Karlsruher Institut für Technologie

Master's Thesis

On Subgraphs With Minimum Degree Restrictions

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Abstract

For a graph H, the Ramsey number $R(H)$ is defined as the minimum integer m such that any edge coloring of K_m in two colors contains a monochromatic subgraph H. The extremal number $ex(n, H)$ for an integer n is defined as the maximum number of edges in a graph on n vertices that does not contain H as a subgraph. Both definitions are based on the existence of a fixed subgraph H. In this thesis, we summarize various results of a similar type that replace the existence of H with the existence of any subgraph from a class of graphs with certain minimum degree properties. Furthermore, we introduce the notion of the *minimum degree Ramsey number* $R_r^{\mathscr{D}}(n)$, defined as the minimum integer m such that any edge coloring of K_m in r colors contains a monochromatic subgraph H with $\delta(H) \geq n$. With the help of a coloring algorithm from Klein and Schönheim [KS92], we determine

$$R_r^{\mathscr{D}}(n) = \left\lfloor \left(r + \sqrt{r(r-1)} \sqrt{1 + \frac{1}{4r(r-1)(n-1)^2}} \right) (n-1) + \frac{3}{2} \right\rfloor.$$

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Contents

1 Introduction

In this thesis, we handle results from two prominent areas of graph theory. The first one is extremal graph theory. Extremal graph theory deals with the question of how global assumptions about a graph, for example about the average degree or the chromatic number, force the existence of a concrete subgraph. In the simplest form, extremal graph theory asks how many edges a graph on a fixed number of vertices must contain to guarantee the existence of a given subgraph. The second branch called Ramsey theory raises a superficially similar question: What structures must be found in a graph if it is big enough? The basic question is how large does a complete graph have to be such that any edge coloring in a fixed number of colors contains a given subgraph. The common factor in both areas is the existence of the given subgraph. In this thesis we look into what happens if we do not require the existence of a fixed subgraph, but instead just require the existence of any subgraph from a class of graphs that is defined by some minimum degree based properties. There are various approaches in the literature on how to exactly define such "minimum degree based properties" that yield different interesting results. Based on that, we structure the thesis into three main sections, two which summarize results from the literature alongside explanations and insights and one which introduces new concepts alongside extensive proofs. Furthermore, we summarize all basic definitions needed to understand this thesis in Chapter 2.

In Chapter 3, we summarize results that focus on the existence of subgraphs with some kind of restriction regarding their minimum degree in an extremal graph theory setting. We start by mentioning basic results that yield the existence of subgraphs classified by their minimum degree. In Section 3.2, we calculate the number of edges a graph on n vertices may have that guarantees the existence of a (true) subgraph of fixed minimum degree. In the following section, we look into full subgraphs where the required minimum degree scales with the order of the subgraph. Finally we mention relatively full subgraphs, where the minimum degree for each vertex in the subgraph must be at least a fraction of the degree in the original graph.

The next Chapter summarizes results of a Ramsey type where we look for the existence of monochromatic subgraphs with minimum degree restrictions. For the quasi Ramsey number in Section 4.2, there must exist a subgraph of order at least k whos minimum degree must not be smaller than a fraction of its order. In the next section this requirement is strengthened in fixed quasi Ramsey numbers by requiring the subgraph to have order exactly k. Furthermore, we look into full subgraphs in a Ramsey type fashion by defining the co-fullness as fullness of the complement of a graph. The last section of this chapter deals with k-cores which are very closely related to the minimum degree Ramsey number that we define in the next chapter.

Finally, we introduce new results and definitions in Chapter 5, alongside explaining some current results with greater detail. There are two main parts, the first is on bipartite quasi Ramsey numbers, where we replace the complete base graph in quasi Ramsey numbers by a balanced bipartite base graph. The second part introduces the notion of the minimum degree Ramsey number $R_r^{\mathscr{D}}(n)$, which is inspired by the quasi Ramsey number. We drop the requirement of the subgraph having a certain minimum order and just look for a subgraph with minimum degree at least n. Here, we first look at the two colored case for $r = 2$, where we calculate tight bounds. For the multicolored case with $r \geq 2$, we use an algorithm from Klein and Schönheim [KS92] that we explain in detail. Finally, an implementation of the algorithm of Klein and Schönheim that prints every intermediate step can be found in the Appendix.

2 Definitions

In this chapter, we introduce all basic notations and definitions concerning graphs which are used throughout the thesis. For most of the thesis, we are sticking to the definitions and notations of the book "Graph Theory" by R. Diestel [Die12], with some minor additions if necessary.

2.1 Basic Definitions

A graph is an ordered pair $G = (V, E)$ where V is a finite set of so called *vertices* and $E \subseteq \binom{V}{2}$ a set of unordered pairs of elements of V called *edges*. For simplicity, an edge $\{x, y\}$ is abbreviated by xy. The notation $V(G)$ refers to the vertex set of the graph G. The *order* of a graph G, denoted by $|G|$, is the cardinality of the set $V(G)$. Analogously, $E(G)$ refers to the graphs edge set whereas the number of edges in $E(G)$ is the *size* of the graph G, denoted by $\|G\|$. A graph is always simple, undirected, and finite unless specifically mentioned otherwise. Graphs can be depicted in diagrammatic form as a set of dots for the vertices, joined by lines or curves for the edges.

We define $G' = (V', E')$ to be a *subgraph* of a given graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. This subgraph relation will be denoted as $G' \subset G$, we say that G' is *contained* in G. If G' is a subgraph of G on the vertex set V' and G' contains all edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph*, denoted by $G' = G[V']$. The union $\tilde{G} = G \cup G'$ of two graphs $G = (V, E)$, $G' = (V', E')$ is defined as $\tilde{G} := (V \cup V', E \cup E')$. The union is called *disjoint* if the vertex sets V, V' are disjoint.

For a graph G, the *complement* is denoted by $\overline{G} = (\overline{V}, \overline{E})$. The complement $\overline{G}$ shares the same vertex set $\overline{V} = V$ but the edge set consists of all edges that are not in E, i.e. $\overline{E} = \{xy \in \binom{V}{2} \mid xy \notin E\}$.

If $\{u, v\} \in E(G)$, these two vertices *share* an edge or are *joined* by an edge and are said to be *adjacent* or *neighbours*. For a given $v \in V(G)$, the set of all its neighbours is denoted by $N(v)$. For a fixed v, this implies that $v \notin N(v)$, i.e. v is no neighbour of itself. The *closed neighbourhood* $N[v]$ is $N(v) \cup \{v\}$. If $v \in e$ for a vertex $v \in V$ and an edge $e \in E$ then v is *incident* in e. For an edge $e = xy$, the two vertices x, y incident in e are called its *endpoints* or *ends*. If a vertex v is *deleted* from V, all its incident edges are removed from E as well.

The *degree* of a vertex $v \in V(G)$ is the number of edges that end in v and is denoted by $d(v)$. For simple graphs, the *degree* of v is the size of its neighbourhood $N(v)$. For a subgraph H, we define $d_H(v)$ to be the number of edges that start in v and end in H. The *maximum degree* of G is $\Delta(G) := \max\{d(v) \mid v \in V(G)\}$. Analogously, the *minimum degree* of G is $\delta(G) := \min\{d(v) \mid v \in V(G)\}$. A vertex is called *isolated* if it has degree 0. If all vertices of a given graph have the same degree k, the graph is called *k-regular* or just *regular.* A set of vertices in which all vertices are pairwise not adjacent is called an *independent set.* The order of a largest independent set in a graph is denoted by $\alpha(G)$, the *independence number.*

A graph G on n vertices is called *complete*, *complete graph* or K_n if any two different vertices are adjacent. If any two vertices in a graph on n vertices are not adjacent, the graph is an *empty graph* E_n. If the vertex set of a graph can be partitioned in two non-empty sets A and B such that A and B induce an independent set, the graph is called *bipartite.* If $||A| - |B|| \leq 1$, the bipartite graph is *balanced.* $K_{n,m}$ denotes the *complete bipartite graph* with independent sets A and B of sizes n and m, where every vertex in A is adjacent to every vertex in B. A *walk* is a sequence $v_0, e_1, v_1, \ldots, v_k$ of vertices v_i and edges e_i such that for $1 \leq i \leq k$ the edge e_i has endpoints v_{i-1} and v_i. The length of a walk is its number of edges. A *path* is a graph $P_k = (V, E)$ with vertex set $V = \{v_0, \ldots, v_k\}$, $v_i \neq v_j$ for $i \neq j$ and $i, j = 0, \ldots, k$ and edge set $E = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$ where k indicates the *length* of the path, i.e. the number of edges. A path on k vertices is called a *k-path.* The vertices v_0 and v_k are *linked* or *connected* by P and called the *endpoints* or *ends* of P. A *cycle* C is formed by a path and an extra vertex v_{k+1} that is adjacent only to v_0 and v_k, i.e. $C = (\{v_o, \ldots, v_k\} \cup \{v_{k+1}\}, \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\} \cup \{v_kv_{k+1}, v_{k+1}v_0\})$. The length of a cycle is the number of its edges, a *k-cycle* is a cycle on k vertices called C_k. A graph is *acyclic* if it contains no cycle. An acyclic graph is called a *forest.* A *tree* is a connected forest. Every vertex of degree one in a forest is a *leaf.*

We define the *distance* between two vertices in a graph as the number of edges in a shortest path connecting them. If no such path exists, the distance is defined as infinite. For any cycle C, we define the *k-th power* C^k to be a graph that has the same vertex set, where two vertices are adjacent if their distance in C is at most k.

A graph G is *connected* if any two vertices in G are linked by a path. A vertex that

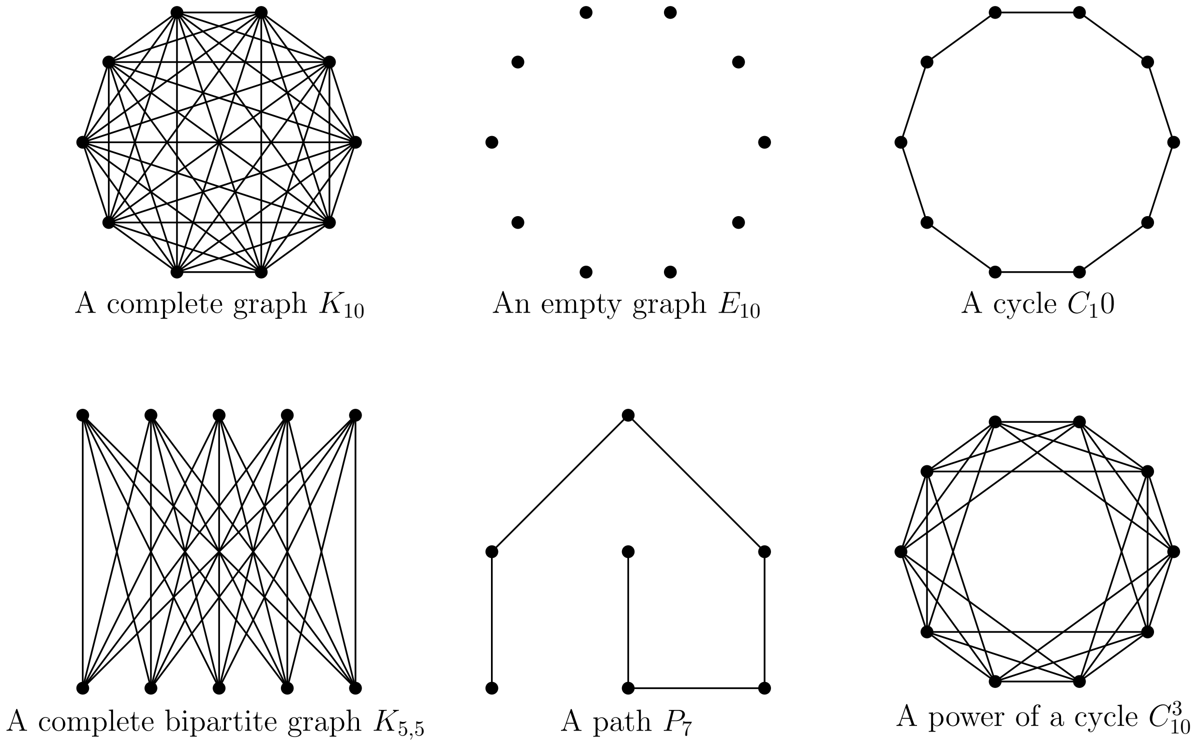


Figure 1: Examples for special types of graphs.

disconnects the graph upon removal is called a *cut vertex*. A graph G on at least $k+1$ vertices has connectivity $\kappa(G) = k$ if it is *k-connected* for $k \in \mathbb{N}$, i.e. after the removal of any $k-1$ vertices, the graph is still connected. A *connected component* of a graph is a maximal connected subgraph. A *separating cycle* of a given graph G with k connected components is a cycle in G such that $G - C$ has more than k connected components.

