangular shape of our quantizer. Assuming A is diagonizable, we can apply the quantizer separately on each independent mode, and allocate a different amount of quantization regions for each mode based on its growth rate. The 2-norm growth rate of each mode is $e^{T_s \mathfrak{Re}\lambda(A_l)} \equiv |\eta_j|$, where $\mathfrak{Re}\lambda(A_l)$ is the real part of the eigenvalues of mode l, and η_j is any corresponding discrete eigenvalue. While for modes whose eigenvalues have non zero imaginary part the ∞-norm is in general bigger then the 2-norm, we can rotate the orientation of the quantizer and use the 2-norm to determine the change in zoom-factor between two sampling times. This is explained in [8].

Overall the minimum data rate required for our scheme is

$$R = \sum_{|\eta_j|\geq 1} \log_2\left(\tilde{N}_j\right)/T_s \tag{14}$$

where $\tilde{N}_j$ is the smallest odd integer such that

$$N_j^P (N_j - 2) > |\eta_j|^{(P+1)} \tag{15}$$

holds. We arrived at (15) using the fact that α can be made arbitrarily small without compromising the ISS property. If we had $\tilde{N}_j = |\eta_j|$ then we would achieve (13). As is done in [8] we can remove the restriction to odd integers in (15) by using different number of quantization regions at each sampling times, and then measure the average data rate. We can also take P to be large enough so that the "-2" term, which is the penalty we need to pay to handle the disturbance, becomes negligible. Thus, we can apply our scheme using a data rate which is arbitrarily close to the minimum data rate (13).

VII. Extension to nonlinear systems

The crucial properties of linear systems which will be used in the proof of Theorem 1 are (a) that the continuous, unquantized, closed-loop system is ISS with respect to the estimation error and the disturbance, and (b) that the estimation error grows independently of the state as described by (12). Both properties hold not only for linear systems, and specifically for any nonlinear system,

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}), \tag{16}$$

there exists an upper bound on the estimation error that grows independently of the state, if the system has the Lipschitz property: Given $l_x > 0$ and $l_w > 0$ there exist L_x and L_w such that

$$|f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) - f(\hat{\boldsymbol{x}}, \boldsymbol{u}, 0)| \leq L_x |\boldsymbol{x} - \hat{\boldsymbol{x}}| + L_w |\boldsymbol{w}| \tag{17}$$

$\forall |\boldsymbol{x}| < l_x, \forall |\hat{\boldsymbol{x}}| < l_x, \forall |\boldsymbol{w}| < l_w$. Note that for linear systems the Lipschitz property holds globally - there exist $L_x < \infty$ and $L_w < \infty$ which work for $l_x = \infty$ and $l_w = \infty$. With this property we have

$$\left|\boldsymbol{e}^-((k+1)T_s)\right| \leq \bar{\Lambda} \left|\boldsymbol{e}(kT_s)\right| + \bar{W}_{kT_s} \tag{18}$$

where $\bar{\Lambda} = e^{T_s L_x}$, $\bar{W}_t \doteq \int_0^{T_s} e^{(T_s - \tau)L_x} L_w |\boldsymbol{w}(t+\tau)|\, d\tau$, which can be further bounded using $\bar{W}_t \leq \bar{\Gamma} \|\boldsymbol{w}\|_{[t, t+Ts]}$, $\bar{\Gamma} \doteq \| \int_0^T e^{(T_s - \tau)L_x} L_w \|$.