A *coloring* of a graph G with k colors is a map $c : V(G) \longrightarrow \{0, \ldots, k-1\}$ such that for any vertex $v \in V(G)$, $c(v)$ yields its color. The set of all vertices of the same color forms a *color class*. A *proper coloring* is a coloring in which each color class induces an independent set. The *chromatic number* $\chi(G)$ is the minimum number of colors such that G can be colored properly with that many colors. If the coloring is not proper, it is called *improper*.

Analogously, an *edge coloring* of a graph G with k colors is a map $c : E(G) \longrightarrow \{0, \ldots, k-1\}$ such that for any edge $e \in E(G)$, $c(e)$ yields its color. A *proper coloring* is a coloring in which any two edges that share a vertex receive different colors. We further define the *edge chromatic number* $\chi'(G)$ as the smallest k such that there is a proper k-edge-coloring of G, but no proper $(k-1)$-edge-coloring. For a color b and a vertex x in an edge-colored graph, let $N_b(x)$ denote the *neighbourhood of x in color b*, i.e. the set of vertices adjacent to x via an edge of color b. We define the *edge-coloring incidence matrix* of a edge-colored graph G on n vertices in colors $1, \dots, r$ as an $n \times n$ matrix M with $M_{i,j} = i$ if and only if $c(v_i v_j) = i$ for an edge $v_i v_j \in E(G)$ and $M_{i,j} = 0$ if $v_i v_j \notin E(G)$.

An *edge contraction* is an operation which removes an edge e from a graph while simultaneously merging the two vertices u, v it used to connect. More specifically, u and v are merged into a new vertex w where the edges incident to w each correspond to an edge incident to either u or v. A graph H is a *minor* of another graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices. The order in which a sequence of such contractions and deletions is performed on G does not affect the resulting graph H.

A *random graph $G(n,p)$* is defined as a complete graph on n vertices where each edge is deleted independently with probability $1-p$. We say that *almost every $G(n,p)$* has some property if the probability that $G(n,p)$ has this property tends to 1 as $n \to \infty$, where $p = p(n)$ may vary as a function of n.

We define a *hypergraph H* as a pair $H = (V, E)$ where V is a set of vertices and E is a set of non-empty subsets of V of size at least two called *hyperedges*. A hypergraph is *k-uniform* if all its hyperedges contain k vertices each, so they have *size k*. An edge e in a hypergraph H is colored properly if there are two vertices x_1 and x_2 in e of different color. The *hypergraph chromatic number* $\chi(H)$ is the minimum number of colors such that every hyperedge $e \in E(H)$ is colored properly.

2.2 Ramsey Numbers, Extremal Numbers and Related Definitions

The *density* of a graph G on n vertices is defined as $\|G\|/\binom{n}{2}$. A *full subgraph* of a graph G of density μ is an m-vertex subgraph H of minimum degree at least $\mu(m-1)$. Let $f(G)$ denote the largest number of vertices in a full subgraph of G, i.e. the order of the largest subgraph H of G such that H has minimum degree at least $(\|G\|/\binom{n}{2})(|H|-1)$. If $\mu\binom{n}{2}$ is a non-negative integer, define $f(n,\mu) := \min\{f(G) \mid |V(G)| = n,\ |E(G)| = \mu\binom{n}{2}\}$.

A subgraph H of a graph G is *co-full* if $V(H)$ induces a full subgraph of $\overline{G}$. Equivalently, an induced subgraph H of a graph G with density μ with $|H| = m$ is co-full if H has maximum degree at most $\mu(m-1)$. We define $g(n)$ to be the largest integer m such that G has a full subgraph with at least m vertices or a co-full subgraph with at least m vertices, i.e. $g(G) = \max\{f(G), f(\overline{G})\}$. We further set $g(n) = \min\{g(G) \mid |V(G)| = n\}$.

We define a subgraph H of a graph G to be *relatively q-full* if $d_H(v) \geq q d_G(v)$ for every $v \in V(H)$.

For a graph G of density μ, we define the *positive* and *negative discrepancy* of G as $\mathrm{disc}^+(G) = \max_{X \subseteq V(G)}(\|X\| - \mu\binom{|X|}{2})$ and $\mathrm{disc}^-(G) = \max_{X \subseteq V(G)}(\mu\binom{|X|}{2} - \|X\|)$. We then define the *discrepancy* of G as $\mathrm{disc}(G) = \max\{\mathrm{disc}^+(G), \mathrm{disc}^-(G)\}$. To determine how "random like" G is, we say that G is *(μ, j)-jumbled* if, for every $X \subseteq V(G)$, it holds that $\|X\| - \mu\binom{|X|}{2} \leq j|X|$.

The *Ramsey number* $R(n)$ for a integer n is defined as the minimum order m of K_m such that any edge coloring of K_m in two colors contains a monochromatic copy of K_n. For any given integers p and q, $R(p, q)$ is defined as the minimum order m of a complete graph K_m such that any coloring of the edges of K_m in red and blue contains either a red K_p or a blue K_q. Likewise, for two graphs H and I, $R(H, I)$ is the minimum order m such that any edge-two-coloring of K_m contains either a red copy of H or a blue copy of I.

We generalize this notation in the following way. For any real number $\gamma \in [0, 1]$, define the *quasi Ramsey number* $R_\gamma(n)$ as the minimum order m such that any two-coloring of the edges of K_m contains a monochromatic subgraph H of order at least n with $\delta(H) \geq \gamma(|V(H)| - 1)$. The cases for $\gamma = 0$ and $\gamma = 1$ are special, since $R_0(n) = n$ and $R_1(n) = R(n)$. The *fixed quasi Ramsey number* $R^*_\gamma(n)$ is analogously defined as the minimum order m such that any two-coloring of the edges of K_m contains a monochromatic subgraph of order exactly n and $\delta(H) \geq \gamma(|V(H)| - 1)$.

Similarly, we define the *bipartite quasi Ramsey number* $R^{\mathrm{bip}}_\gamma(n)$ to be the minimum integer m such that any edge two-coloring of $K_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}$ contains a monochromatic subgraph H of order at least n with $\delta(H) \geq \gamma|V(H)|/2$.

We want to classify graphs based on their minimum degree, thus we define for any $n \in \mathbb{N}$ the class $\mathscr{D}_n$ as the class of all graphs with minimum degree at least n. This

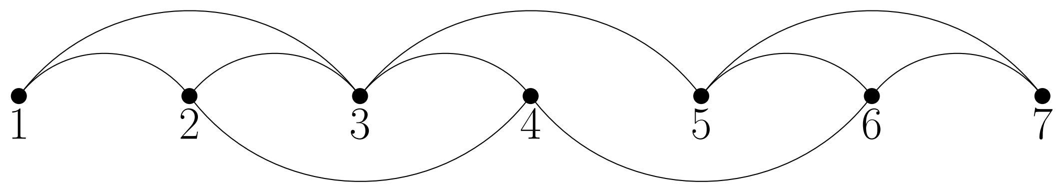


Figure 2: An example of a 2-degenerate graph.

definition implies that $\mathscr{D}_{n+1} \subset \mathscr{D}_n$. For any class of graphs $\mathscr{D}$ and any integer $r \geq 2$, we define the *minimum degree Ramsey Number* $R_r^{\mathscr{D}}(n)$ as the smallest integer m such that any r-coloring of K_m contains a monochromatic subgraph G with $G \in \mathscr{D}_n$. We may omit r by defining $R^{\mathscr{D}}(n) := R_2^{\mathscr{D}}(n)$.

A graph G is called *q-degenerate* if there exists an left to right ordering of the vertices $V(G)$ such that each vertex sends at most q edges to the right, an example can be seen in Figure 2.

A *packing* into a graph G is a set of graphs G_i for $i = 1, \ldots, k$ such that there exist injective mappings of the vertex sets $V_i = V(G_i)$ into $V(G)$, $V_i \to V$, where the images of the edge sets do not pairwise intersect. If any edge of G is contained in a edge set $E_i = E(G_i)$, the packing is called a *perfect packing*. A *matching* M in G is a set of pairwise non adjacent edges of $E(G)$. It is *maximal* if any edge in G has a non-empty intersection with at least one edge in M. A *decomposition* of a graph G is a collection of subgraphs $M_1, \ldots, M_k$ on the same vertex set as G with $E(M_1) \cup \ldots \cup E(M_k) = E(G)$ where $e \in E(M_i)$ implies $e \notin E(M_j)$ for $j \neq i$.

Furthermore, we define for a given family of graphs $\mathcal{H}$ the *extremal number* as $\mathrm{ex}(n, \mathcal{H}) := \max\{\|G\| : |G| = n,\ H \not\subset G\ \forall H \in \mathcal{H}\}$. If $\mathcal{H} = \{H\}$, we write $\mathrm{ex}(n, H)$ instead of $\mathrm{ex}(n, \mathcal{H})$. A graph is called *extremal* with respect to $\mathcal{H}$ if $H \notin G$ for all $H \in \mathcal{H}$ and $|G| = \mathrm{ex}(n, \mathcal{H})$.

2.3 Big O Notation

Because of the asymptotic nature of our results, we use big O notation as in Table 1 to simplify.

Notation	Description	Limit Definition
$f(n) = o(g(n))$	f is dominated by g asymptotically	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
$f(n) = O(g(n))$	$\lvert f\rvert$ is bounded from above by g	$\limsup_{n\to\infty} \frac{\lvert f(n)\rvert}{g(n)} < \infty$
$f(n) = \Omega(g(n))$	f is bounded from below by g asymptotically	$\liminf_{n\to\infty} \frac{f(n)}{g(n)} > 0$
$f(n) = \omega(g(n))$	f dominates g asymptotically	$\lim_{n\to\infty} \lvert\frac{f(n)}{g(n)}\rvert = \infty$
$f(n) = \Theta(g(n))$	f is bounded above and below by g asymptotically	$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
$f(n) \sim g(n)$	f is equal to g asymptotically	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

Table 1: Big O notation used in this thesis.

3 Summary of Extremal Graph Theory Type Results

In this chapter, we summarize results regarding the existence of minimum degree restricted subgraphs that are of a extremal graph theory type flavor, i.e. which global assumptions and restrictions about a graph yield the existence of any subgraph from a class of subgraphs defined by certain minimum degree requirements. Most of the results are taken from literature, excluding some easy results which are proven directly.

As an introduction, we give a quick overview of basic extremal graph theory in Section 3.1, starting with theorems of Mantel [Man07] and Turán [Tur41]. We further explain the relation of the extremal number $\mathrm{ex}(n, H)$ and the chromatic number $\chi(H)$ as shown by Erdős and Stone [ES+46]. Finally, we present a result from Diestel [Die12] that yields the existence of subgraphs with minimum degree that is dependent on the chromatic number of the original graph.

In Section 3.2, we present tight results for $\mathrm{ex}(n, \mathscr{D}_k)$, the maximum size of a graph G of order n that does not contain a subgraph of minimum degree at least k. Furthermore, we talk about a conjecture of Erdős, Faudree, Rousseau and Schelp [ELS92] on the existence of true subgraphs with minimum degree at least k. We show the progress made towards that conjecture by Mousset, Noever and Škorić [MNŠ17] in 2017 and the final confirmation of the conjecture by Sauermann [Sau19] in 2019.

Section 3.3 focuses on results obtained by Erdős, Łuczak and Spencer [ELS92] and extended by Falgas-Ravry, Markström and Verstraëte [FRMV18] on full subgraphs. Recall a subgraph H of a graph G with density μ is said to be full if $\delta(H) \geq \mu(|H| - 1)$. The main focus is $f(G)$, the order of the largest full subgraph of a graph G. There are two kinds of results in this section, some focus on random graphs and almost surely hold true asymptotically in the order of G. The others yield bounds on $f(G)$ for all graphs with fixed order and number of edges ($\tilde{f}(n, s)$) or fixed order and density ($f(n, \mu)$). Furthermore, we explain the relation between discrepancy and full subgraphs and how this can be used to obtain better bounds on $f(G)$. Finally, we mention some open problems and further reading.

Further results that are related to full subgraphs can be found in Section 3.4, where we summarize results on relatively full subgraphs. Introduced by Falgas-Ravry, Markström and Verstraëte [FRMV18], in a relatively full subgraph the minimum degree of any vertex has to be at least a fixed fraction of the degree of that vertex in the original graph. The bounds in this section are very tight, which leaves little room for further research.

3.1 Basic Extremal Graph Theory

We start with a quick introduction into basic extremal graph theory. Extremal graph theory studies extremal (maximal or minimal) graphs which satisfy a certain property. Extremality can be taken with respect to different graph invariants such as order, size or girth. More abstractly, it studies how global properties of a graph influence local substructures of the graph. It may be roughly subdivided into the two areas of "dense" extremal graph theory and "sparse" extremal graph theory based on the proportion of edges required related to the number of vertices. Classes of graphs where the number of edges for each graph $G = (V, E)$ grows at most linearly with the number of vertices are called *sparse* graphs. For example, to prove the existence of given minors H in a graph $G = (V, E)$ it is sufficient to require linear growth of $|E|$ in relation to $|V|$. In this thesis, we are more interested in results for *dense* graphs since most results that tackle questions about the existence of fixed subgraphs require quadratic growth of $|E|$ in relation to $|V|$. One of the first results here is a theorem by Mantel from 1907 [Man07].

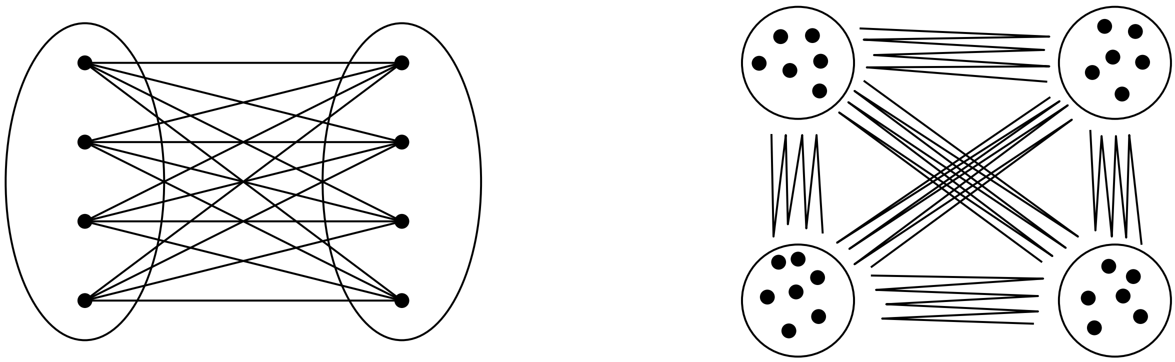
Theorem 3.1 (Mantel [Man07]). *If the size of a graph G of order $n \geq 3$ is at least $\lfloor n^2/4 \rfloor + 1$, then G contains a triangle.*

This result was generalized by Turán in 1941 [Tur41] which resulted in the famous Turán Theorem. A *Turán graph* $T_{r-1}(n)$ for integers n and r is defined as the unique complete $(r-1)$–partite graph of order n whose partite sets differ by at most one in size. By the Pigeonhole Principle, it is easy to see that $T_{r-1}(n)$ does not contain K_r as a subgraph since in any set of n vertices at least two must be in the same partition class and have no edge between them. The size of $T_{r-1}(n)$ is denoted by $t_{r-1}(n)$. An example for Turán graphs can be seen in Figure 3.

Theorem 3.2 (Turán [Tur41]). *For all integers $r \geq 1$ and $n \geq 1$, any graph with n vertices, $ex(n, K_r)$ edges and $K_r \not\subset G$ is $T_{r-1}(n)$.*

In 1946, Erdős and Stone [ES$^+$46] proved that adding only a small amount of edges to $T_{r-1}(n)$ yields the existence of many subgraphs isomorphic to K_r, by showing that the resulting graph must contain a complete r-partite subgraph where each partition set has order s. We denote such a graph as K_s^r.

Theorem 3.3 (Erdős and Stone [ES$^+$46]). *For all integers $r > s \geq 1$ and any $\epsilon > 0$, there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least $t_{r-1}(n) + \epsilon n^2$ edges contains a K_s^r.*



(a) The graph $T_2(8)$ (b) Sketch of the graph $T_4(25)$

Figure 3: Turán graph on eight vertices without K_3 (a) and Turán graph on 25 vertices without K_5 (b).

Although the original proof did not use it, today Theorem 3.3 can be proven more easily by using Szemerédis Regularity Lemma [Sze75], which states that every large enough graph can be divided into subsets of about equal size so that the edges between different subsets behave almost randomly. To understand the lemma that has much use all throughout extremal graph theory, we need a few definitions. Therefore, let G be a graph with vertex set V.

- Let X, Y be disjoint subsets of V. The *density* of the pair (X, Y) is defined as:

$$d(X,Y) := \frac{|E(X,Y)|}{|X||Y|}$$

  where $E(X, Y)$ denotes the set of edges having one end vertex in X and one in Y.

- For $\epsilon \geq 0$, a pair of vertex sets X and Y is called *ϵ-regular* if, for all subsets $A \subseteq X,\ B \subseteq Y$ satisfying $|A| \geq \epsilon|X|,\ |B| \geq \epsilon|Y|$, we have

$$|d(X,Y) - d(A,B)| \leq \epsilon.$$

- A partition of V into k sets $V_1, \dots, V_k$ is called an *ϵ-regular partition* if for all i, j we have $||V_i| - |V_j|| \leq 1$ and all except at most ϵk^2 of the pairs $V_i, V_j, i < j$ are ϵ-regular.

Then we can state the Regularity Lemma as

Lemma 3.4 (Szemerédi [Sze75]). *For every $\epsilon > 0$ and positive integer m, there exists an integer M such that if G is a graph with at least M vertices, there exists an integer k in the range $m \leq k \leq M$ and an ϵ-regular partition of the vertex set of G into k sets.*

What makes Theorem 3.3 famous is a corollary, that can be derived from it, which connects the chromatic number $\chi(H)$ of any graph H to the extremal number $\mathrm{ex}(n, H)$.

Corollary 3.5. *For any graph H with at least one edge*

$$\lim_{n\to\infty} ex(n, H)\binom{n}{2}^{-1} = \frac{\chi(H) - 2}{\chi(H) - 1}.$$

This implies that we can asymptotically bound the number of edges in a H-free graph on n vertices for any graph H solely based on the chromatic number $\chi(H)$. One could think that we could use this result to prove an extremal number where, instead of avoiding the existence of a fixed subgraph H, we try to avoid any subgraph from the class $\mathscr{D}_k$, the class of all graphs with minimum degree at least k for $k \in \mathbb{N}$. This will not work since Corollary 3.5 is applicable only for fixed graphs H, thus using it for any graph $H \in \mathscr{D}_k$ (even if $\chi(H) = \min_{H' \in \mathscr{D}_k}(\chi(H'))$ will not guarantee the non-existence of subgraphs $H' \neq H$ with $H \in \mathscr{D}_k$. Still, there is a connection between the chromatic number of a graph and the existence of a subgraph with certain minimum degree, as the following Proposition shows.

Proposition 3.6 (Diestel [Die12]). *Every graph G has a subgraph H with $\delta(H) \geq \chi(G) - 1$.*

Proof. Let $[v_1, \ldots, v_n]$ be an ordering of the vertices of G such that $d_{G[v_1,\ldots,v_{i-1}]}(v_i)$ is minimal in $\{v_1, \ldots, v_i\}$. This is well defined and can be constructed iteratively. Therefore, pick v_n as a vertex with minimal degree in G, then pick v_{n-1} as a vertex with minimal degree in $G - v_n$ and continue inductively until all vertices are picked. Define $l(G)$ as the smallest natural number k such that G has an ordering as above where every vertex is incident in less than k edges to earlier neighbours, i.e. $l(G) = \min\{k \in \mathbb{N} \mid k > \max_{i=2,\ldots,n}(d_{G[v_1,\ldots,v_{i-1}]}(v_i))\}$. By construction, we get that $l(G) \leq \max_{H\subseteq G}(\delta(H)) + 1$. On the other hand, for every $H \subseteq G$, we have that $l(G) \geq l(H)$ and $l(H) \geq \delta(H) + 1$ since in any ordering of H the number of earlier neighbours of the last vertex v is simply $d_H(v) \geq \delta_H(v)$. Thus we get $\chi(G) \leq l(G) = \max\{\delta(H) \mid H \subseteq G\} + 1$, which implies the proposition. $\square$

3.2 Subgraphs of Fixed Minimum Degree

In this section, we summarize results that find the maximum size $s(n,k)$ such that there exists a graph of order n that does not contain a subgraph with minimum degree at least k, i.e. $s(n,k) = \mathrm{ex}(n, \mathscr{D}_k)$. For this section, we call such a subgraph a "bad" subgraph. The following Proposition yields the upper bound on $\mathrm{ex}(n, \mathscr{D}_k) < (k-1)(n-k+2)+\binom{k-2}{2}$.

Proposition 3.7. *Every graph on $n \geq k-1$ vertices with at least*

$$(k-1)(n-k+2)+\binom{k-2}{2}$$

edges contains a subgraph of minimum degree at least k.