Thus following the proof of Theorem 1 we can arrive at the following theorem, which uses an (ϵ, δ) ISS (local-ISS) definition [16, §4] because the Lipschitz property may not hold globally:

Theorem 2: Consider a system (16) which has the Lipschitz property (17), and for which there exists a static feedback with the ISS property (2). Assume l_x and l_w are given. Then applying the same controller described above for the linear system, with any choice of design parameters such that (9) holds with $\bar{\Lambda}$ in place of Λ, will make the closed-loop system (ϵ,δ) ISS:

$$|\boldsymbol{x}(t)| \leq \bar{\beta}_{cl}\left(|\boldsymbol{x}_0|, t\right) + \bar{\gamma}_{cl}\left(\|\boldsymbol{w}\|_{[0,t]}\right) \tag{19}$$

$\forall |\boldsymbol{x}(0)| \leq \delta, \forall \|\boldsymbol{w}\|_{[0,t]} \leq \epsilon$. Here $\bar{\beta}_{cl}$, $\bar{\gamma}_{cl}$ are defined as in the proof of Theorem 1, with $\bar{\Gamma}$, $\bar{\Lambda}$ replacing Γ, Λ, and ϵ, δ are any pair such that $\bar{\beta}_{cl}(\delta) + \bar{\gamma}_{cl}(\epsilon) \leq l_x$ and $\epsilon \leq l_w$.

VIII. Conclusion

In this paper we showed that it is possible to achieve input-to-state stability even in the presence of a disturbance whose bound is unknown, with as few as 3 quantization regions per dimension. We also showed that we are able to approach the minimum data rate needed to stabilize an unperturbed system, without compromising the ISS property with respect to a completely unknown disturbance. The quantizer we used should be very easy to implement in practice.

Our future work is to extend this method to handle also linear and nonlinear systems with output feedback and delays.

Appendix
Proof of Theorem 1

Because using a stabilizing state feedback gain, K, makes the system (11) ISS with respect to the estimation error (and the disturbance), all we need to show is that the described controller makes the relation between the disturbance and the estimation error ISS. The main result will then follow as we have a cascade of two ISS systems. We present the main key steps in proving the ISS property, but we omit the proofs due to paper length limitations. Complete proofs are contained in the full version of this paper which is currently available (until a journal version will be published) in our website: http://decision.csl.uiuc.edu/~ysharon/cdc07_full.pdf.

Lemma 1: If the controller is in *zoom-in/escape detection* mode at $t = (k+1)T_s$, and $\mu_k > \frac{1}{\alpha}\max_{m\in\{k-P,\dots,k\}} W_{mT_s}$, where

$$W_t \doteq \left| \int_0^{T_s} \exp\left(A(T_s-\tau)\right) D\boldsymbol{w}(t+\tau) d\tau \right|$$

then the controller does not switch to the *zoom-out* mode and

$$\|\boldsymbol{e}\|_{[(k-P)T_s,(k+1)T_s]} \le N\mu_{k-P+1}. \tag{20}$$

For the sequel we define $\Gamma \doteq \| \int_0^{T_s} \exp\left(A(T_s-\tau)\right) D\|$.

Lemma 2: Whenever the controller switches to the *zoom-out* mode after being in the *zoom-in/escape detection* mode at $t = (k_1+1)T_s$, the controller will switch back to *zoom-in/measurement update* mode before the norm of the estimation error $|\boldsymbol{e}(t)|$ reaches $\tilde{\delta}\left(|\boldsymbol{w}|_{[(k_1-P)T_s,t]}\right)$, $t > (k_1+1)T_s$, where $\tilde{\delta}$ is a continuous, non decreasing and unbounded function:

$$\tilde{\delta}(\nu) \doteq s\cdot\max\left\{1, \left(\frac{\eta_1\Gamma\Omega_{\text{out}}^2\nu}{s}\right)^{\frac{1}{1-log(\Lambda)/log(\Omega_{\text{out}})}} /\Omega_{\text{out}}\right\}.$$

Note that $\tilde{\delta}$ is not a class $\mathcal{K}_\infty$ function as $\tilde{\delta}(0) = s > 0$.

Lemma 3: Let k_1 be defined so that $(k_1+1)T_s$ is the first switch to *zoom-out* from *zoom-in/escape detection*. After this first switch, the norm of the estimation error is bounded as $\|\boldsymbol{e}\|_{[(k_1-P)T_s,\infty)} \le \delta(\|\boldsymbol{w}\|)$, where δ is a class $\mathcal{K}_\infty$ function.