Proof. Let the statement above be the induction hypothesis. For $n = k-1$, the induction hypothesis is trivially true because there exists no graph on $k-1$ vertices with the required number of edges since

$$\binom{k-1}{2} = \binom{k-2}{2} + \binom{k-2}{1} < \binom{k-2}{2} + (k-1).$$

For any $n \geq k$, given a graph with at least $(k-1)(n-k+2)+\binom{k-2}{2}$ edges that does not have minimum degree at least k, we can delete a vertex of degree at most $k-1$. We then obtain a graph with $n-1$ vertices and at least

$$(k-1)(n-k+2)+\binom{k-2}{2} - (k-1) = (k-1)((n-1)-k+2)+\binom{k-2}{2}$$

edges. Thus we can apply the induction hypothesis to find a subgraph of minimum degree at least k. □

Erdős, Faudree, Rousseau and Schelp observed in 1990 [EFRS90] that for each $n \geq k+1$ there exist graphs on n vertices with $(k-1)(n-k+2)+\binom{k-2}{2}$ edges that do not have any subgraphs of minimum degree at least k on fewer than n vertices. To see that, we define the wheel graph $W(1,n)$ as a combination of K_1 and C_{n-1}, whereas every vertex in C_{n-1} is connected to K_1. Then, the minimum degree in $W(1,n)$ is 3 but any proper subgraph has minimum degree less than 3. More generally, we define the *wheel graph* $W(k-2,n) = K_{k-2} + C_{n-k+2}$ for $k \geq 3$ such that any vertex in K_{k-2} is connected to any vertex in C_{n-k+2}. The resulting wheel graph has minimum degree k, but no proper subgraph of minimum degree at least k. Examples can be seen in Figure 4.

We can fit this construction to our definition which includes all subgraphs, especially G itself. Since any such wheel graph $W(k-2,n)$ has order n and size $(n-k+2)(k-2)+(n-k+2)+\binom{k-2}{2} = (k-1)(n-k+2)+\binom{k-2}{2}$, we remove one edge from the cycle to obtain a graph on $(k-1)(n-k+2)+\binom{k-2}{2}-1$ edges that does not contain a subgraph of minimum degree at least k, since any such subgraph can not contain a vertex from the now incomplete cycle. This together with Proposition 3.7 implies

$$\mathrm{ex}(n,\mathscr{D}_k) = (k-1)(n-k+2)+\binom{k-2}{2}-1.$$

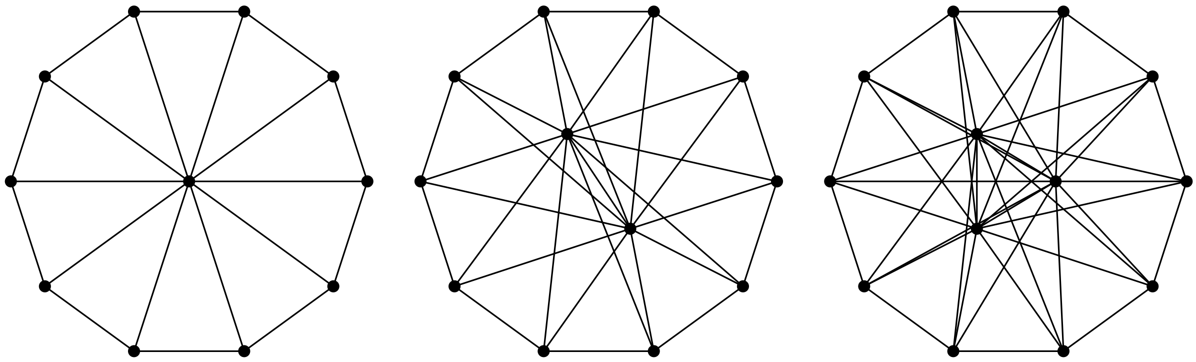


Figure 4: The wheel graphs $W(1,10)$, $W(2,11)$, $W(3,12)$ with minimum degrees 3, 4 and 5.

One may be inclined to ask questions about further properties of these subgraphs, for example to impose a maximum order on the subgraphs that fulfill the minimum degree requirement. Thus Erdős, Faudree, Rousseau and Schelp conjectured that adding a single edge implies the existence of a subgraph H of minimum degree at least k that has only a fraction of the order of the original graph G, i.e. $|H| = (1-\epsilon_k)|G|$ for $\epsilon_k > 0$. They made first progress towards that conjecture in Theorem 3.8, where they show the existence of a bad subgraph with order strictly smaller than $\|G\|$, although not on a fractional scale.

Theorem 3.8 (Erdős, Faudree, Rousseau and Schelp [EFRS90])**.** *For the integer $k \geq 2$, let G be a graph with n vertices and $(k-1)(n-k+2)+\binom{k-2}{2}+1$ edges. Then, G contains a subgraph H of order at most $n - \lfloor \sqrt{n}/\sqrt{6k^3} \rfloor$ with $\delta(H) \geq k$.*

They were also able to determine the correct order of magnitude of the number of edges needed in a graph of order n to ensure the existence of small subgraphs with minimum degree k.

Theorem 3.9 (Erdős, Faudree, Rousseau and Schelp [EFRS90]). *Let the integer $k \geq 2$ and $0 < \epsilon < 1$ be given. Then, any graph of order n and size $\lceil kn/\epsilon \rceil$ has a subgraph H of order at most $\lceil \epsilon n \rceil$ with $\delta(H) \geq k$.*

More progress towards the conjecture was made by Mousset, Noever and Škorić in 2017 [MNŠ17], where they proved that one can remove at least $\Omega(n/\log n)$ vertices and still find a subgraph of minimum degree k. The theorem holds vacously true for $n = k+1$ since there exists no such graph G as $(k-1)(n-k+2) + \binom{k-2}{2} + 1 = \binom{k+1}{2} + 1$ for $n = k+1$.

Theorem 3.10 (Mousset, Noever and Škorić [MNŠ17]). *For $k \geq 2$, let G be a graph on $n \geq k+1$ vertices and $(k-1)(n-k+2) + \binom{k-2}{2} + 1$ edges. Then G contains a subgraph of order at most $n - n/(4(k+1)^5 \log_2 n)$ and minimum degree at least k.*

Finally, a more general Theorem by Sauermann in 2019 [Sau19] proved the conjecture.

Theorem 3.11 (Sauermann [Sau19]). *Let $k \geq 2$ and let $1 \leq t \leq \frac{(k-2)(k+1)}{2} - 1$ be an integer. Then every graph on $n \geq k-1$ vertices with at least $(k-1)n - t$ edges contains a subgraph on at most*

$$\left(1 - \frac{1}{\max(10^4k^2, 100kt)}\right) n$$

vertices and with minimum degree at least k.

Theorem 3.11 implies the conjecture of Erdős with

$$\epsilon_k = \frac{1}{\max\left(10^4k^2, 100k\left(\frac{(k-2)(k+1)}{2} - 1\right)\right)} > \frac{1}{10^4k^3}$$

if we choose $t = \frac{(k-2)(k+1)}{2} - 1$, since

$$(k-1)n - t = (k-1)n - \frac{(k-2)(k+1)}{2} + 1 = (k-1)(n-k+2) + \binom{k-2}{2} + 1.$$

Furthermore, Theorem 3.11 implies for $t = 1$ that every graph on $n \geq k-1$ vertices with at least $(k-1)n - 1$ edges contains a subgraph on at most

$$\left(1 - \frac{1}{10^4k^2}\right) n$$

vertices with minimum degree at least k. This shows the presence of one additional edge implies the existence of a subgraph with minimum degree at least k on $(1-\epsilon)n$ vertices with $\epsilon = \Omega(k^{-3})$ while the presence of $(k-2)(k+1)/2$ additional edges already gives $\epsilon = \Omega(k^{-2})$.

The approach used by Sauermann [Sau19] to prove Theorem 3.11 is to iteratively assign colors to some vertices such that for every color the subgraph remaining after deleting all vertices of that color has minimum degree at least k. Then, she ensures that sufficiently many vertices get colored while the number of colors is fixed, thus she can find a significantly smaller subgraph with minimum degree at least k. The proof relies on and extends the ideas of Mousset, Noever and Škorić from [MNŠ17] used in their proof of Theorem 3.10.

3.3 Full Subgraphs

A different approach on subgraphs with minimum degree restrictions was taken in 1992 by Erdős, Łuczak and Spencer [ELS92]. They introduced the notion of *full subgraphs.* For them, a subgraph $H \subset G$ on m vertices is "full" if each vertex of H has at least $\lceil (m-1)/2 \rceil$ neighbours, i.e. at least half of all possible neighbours. This definition does not depend on the density of the original graph G and is thus in general not the same as our definition of *fullness* that is used for $f(G)$. Although, it is equivalent if and only if G has density $\frac{1}{2}$. Remember, we define $f(G)$ as the largest number of vertices in a subgraph $H \subset G$ with $\delta(H) \geq \mu(|H|-1)$, where μ is the density of G. Nevertheless, they study the behaviour of the order of the largest "full" subgraph of G, which we therefore denote with $\tilde{f}(G)$ for graphs G with n vertices and M edges called (n, M)-graphs, where n and M are chosen such that G has a density close to $\frac{1}{2}$. This makes the results comparable to results for our notion of full subgraphs.

They start off with a result which characterizes the "typical" structure of almost all (n, M)-graphs, i.e. *almost every* $(n, M(n))$-graph has some property if the fraction of $(n, M(n))$-graphs without this property tends to 0 as n tends to infinity. The behaviour around $M(n) = \lfloor n^2/4 \rfloor$ depends heavily on a small factor $|s| \leq 0.01n^2$.

Theorem 3.12 (Erdős, Łuczak and Spencer [ELS92])**.** *Let $n \in \mathbb{N}$ and $M(n) = \lfloor n^2/4 \rfloor + s(n)$ and $|s(n)| \leq 0.01n^2$.*

(i) If $sn^{-3/2} \to -\infty$, then there are some positive constants c_1, c_2 such that for almost every (n, M) graph G we have

$$c_1(n^4/s^2)\log(s^2n^{-3}) \leq \tilde{f}(G) \leq c_2(n^4/s^2)\log(s^2n^{-3}).$$

(ii) If $s^2n^{-3} = O(1)$, then for some constants $0 < c_3 < c_4 < 1$ almost every (n, M) graph G is such that

$$c_3 n \leq \tilde{f}(G) \leq c_4 n.$$

(iii) If $sn^{-2/3} \to \infty$, then there exist constants $c_5 > c_6 > 0$ such that for almost every (n, M) graph the following inequality holds

$$n\exp\left(-\frac{c_5 s^2}{n^3}\right) \leq n - \tilde{f}(G) \leq n\exp\left(-\frac{c_6 s^2}{n^3}\right).$$

They also proved a theorem concerning the extremal case, that is for any n and M the smallest possible value of $\tilde{f}(G)$ achieved on all (n, M) graphs around $M(n) = \lfloor n^2/4 \rfloor$.

Theorem 3.13 (Erdős, Łuczak and Spencer [ELS92])**.** *Let* $\tilde{f}(n, s) = \min\{\tilde{f}(G) \mid G$ *is* $(n, \lfloor n^2/4 \rfloor + s)$ *graph*$\}$ *and* $|s| \leq 0.01n^2$.

(i) If $s \leq -n/3$, *then*

$$\max\left(\frac{n}{2\sqrt{|s|}}, \log n / \log \frac{2n^2}{n^2 + 4s}\right) \leq \tilde{f}(n, s) \leq \frac{n^2}{|s|} \log \frac{4|s|}{n}.$$

(ii) If $s \geq -n/3$, *then*

$$\sqrt{4s + 2n} - 2 \leq \tilde{f}(n, s) \leq 2\sqrt{s + (n^{4/3}/3) \log^{2/3} n} + 2n^{2/3} \log^{1/3} n.$$

Theorem 3.12 can be formulated in terms of a random graph $G(n, p)$, which yields the equivalent Theorem 3.14. It is well known that asymptotic properties of $G(n, p)$ are similar to a graph chosen at random from the family of all (n, M) graphs, provided $M = \binom{n}{2}p$ (see [BT85]).

Theorem 3.14 (Erdős, Łuczak and Spencer [ELS92])**.** *Let* $p(n) = \frac{1}{2} + \epsilon(n)$, *where* $|\epsilon(n)| \leq 0.002$.

(i) If $\epsilon\sqrt{n} \to -\infty$, *then for some positive constants* c_1^*, c_2^* *almost every* $G(n, p)$ *is such that*

$$c_1^* \epsilon^{-2} \log n\epsilon^2 \leq \tilde{f}(G(n, p)) \leq c_2^* \epsilon^{-2} \log n\epsilon^2.$$

(ii) When $\epsilon^2 n = O(1)$, *then for some constants* $0 < c_3^* < c_4^* < 1$ *for almost every* $G(n, p)$ *we have*

$$c_3^* n \leq \tilde{f}(G(n, p)) \leq c_4^* n.$$

(iii) If $\epsilon\sqrt{n} \to \infty$, *then for some constants* $c_5^* > c_6^* > 0$ *and almost every* $G(n, p)$ *we have*

$$\exp(-c_5^* \epsilon^2 n) \leq n - \tilde{f}(G(n, p)) \leq \exp(-c_6^* \epsilon^2 n).$$

The proof of Theorem 3.14 uses an interesting notion of *expandable* subsets. They call a subset S with $|S| = l$ *expandable*, if there exists a vertex outside S, which is adjacent to at least $\lceil (l + 1)/2 \rceil$ vertices from S. A key insight is the following observation: If every subset on l vertices of a graph G is expandable, then G contains a full subgraph of size l. Indeed, if we take any subset S_0 of G with l vertices and $G[S_0]$ were not full, we can replace a vertex of minimum degree in $G[S_0]$ by a vertex $v \notin S_0$ such that $|N(v) \cap S_0| \geq (l+1)/2$ to obtain S_1. If $G[S_1]$ is not full either, we can iterate this process. Since each step increases the density of the subgraph, we end up with a full subgraph on l vertices after less than $\binom{l}{2}$ steps. The final key to the theorem is to show that there exist integers l such that in almost every graph $G(n,p)$ every subset on l vertices is expandable.

In 2018, Falgas-Ravry, Markström and Verstraëte [FRMV18] generalized the notion of full subgraphs to the $f(G)$ notion we use in this thesis by defining a *full subgraph* of a graph G of density μ to be a subgraph H of minimum degree at least $\mu(|H|-1)$. This can be seen as a particularly "rich" subgraph with a minimum degree that is at least as big as the expected average degree of an m-vertex subgraph H, where m vertices of G are selected uniformly at random. By considering a complete multipartite graph with parts of equal size, one can easily see that we can in general not expect to find m-vertex subgraphs of higher minimum degree. Fixing μ and n such that $\mu\binom{n}{2}$ is a nonnegative integer, they define

$$f(n,\mu) := \min\left\{f(G) : |V(G)| = n,\ |E(G)| = \mu\binom{n}{2}\right\}.$$

While the class of graphs relevant for $\tilde{f}(n,s)$ is determined by order n and absolute deviation s of the size from $\lfloor \frac{n^2}{4} \rfloor$, the class of graphs relevant for $f(n,\mu)$ is determined by order n and density μ. It is easy to see that by picking $\mu = (\lfloor \frac{n^2}{4} \rfloor + s)/\binom{n}{2}$ the definitions are equivalent. Thus, they improve the lower bounds from Theorem 3.13 for general μ. If the density of a graph is $\mu = \frac{r}{r-1} + cn^{-\frac{2}{3}}$ then Theorem 3.15 yields the precise order of magnitude for $f(\mu,n)$. Since the proof of $f(n,p) \geq \frac{1}{4}(1-p)^{\frac{2}{3}} n^{\frac{2}{3}} - 1$ for $p = p_n : n^{-\frac{2}{3}} < p_n < 1 - n^{-\frac{1}{7}}$ in the most recent version of their paper [FRMV18] contains a small but easily fixable mistake, we will give a corrected version here.

Theorem 3.15 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *For all $\mu = \mu_n$ such that $n^{-\frac{2}{3}} < \mu_n < 1 - n^{-\frac{1}{7}}$,*

$$f(n,\mu) \geq \frac{1}{4}(1-\mu)^{\frac{2}{3}} n^{\frac{2}{3}} - 1.$$

Moreover, for each $c \geq 1$ if $\mu = \frac{r}{r-1} + cn^{-\frac{2}{3}}$ for some $r \in \mathbb{N}$, then $f(n,\mu) = \Theta(n^{\frac{2}{3}})$.

Proof. Proof of $f(n,p) \geq \frac{1}{4}(1-p)^{\frac{2}{3}}n^{\frac{2}{3}} - 1$ for $p = p_n : n^{-\frac{2}{3}} < p_n < 1 - n^{-\frac{1}{7}}$. Let G be a n-vertex graph of density μ. We shall repeatedly delete vertices of minimum degree to obtain a sequence of subgraphs $G = G_1, G_2, G_3, \ldots$ with G_i having $n-i+1$ vertices. Let $m = \lceil \frac{n}{2} \rceil$ and $d_i = \lceil \mu(n-i) \rceil$. Note that d_i is the minimum degree required for G_i to be full.

Let t be the positive integer that fulfills $(1-\mu)^{-\frac{2}{3}}n^{\frac{1}{3}} \leq 2^t < 2(1-\mu)^{-\frac{2}{3}}n^{\frac{1}{3}}$, and let r_i be the remainder when d_i is divided by 2^t. Since $\frac{1-\mu}{2}m \leq \frac{1-\mu}{1+\mu}m$ we have for at least $\frac{1-\mu}{2}m$ of the values i with $1 \leq i \leq m$ that $r_i \leq (1-\mu)2^t$. At stage $i \leq m$ of the algorithm, we delete a vertex of minimum degree from G_i. We continue with a case distinction.

1. If for some $i \leq m$ such that $r_i \leq (1-\mu)2^t$, all $n-i+1$ vertices in the graph G_i have degree at least $d_i - r_i + 1$, then by Theorem 3.23 G_i has a $\frac{1}{2^t}$-full subgraph H on N vertices where

$$\left\lfloor \frac{n-i+1}{2^t} \right\rfloor \leq N \leq \left\lceil \frac{n-i+1}{2^t} \right\rceil + 1 \leq \frac{n-i}{2^t} + 2.$$

Write $d_i = q2^t + r_i$. The minimum degree in H is

$$D \geq \left\lceil \frac{d_i - r_i + 1}{2^t} \right\rceil = q + 1.$$

For H to be a full subgraph of G, we require $D \geq \mu(N-1)$. Now

$$\mu(N-1) \leq \mu\left(\frac{n-i}{2^t} + 1\right) \leq \frac{d_i}{2^t} + \mu = q + \frac{r_i}{2^t} + \mu \leq q + 1$$

since $r_i \leq (1-\mu)2^t$. As this is at most our lower bound on D, H is a full subgraph of G. Our choice of t ensures

$$|V(H)| \geq \left\lfloor \frac{m}{2^t} \right\rfloor \geq \frac{(1-\mu)^{\frac{2}{3}}n^{\frac{2}{3}}}{4} - 1.$$

2. If for a stage $i \leq m$ where $r_i > (1-p)2^t$, we can not remove a vertex of degree at most $\lceil \mu(n-i) \rceil - 1$, then G_i is a full subgraph on at least m vertices.

3. Finally, suppose that at every stage $i \leq m$ of the greedy algorithm where $r_i \leq (1-p)2^t$ we could remove a vertex of degree at most $\lceil \mu(n-i) \rfloor - r_i$ and that at every other stage $i \leq m$ we could remove a vertex if degree at most $\lceil \mu(n-i) \rceil - 1$. Set $I = \{i \leq m \mid r_i \leq (1-\mu)2^t\}$. We know that $|I| \geq \frac{(1-p)m}{2}$. We see, I can be divided

into intervals of consecutive indices i of length at most $(1-\mu)2^t \cdot \frac{1}{\mu}$. Over each of these intervals, r_i takes each of the values $1, 2, \ldots, \lfloor (1-\mu)2^t \rfloor$ at least $\frac{1-\mu}{\mu}$ times. Indeed, suppose $r_{i-1} = j+1$ and $r_i = j$ for some $j \geq 1$. Then, there is a k with $\frac{1-\mu}{\mu} \leq k \leq \frac{1}{\mu}$ such that $r_{i'} = j$ for $i' \in [i, i+1, \ldots, i+k-1]$ and $r_{i+k} = j-1$. An illustration can be seen in Figure 5.

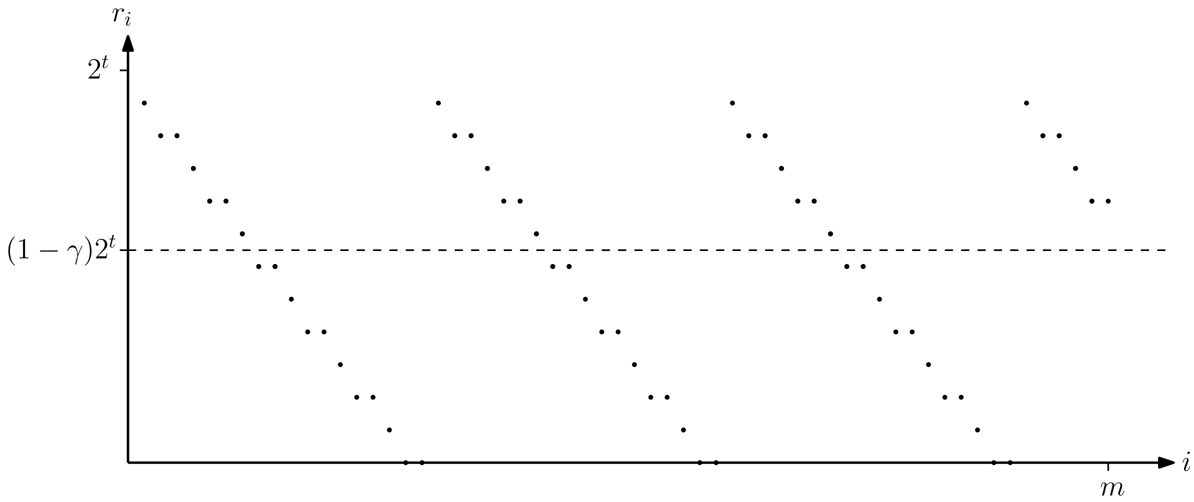


Figure 5: Illustration of r_i's in the proof of Theorem 3.15. Indices of points below the dashed line are in I.