Lemma 4: Consider the first *zoom-in* sequence. Define J_1T_s to be the sampling time of the first *zoom-in/escape detection* mode in that sequence for which $\mu_{J_1+P} < \frac{1}{\alpha}\Gamma\|\boldsymbol{w}\|_{[0,(J_1+P)T_s]}$. If $\mu_{J_0+P} < \frac{1}{\alpha}\Gamma\|\boldsymbol{w}\|_{[0,(J_1+P)T_s]}$ where J_0T_s is the last sampling time in the *zoom-out* sequence preceding the first *zoom-in* sequence, then define $J_1 = J_0$. The following holds:

$$\|\boldsymbol{e}\|_{[J_1T_s,(k_1-P)T_s)} < \delta\left(\|\boldsymbol{w}\|_{[0,(J_1+P)T_s]}\right) \tag{21}$$

Lemma 3 and Lemma 4 bound the estimation error for $t \ge J_1T_s$. The following two lemmas will bound the estimation error for $t < J_1T_s$. In the first lemma we consider the case $|\boldsymbol{e}_0| > (N-2)\mu_0$ for which the controller will not switch to *zoom-in* immediately at $t=0$. In the second lemma we will consider the complementary case.

Lemma 5: If $|\boldsymbol{e}_0| > (N-2)\mu_0$ then the estimation error for all $t\in[0,J_1T_s]$ is bounded as

$$|\boldsymbol{e}(t)| < \tilde{\beta}_e\left(|\boldsymbol{e}_0|, t\right) + \tilde{\delta}_e\left(\|\boldsymbol{w}\|_{[0,t]}\right) \tag{22}$$

where $\tilde{\beta}_e$ is a class $\mathcal{KL}$ function and $\tilde{\delta}_e$ is a class $\mathcal{K}_\infty$ function.

Lemma 6: If $|\boldsymbol{e}_0| \le (N-2)\mu_0$ then the estimation error for all $t\in[0,J_1T_s]$ is bounded as

$$|\boldsymbol{e}(t)| < \hat{\beta}_e\left(|\boldsymbol{e}_0| + \frac{1}{\Lambda-1}\Gamma\|\boldsymbol{w}\|_{[0,t]}, t\right) \tag{23}$$

where $\hat{\beta}_e$ is a class $\mathcal{KL}$ function.

Combining Lemmas 3–6 we arrive at the following corollary, which basically says that we have ISS from the disturbance, $\boldsymbol{w}$, to the estimation error, $\boldsymbol{e}$:

Corollary 1: For all $t>0$ we have:

$$|\boldsymbol{e}(t)| \le \beta_e\left(|\boldsymbol{e}_0|, t\right) + \delta_e\left(\|\boldsymbol{w}\|_{[0,t]}\right) \tag{24}$$

where $\beta_e\left(|\boldsymbol{e}_0|, t\right) \doteq \max\left\{\hat{\beta}_e\left(2|\boldsymbol{e}_0|, t\right), \tilde{\beta}_e\left(|\boldsymbol{e}_0|, t\right)\right\}$ and $\delta_e(\nu) \doteq \max\left\{\hat{\beta}_e\left(\frac{2\Gamma\nu}{\Lambda-1}, 0\right), \tilde{\delta}_e(\nu), \delta(\nu)\right\}$.

Proof of Theorem 1: Since K is a stabilizing state feedback controller, for (11) we have (see [15, §4.9]):

$$|\boldsymbol{x}(t)| \le \beta\left(|\boldsymbol{x}_0|, t\right) + \gamma_e\left(\|\boldsymbol{e}\|_{[0,t]}\right) + \gamma_w\left(\|\boldsymbol{w}\|\right).$$

This together with (24) shows that the closed-loop system can be regarded as a cascade connection of two ISS systems, and thus we can use a variation of [11, Proposition 7.2] to arrive at $|\boldsymbol{x}(t)| \le \beta_{cl}\left(|\boldsymbol{x}_0|, t\right) + \gamma_{cl}\left(\|\boldsymbol{w}\|\right)$.

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