By considering the sum over r_i on these intervals and using $m = \lceil \frac{n}{2} \rceil$, $(1-\mu)^{-\frac{2}{3}} n^{\frac{1}{3}} \leq 2^t \leq 2(1-\mu)^{-\frac{2}{3}} n^{\frac{1}{3}}$, we get that

$$
\begin{aligned}
\alpha := \operatorname{disc}^+(G) &\geq \left(\mu \binom{n}{2} - \sum_{i=1}^{m} (\lceil \mu(n-i) \rceil - 1) + \sum_{i \in I} (r_i - 1) \right) - \mu \binom{m}{2} \\
&\geq \left\lfloor \Big(\frac{(1-\mu)m}{2} \Big) \Big/ \Big(\frac{(1-\mu)2^t}{\mu} \Big) \right\rfloor \cdot \sum_{j=1}^{\lfloor (1-\mu)2^t \rfloor} \frac{1-\mu}{\mu} j \\
&\geq \left\lfloor \frac{\mu(1-\mu)^{\frac{2}{3}} n^{\frac{2}{3}}}{8} \right\rfloor \Big(\frac{1-\mu}{2\mu} \Big) \Big(\left\lfloor (1-\mu)^{\frac{1}{3}} n^{\frac{1}{3}} \right\rfloor \Big) \Big(\left\lfloor (1-\mu)^{\frac{1}{3}} n^{\frac{1}{3}} \right\rfloor + 1 \Big) \\
&\geq \frac{(1-\mu)^{\frac{7}{3}}}{32} n^{\frac{4}{3}}.
\end{aligned}
$$

Then by using Theorem 3.17, we have

$$
f(G) \geq \sqrt{\frac{2\alpha}{1-\mu}} \geq \frac{(1-\mu)^{\frac{2}{3}}}{4} n^{\frac{2}{3}}.
$$

□

They further show if $\mu \leq n^{-\frac{2}{3}}$, then the value of $f(n,\mu)$ can be calculated almost exactly as $|f(n,\mu) - \mu^{\frac{1}{2}}n| \leq 1$. Looking back at the original question posed by Erdős, Łuczak and Spencer [ELS92] we get for $\mu = \frac{1}{2}$

$$\frac{1}{4^{\frac{4}{3}}}n^{\frac{2}{3}} - 1 \leq f\left(n, \frac{1}{2}\right) = \tilde{f}\left(n, \frac{n^2}{4} - \left\lfloor \frac{n^2}{4} \right\rfloor - \frac{n}{4}\right) \leq \left(2 + \frac{2}{\sqrt{3}}\right) n^{\frac{2}{3}} (\log n)^{\frac{1}{3}}.$$

Finally, we want to mention that Falgas-Ravry, Markström and Verstraëte [FRMV18] doubt that the argument used to prove Theorem 3.15 yields an asymptotically tight lower bound on $f(n,\mu)$. If $\mu = 1 - o(n^{-\frac{1}{7}})$ the lower bound can be superseded by that given in Theorem 3.17. Aside from the cases in Theorem 3.15, determining the order of magnitude of $f(n,\mu)$ for general μ poses an open problem.

Problem 3.16 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *For each $\mu \in (0,1)$, determine the order of magnitude of $f(n,\mu)$.*

3.3.1 Results Relating to Discrepancy

Recall the definition of positive discrepancy

$$\operatorname{disc}^+(G) = \max_{X \subseteq V(G)} (\|X\| - \mu\binom{|X|}{2}),$$

that is the maximum positive difference between the actual size and the expected size of a subgraph of G. It is not surprising that there should be a relation between full subgraphs (which have unexpectedly high minimum degree) and subgraphs with large positive discrepancy (which have unexpectedly many edges). Falgas-Ravry, Markström and Verstraëte [FRMV18] are the first to use this relation to bound the size of $f(G)$ from below, depending on the positive discrepancy of G. Indeed, a subgraph H of maximum discrepancy is always a full subgraph, because otherwise deleting a vertex of minimum degree from H would strictly increase the discrepancy. A greedy algorithm yields the following theorem.

Theorem 3.17 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *Let G be a graph of density μ with* $\operatorname{disc}^+(G) = \beta > 0$. *Then*

$$f(G) \geq (1-\mu)^{-\frac{1}{2}}(2\beta)^{\frac{1}{2}}.$$

The obtained result is best possible and in some cases better than Theorem 3.15. Consider the graph G consisting of a clique with $\binom{m}{2}$ edges and $n-m$ isolated vertices, obviously $f(G)=m$. A quick calculation shows that with $\operatorname{disc}^+(G)=\binom{m}{2}\left(1-\frac{\binom{m}{2}}{\binom{n}{2}}\right)$, Theorem 3.17 yields $f(G)\geq\left\lceil\sqrt{m(m-1)}\right\rceil=m$. On the other hand, if G is any graph on n vertices obtained by adding or removing $o(n^{\frac{4}{3}})$ edges in a complete multipartite graph with a bounded number of parts of balanced size, then $\operatorname{disc}^+(G)=o(n^{\frac{4}{3}})$ and the lower bound in Theorem 3.17 is superseded by Theorem 3.15.

In a random or pseudo-random setting, Falgas-Ravry, Markström and Verstraëte [FRMV18] are able to improve the lower bound on the size of a largest full subgraph by drawing from previous results on discrepancy and jumbledness. Jumbledness is a way to define how "random like" the edges of a graph are distributed. As a quick reminder, a graph G is (μ,j)-jumbled if for every $X\subseteq V(G)$ it holds that $\|X\|-\gamma\binom{|X|}{2}\leq j|X|$. If this condition is true for small j the graph is "well jumbled" and the following Theorem improves on Theorems 3.15 and 3.17.

Theorem 3.18 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *Suppose G is a (p,j)-jumbled graph of density p. Then*

$$f(G)\geq\frac{\operatorname{disc}^+(G)}{j}.$$

Extending results from Erdős and Spencer [ES72], who proved that for $p=\frac{1}{2}$ asymptotically almost surely $\operatorname{disc}^+(G(n,p))=\Theta(p^{\frac{1}{2}}(1-p)^{\frac{1}{2}}n^{\frac{3}{2}})$ and from Krivelevich and Sudakov [KS06] on jumbledness ($G(n,p)$ asymptotically almost surely has $|\operatorname{disc}(G)|=O(\sqrt{p(1-p)n}|X|)$), Theorem 3.18 shows that $f(G(n,p))=\Omega(n)$ asymptotically almost surely for all $p\in(0,1)$. Since random graphs are not the focus of this thesis, we mention the results from Riordan and Selby [RS00] for the sake of completeness. They showed $f(G(n,p))\leq cn+o(n)$ asymptotically almost surely for $c\approx 0.851\ldots$. Thus Falgas-Ravry, Markström and Verstraëte pose the following open problem.

Problem 3.19 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *For each fixed $p\in(0,1)$, prove the existence and determine the value of a real number $c=c_p$ such that for all $\delta>0$, $\mathbb{P}(|f(G(n,p)-c_pn|>\delta n)\to 0$ as $n\to\infty$.*

We close this section by mentioning algorithmic approaches on detecting dense subgraphs, found and collected in [BCC+10] and [APP+08].

3.4 Relatively Full Subgraphs

Another introduction of Falgas-Ravry, Markström and Verstraëte [FRMV18] is the notion of relatively full subgraphs. In a relatively full subgraph, the minimum degree of any vertex has to be at least a fixed fraction of the degree of that vertex in the original graph. They define a *relatively q-full subgraph H* of a graph G such that $d_H(v) \geq q d_G(v)$ for every $v \in V(H)$. They obtain the following result for $q = \frac{1}{2}$.

Theorem 3.20 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *Let G be an n-vertex graph. Then G contains a relatively half-full subgraph with $\lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{2} \rfloor + 1$ vertices.*

This is best possible in the sense that the smallest non empty relatively half-full subgraph of K_n has $\lfloor \frac{n}{2} \rfloor + 1$ vertices and the smallest relatively half-full subgraph of $K_{n,n}$ has $n+1$ vertices when n is odd. This can be seen in Figure 6.

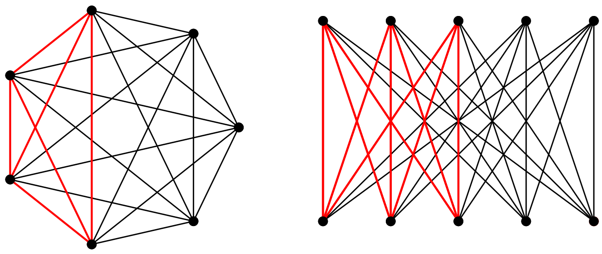


Figure 6: Relatively half full subgraphs of K_7 and $K_{5,5}$ in red.

If G is a d-regular graph for an integer d, Theorem 3.20 yields the existence of a full subgraph H of G on roughly half the vertices of G since for every vertex $v \in H$, we have $d_H(v) \geq \frac{1}{2} d_G(v) = \frac{d}{2}$. Note that G itself is full, since any regular graph is a full subgraph of itself.

Corollary 3.21 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *Let G be an n-vertex d-regular graph. Then G contains a full subgraph with $\lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{2} \rfloor + 1$ vertices.*

It should be mentioned that, while this is sufficient for full subgraphs, one can do better for small integers d relative to n. Alon [Alo97] showed that any d-regular graph on n vertices contains a subgraph on $\lceil \frac{n}{2} \rceil$ vertices with minimum degree at least $\frac{1}{2}d + cd^{\frac{1}{2}}$, which exceeds the requirement for a full subgraph by an additional factor of $cd^{\frac{1}{2}}$. However, this result does not hold true for large d. For general q Falgas-Ravry, Markström and Verstraëte prove the following Theorem.

Theorem 3.22 (Falgas-Ravry, Markström and Verstraëte [FRMV18]). *Let G be a graph on n vertices. Then for every $q \in [0, 1]$, G contains one of the following*

(i) a relatively q-full subgraph on $\lceil qn \rceil$ vertices, or

(ii) a relatively $(1-q)$-full subgraph on $\lfloor (1-q) \rfloor$ vertices, or

(iii) a relatively q-full subgraph on $\lceil qn \rceil$ vertices and a relatively $(1-q)$-full subgraph on $\lfloor (1-q)n \rfloor + 1$ vertices.

Using Theorem 3.22, they also prove an extension of Theorem 3.20 to relatively $\frac{1}{r}$-full subgraphs for integers $r \geq 3$.

Theorem 3.23 (Falgas-Ravry, Markström and Verstraëte [FRMV18]). *Let G be a graph on n vertices, and let $r \in \mathbb{N}$. Then G contains a relatively $\frac{1}{r}$-full subgraph on $\lfloor \frac{n}{r} \rfloor$, $\lceil \frac{n}{r} \rceil$ or $\lceil \frac{n}{r} \rceil + 1$ vertices.*

This is best possible in the sense that if $r \geq 3$, the complete graph K_n for some $n \geq r+2$ with $n \equiv 2 \mod r$ contains a smallest non-empty relatively $\frac{1}{r}$-full subgraph on exactly $\lceil \frac{n-1}{r} \rceil + 1 = \lceil \frac{n}{r} \rceil + 1$ vertices. For further work, they raise the question whether Theorem 3.23 can be extended to cover q-fullness for other q.

Problem 3.24 (Falgas-Ravry, Markström and Verstraëte [FRMV18]). *Determine whether there exists a constant c such that for every $q \in [0, \frac{1}{2}]$ every graph on n vertices has a relatively q-full subgraph with at least $\lfloor qn \rfloor$ vertices and at most $\lfloor qn \rfloor + c$ vertices.*

A cycle of length n shows for $q \geq \frac{1}{2}$ that there exist n-vertex graphs with no non-empty relatively q-full subgraphs on fewer than n vertices since any true, non-empty subgraph contains a vertex of degree one. One might try to circumvent this example by requiring a weaker degree condition. Define a subgraph H of a graph G to be *weakly relatively q-full* if $d_H(v) \geq \lfloor q d_G(v) \rfloor$ for all $v \in V(H)$. However, even with this notion of q-fullness a natural generalisation of Theorem 3.23 fails for rational $q > \frac{1}{2}$:

Consider C_n^2, the second power of a cycle of length n. If x is a vertex in a weakly relatively $\frac{3}{4}$-full subgraph H, then all but at most one of its neighbours must also belong to H. Since all vertices of distance at most two in the original cycle are connected, vertices not in H must lie at distance at least five apart in the original cycle. This implies that H must contain at least $\frac{4}{5}n$ vertices, rather than the $\frac{3}{4}n + O(1)$ we might have hoped for. An example can be seen in Figure 7.

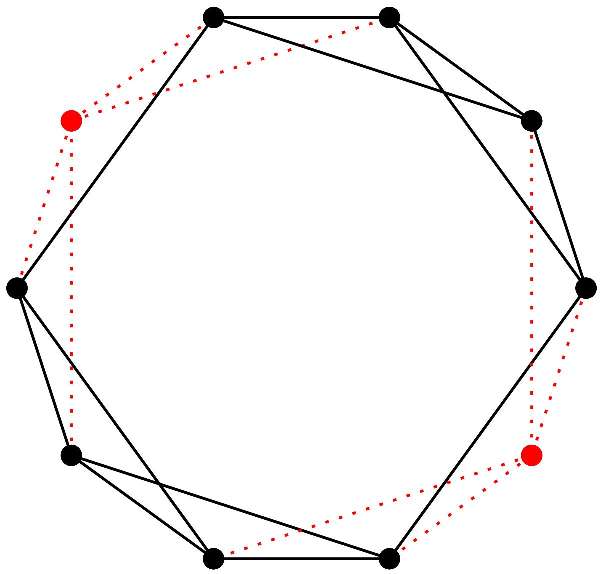


Figure 7: Relatively $\frac{3}{4}$-full subgraph of C_{10}^2 in black with excluded vertices in red.

Finally, it might be interesting to determine whether powers of paths or cycles provide the worst-case scenario for finding weakly relatively q-full subgraphs when $q > \frac{1}{2}$. This poses an open problem.

Problem 3.25 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *Let $q \in (\frac{1}{2}, 1)$. Determine whether there exists a constant $c_q < 1$ such that every graph on n vertices has a weakly relatively q-full m-vertex subgraph where $\lfloor qn \rfloor \leq m \leq c_q n$.*

4 Summary of Ramsey Type Results

In this chapter, we summarize results regarding the existence of minimum degree restricted subgraphs that are of a Ramsey theory type flavour, i.e. what is the size and minimum degree of monochromatic graphs one can expect by coloring the edges of a complete graph. Despite the superficial similarity to extremal problems, the Ramsey type problems in this chapter require different approaches, since any edge in the colored (complete) base graph appears in a monochromatic subgraph.

We start with a quick introduction into Ramsey theory in Section 4.1, where we state Ramseys Theorem. We then argue how and how not we can use classic Ramsey numbers for our purposes of finding monochromatic subgraphs with high minimum degree.

There are different ways to further specify "monochromatic subgraphs with high minimum degree". One of them by Erdős and Pach [EP83] results in *quasi Ramsey numbers*, which we handle in Section 4.2. Denoted by $R_\gamma(k)$, they represent the minimum order n of a complete graph G such that in any edge-two-coloring of G there exists a monochromatic subgraph of order at least k and minimum degree depending on $\in (0, 1)$. The asymptotics heavily depend on γ with a special point of interest around $\gamma = \frac{1}{2}$ that is further examined by Kang, Pach, Patel and Regts [KPPR15].

Section 4.3 summarizes results about a stricter version of quasi Ramsey numbers called *fixed quasi Ramsey numbers* denoted by $R^*_\gamma(k)$, where the required subset H of minimal degree at least $\gamma(|H| - 1)$ has order exactly k. This version is often handled alongside the (variable) quasi Ramsey numbers with some additional results from Kang, Long, Patel, Regts in [KLPR17]. The problem shows the same threshold phenomenon around $\gamma = \frac{1}{2}$.

In Section 4.4, we look at further work from Falgas-Ravry, Markström and Verstraëte [FRMV18]. They introduce the notion of *co-fullness*, whereas a graph H is *co-full* if the complement is full. They define a function $g(n)$ that is the order of the largest full or co-full subgraph, which they bound with methods and results from quasi Ramsey numbers.

Results from Caro and Yuster [CY03] are collected in Section 4.5. They search for monochromatic subgraphs of high minimum degree in graphs classified by order and minimum degree. Therefore, they define the function $h_G(d,r)$, which denotes the largest integer t such that in every coloring of the edges of a graph G with r colors there is a monochromatic subgraph H with minimum degree at least d and order at least t. Furthermore, they define $h(n,k,d,r)$ for $n > k > d$ as the minimum over $h_G(d,r)$, where G ranges over all graphs with n vertices and minimum degree at least k. They obtain general bounds on $h(n,k,d,r)$ for all cases and tight bounds whenever k is close to n.

In Section 4.6, we summarize results from Łuczak [Łuc16] and Liu and Person [LP09] on highly connected monochromatic subgraphs. They state bounds on the very general function $m(n,r,s,k)$, which bounds the minimum order of the maximum k-connected subgraph in at most s colors in any edge coloring of K_n with r colors. Since any k-connected graph has minimum degree at least k the results are relevant for us.

Finally Section 4.7 shows results about Ramsey core numbers from Bickle [Bic12] that are a generalization of our minimum degree Ramsey numbers. For positive integers $t_1,\ldots,t_r$ the *Ramsey core number* denoted by $R_r^{\mathscr{D}}(t_1,\ldots,t_r)$ is the minimum number n such that in any edge coloring of K_n in r colors, there is a color i such that the monochromatic subgraph H_i induced by color i contains a subgraph with minimum degree t_i. This subgraph is called a *t_i-core*. The simple minimum degree Ramsey number can be obtained by setting $t_1 = \ldots = t_r$. The main result is using a complicated algorithm of Klein and Schönheim [KS92], which we further inspect in Section 5.2 and in the Appendix.

To compare the different kinds of functions in the following chapter, we have:

- The *quasi Ramsey number* $R_\gamma(k)$ is the minimum $n \in \mathbb{N}$ such that in any edge-two-coloring of the edges of K_n exists a monochromatic subgraph H with $|H| \geq k$ and $\delta(H) \geq \gamma(|H|-1)$.
- The *fixed quasi Ramsey number* $R_\gamma^*(k)$ is the minimum $n \in \mathbb{N}$ such that in any edge-two-coloring of the edges of K_n is a monochromatic H with $|H| = k$ and $\delta(H) \geq \gamma(|H|-1)$.
- The function $g(n)$ is the minimum over all graphs G with order n of the maximum order of a subgraph H such that either $\delta(H) \geq \mu(|H|-1)$ or $\delta(\overline{H}) \geq (1-\mu)(|\overline{H}|-1)$,

whereas μ is the density of G. In short, $g(n) := \min\{g(G) \mid |V(G)| = n\}$ with $g(G) = \min\{f(G), f(\overline{G})\}$, whereas $f(G)$ is the maximum order of a full subgraph of G.

- The function $h_G(d, r)$ is the largest integer t such that in every coloring of the edges of the graph G with r colors there is a monochromatic subgraph H with minimum degree at least d and order at least t. Furthermore, $h(n, k, d, r)$ is the minimum of $h_G(d, r)$ over all graphs G with order n and minimum degree k.

- The function $m(n, r, s, k)$ is the minimum over the maximum order of any k-connected subgraph that contains edges of at most s different colors in any edge coloring of K_n with r colors.

- The Ramsey core number $R_r^{\mathscr{D}}(t_1, \ldots, t_r)$ is the minimum order of K_n such that in any edge coloring exists a color i such that there is a monochromatic subgraph of color i with minimum degree at least t_i.

4.1 Ramsey Theory

In this section, we want to present a quick introduction into Ramsey theory and bridge the gap into current research and how we might use it to find monochromatic subgraphs of high minimum degree. Ramseys Theorem [Ram09] states that for any integer $n \geq 0$, every large enough graph G contains either K_n or $\overline{K}_n$ as an induced subgraph, which is the same as stating that any edge coloring of a large enough complete graph in two colors contains a monochromatic K_n.

Theorem 4.1 (Ramsey [Ram09]). *For every $n \in \mathbb{N}$ there exists an integer m such that every graph of order at least m contains either K_n or $\overline{K}_n$ as an induced subgraph.*

While Theorem 4.1 guarantees the existence of a monochromatic complete subgraph, it does not specify exactly how big m has to be. The smallest integer m associated with the existence of a monochromatic K_n is called the *Ramsey number* $R(n)$. From Theorem 4.1 follows that we can find a monochromatic copy of any graph H in a sufficiently large graph G since for every graph H, there exists an integer n such that $H \subset K_n$. The minimum order of G is denoted by $R(H)$ in that case. Further generalizing, one can easily pose questions on the minimum order of a complete graph such that it contains either a red subgraph H or a blue subgraph J which is denoted by $R(H, J)$. If $H = K_s$ and $J = K_t$, $R(s,t)$ is used equivalently. Surveys and collections of Ramsey numbers can be found in [R$^+$94] and [CG83]. The best asymptotic bounds to date proven by Spencer [Spe75] and Conlon [Con09] are

$$(1 + o(1))\frac{\sqrt{2}n}{e}2^{\frac{n}{2}} \leq R(n) \leq n^{-\frac{c\log n}{\log\log n}}4^n.$$

We are interested in Ramsey numbers for monochromatic subgraphs of large minimum degree. On could think of an easy first approach to bounding those from above by checking the surveys and collections of Ramsey numbers and taking the smallest bounds for fixed subgraphs, which fulfill our minimum degree requirements. This yields a valid upper bound, but it is certainly not tight, since all Ramsey numbers are based on the existence of a specific subgraph for each color class, whereas we are satisfied with the existence of any monochromatic subgraph from a whole class of graphs of minimum degree at least n called $\mathscr{D}_n$. For lower bounds on $R^{\mathscr{D}}(n)$, we can not use classic Ramsey numbers since the requirement of the non-existence of any monochromatic subgraph with minimum degree at least n is much stronger than the non-existence of a fixed monochromatic subgraph (with minimum degree at least n).

4.2 Quasi Ramsey Numbers

The notion of quasi Ramsey numbers was introduced by Erdős and Pach in 1983 in "On a Quasi-Ramsey Problem" [EP83]. Quasi Ramsey numbers are a degree based generalisation of Ramsey numbers where instead of looking for complete monochromatic subgraphs, one looks for "fairly complete" subgraphs in the sense that the minimum degree of the vertices in the monochromatic subgraph is sufficiently large. To recap, we define the *quasi Ramsey number* $R_\gamma(n)$ as the minimum order m such that any two-coloring of the edges of K_m contains a monochromatic subgraph H of order at least n with $\delta(H) \geq \gamma(|V(H)| - 1)$. Erdős and Pach use a comparable definition that requires the monochromatic subgraph H of order n to have $\delta(H) \geq \gamma|V(H)|$, which we will indicate with $\tilde{R}_\gamma(n)$. This definition is slightly more strict, but somewhat inferior to the definition of Kang, Pach, Patel and Regts [KPPR15] that scales nicely into normal Ramsey numbers (by setting $\gamma = 1$ we get $R_\gamma(n) = R(n)$). Obviously we have $R_\gamma(n) \leq R(n)$.

Erdős and Pach were the first to observe the behaviour of $\tilde{R}_\gamma(n)$ for $0 < \gamma < 1$ and noticed a sharp change in behaviour around $\gamma = \frac{1}{2}$. For $\gamma < \frac{1}{2}$, the quasi Ramsey number grows at most linearly. Whereas for $\gamma > \frac{1}{2}$, the quasi Ramsey number grows at least exponentially. Although the following two propositions are originally formulated for $\tilde{R}_\gamma(n)$, a quick recalculation similar to the ones we do in Theorems 5.1 and 5.3 where we look at the quasi Ramsey number with a bipartite base graph shows that they also hold true for $R_\gamma(n)$.

Proposition 4.2 (Erdős, Pach [EP83])**.** *Let $0 < \gamma < \frac{1}{2}$. Then there exists a constant $c(\gamma)$ such that $R_\gamma(n) \leq c(\gamma)n$ for all n.*

Proposition 4.3 (Erdős, Pach [EP83])**.** *Let $\frac{1}{2} < \gamma < 1$. Then there exists a constant $c(\gamma) > 1$ such that $R_\gamma(n) \geq (c(\gamma))^n$ for all n.*

The propositions above yield no result for $\gamma = \frac{1}{2}$. This point of interest is addressed in the following two theorems. Theorem 4.4 yields an upper bound for $R_{\frac{1}{2}}(n)$ of order $\Theta(n \log n)$ by using a graph discrepancy argument. Proven by using a weighted random graph construction, Theorem 4.5 yields a lower bound for $R_{\frac{1}{2}}(n)$ of order $\Theta(\frac{n \log n}{\log \log n})$.

Theorem 4.4 (Erdős, Pach [EP83])**.** *For every $p \geq 0$, there is a constant $C_p > 0$ having the property that, if G is any graph of at least $C_p n \log n$ vertices, then either G or $\overline{G}$ contains a subgraph H satisfying*

- $|V(H)| \geq n$,
- $\delta(H) \geq \frac{|V(H)|}{2} + p|V(H)|^{\frac{1}{2}}(\log |V(H)|)^{\frac{3}{2}}$.

Theorem 4.5 (Erdős, Pach [EP83]). *Let p be any fixed natural number. Then there exists a constant $C'_p > 0$ such that, for each $n \geq n_0(p)$, one can find a graph G which has $C'_p(\frac{n \log n}{\log \log n})$ vertices and satisfies the following condition: If H is any subgraph of G or $\overline{G}$, and $|V(H)| \geq n$, then $\delta(H) < \frac{|V(H)|}{2} - p$.*

Taken together, both theorems imply

$$C' \frac{n \log n}{\log \log n} \leq R_{\frac{1}{2}}(n) \leq Cn \log n.$$

In 2015 Pach together with Kang, Patel and Regts [KPPR15] revised the abrupt change around $\gamma = \frac{1}{2}$. They obtain sharp results by the application of a short discrepancy argument and the analysis of a probabilistic construction similar to the one of Erdős and Pach.

Theorem 4.6 (Kang, Pach, Patel, Regts [KPPR15]). *(i) Let $\nu \geq 0$ and $c \geq \frac{4}{3}$ be fixed. For large enough n and any graph G with at least $n^{c10^6\nu^2+\frac{4}{3}}$ vertices, G or $\overline{G}$ has an induced subgraph H on $l \geq n$ vertices with minimum degree at least $\frac{1}{2}(l-1) + \nu\sqrt{(l-1)\ln l}$.*

(ii) There is a constant $C > 0$ such that, if $\nu(\cdot)$ is a nondecreasing nonnegative function, then for large enough n there is a graph G with at least $Cn^{\nu(n)^2+1}$ vertices such that the following holds. If H is any induced subgraph of G or $\overline{G}$ on $l \geq n$ vertices, then H has minimum degree less than $\frac{1}{2}(l-1) + \nu(l)\sqrt{(l-1)\ln l}$.

Since the construction for Theorem 4.5 remains valid slightly before the abrupt change around $\gamma = 1/2$, they also state a small technical improvement of said theorem.

Theorem 4.7 (Kang, Pach, Patel, Regts [KPPR15]). *For any $\nu > 0$, there exists $C_\nu > 0$ such that for large enough n, there is a graph G with at least $C_\nu n \log n / \log \log n$ vertices satisfying the following. If H is any induced subgraph of G or $\overline{G}$ on $l \geq n$ vertices, then H has minimum degree less than $(\frac{1}{2} - l^{-\nu})(l-1)$.*

For a more fine grained comparison of results Kang, Pach, Patel and Regts introduced the terminology of *t-homogenous* sets, which we want to mention here. A *t-homogenous* set is a vertex subset of a graph that induces either a graph of minimum degree at least

t or a graph where the complement has minimum degree at least t. This can easily be translated to our two color Ramsey definition by labeling edges in the graph as red edges and non-edges in the graph as blue edges. Let $f : \mathbb{Z}^+ \mapsto \mathbb{N}$ be a nondecreasing nonnegative integer function satisfying $f(l) < l$ for all l, then define the *variable quasi-Ramsey number* $\widehat{R}_f(n)$ to be the smallest integer such that any graph of order $\widehat{R}_f(n)$ contains a $f(l)$-homogenous set of order l for some $l \geq n$. We can now summarize results from this section. The calculations for some of the constants can be found in Section 5.1.

- For $f(l) = \gamma(l-1)$ and $0 \leq \gamma < \frac{1}{2}$

$$\widehat{R}_f(n) \leq \frac{1 - \frac{1}{2n}}{\frac{1}{2} - \gamma} n.$$

- For $f(l) = \frac{1}{2}(l-1)$, there exist constants $c_1, c_2 > 0$ such that for large enough n

$$c_1 \frac{n \log n}{\log \log n} \leq \widehat{R}_f(n) \leq c_2 n \log n.$$

.

- For $f(l) = \frac{1}{2}(l-1) + \nu\sqrt{(l-1)\log l}$ with $\nu \geq 0$ exist constants $c_1 \geq 0$, $c_2 \geq \frac{4}{3}$ such that for large enough n

$$c_1 n^{\nu^2+1} \leq \widehat{R}_f(n) \leq n^{c_2 10^6 \nu^2 + \frac{4}{3}}.$$

- For $f(l) = \gamma(l-1)$ and $\frac{1}{2} < \gamma \leq 1$

$$e^{(\gamma - \frac{1}{2})^2 n} \leq \widehat{R}_f(n).$$

Further reserach, for example quasi Ramsey numbers for more than two colors by Kang, Patel and Regts can be found in [KPR19].

4.3 Fixed Quasi Ramsey Numbers

The definition of quasi Ramsey numbers leaves some leeway on the order of the required subgraph, that only needs to fulfill a minimum order requirement. Erdős and Pach [EP83] look into a modification of the quasi Ramsey number that removes this leeway. Denoted by $R^*_\gamma(n)$ the *fixed quasi Ramsey number*, instead of requiring a monocromatic subgraph H with order at least n and sufficient minimum degree, requires H to have exactly n vertices and sufficient minimum degree $\delta(H) \geq \gamma(n-1)$. Per definition

$$R_\gamma(n) \leq R^*_\gamma(n)$$

holds for any n and any γ. They prove the following upper bound on $R^*_\gamma(n)$ which was later superseded by Theorem 4.11.

Theorem 4.8 (Erdős, Pach [EP83]). *There exists a constant $C > 1$ such that, if n and m are natural numbers ($m < n/2$) and G is any graph of at least $C^m n^2$ vertices, then either G or $\overline{G}$ contains a subgraph H satisfying*

(i) $|V(H)| = n$

(ii) $\delta(H) \geq \frac{n}{2} + m$.

Kang, Pach, Patel and Regts [KPPR15] provide a probabilistic thinning lemma that allows them to translate results about the (variable) quasi Ramsey problem into results about the fixed quasi Ramsey problem. Lemma 4.9 roughly says that, in any graph of large minimum degree, they can find an induced subgraph of predefined order that approximately preserves the minimum degree condition in an appropriate way.

Lemma 4.9 (Kang, Pach, Patel, Regts [KPPR15]). *For any $0 < c < 1$ and $\epsilon > 0$, let n be such that*

$$\exp\left(\frac{1}{2}\epsilon^2(n-1)\right) > n.$$

If H is a graph of order $l \geq n$ such that $\delta(H) \geq cl$, then there exists $S \subseteq V(H)$ of order n such that $\delta(H[S]) \geq (c-\epsilon)(n-1)$.

Using this thinning Lemma together with Theorem 4.4, they could prove the following result that yields a subquadratic upper bound on $R^*_\gamma(n)$ for γ slightly smaller than $\frac{1}{2}$.

Theorem 4.10 (Kang, Pach, Patel, Regts [KPPR15]). *There exists a constant $C > 0$ such that, for large enough n and any graph G with at least $Cn \log n$ vertices, G or $\overline{G}$ has an induced subgraph H on exactly n vertices with minimum degree at least $\frac{1}{2}(n-1) - 2\sqrt{(n-1)\log n}$.*

Since $R^*_\gamma(n) \geq R_\gamma(n)$, we know by Theorem 4.7 that the bound above is tight up to a $\log \log n$ factor. We want to mention that, like for the quasi Ramsey number, Kang, Pach, Patel and Regts defined a notation for fixed quasi Ramsey numbers based on t-homogenous sets. For integers t and n with $0 \leq t < n$, they define the *fixed quasi Ramsey number* $\widehat{R}^*_t(n)$ to be the smallest integer such that any graph of order $\widehat{R}^*_t(n)$ contains a t-homogenous set of order exactly n. Contrary to the homogenous set based notion for (variable) quasi Ramsey numbers, their notion for the fixed quasi Ramsey numbers $\widehat{R}^*_t(n)$ is essentially the same as $R^*_\gamma(n)$ with $\gamma = \frac{t}{n-1}$.

In 2017, Kang, Long, Patel and Regts [KLPR17] revisited the fixed quasi Ramsey numbers improving the order of magnitude of the upper bound for $\gamma = \frac{1}{2}$ from Cn^2 to $Cn \log n$. They manage this by using results about graph discrepancy from Erdős [ES74], an application of the celebrated "six standard deviations" result of Spencer [Spe85] and a greedy algorithm that was inspired by similar procedures for max-cut and min-bisection.

Theorem 4.11 (Kang, Long, Patel, Regts [KLPR17]). *There exists constants $C, D > 0$ such that for any graph G on $Cn \log n$ vertices, either G or its complement $\overline{G}$ has an induced subgraph on n vertices with minimum degree at least $\frac{1}{2}(n-1) + D\sqrt{(n-1)/\log n}$.*

Another result can be taken from Chappell and Gimbel [CG11]. They define the *defective Ramsey number* $\overline{R}_t(n)$ as the smallest integer m such that K_m contains a monochromatic subgraph of order n with maximum degree at most t. We may use these results for our fixed quasi Ramsey numbers, since the complement of a graph G with maximum degree t is a graph with minimum degree $|G| - 1 - t$. Therefore we obtain by their results

$$\widehat{R}^*_t(n) \leq (n-t-1)\binom{2(t-1)}{t-1} + \binom{2t}{t} \leq (n-t+3)4^{t-1}$$

for $1 \leq t \leq n-1$. Furthermore, they give an exact formula for $\widehat{R}^*_t(n)$ when t is between 1 and $\frac{n+2}{4}$. If that is the case, then $\widehat{R}^*_t(n) = n + 2t - 2$. The lower bound they give actually remains valid for all $t \leq \frac{n+1}{2}$. The construction of such a graph G is quite simple: Set $P = K_{2(t-1)}$, $Q = E_{2(t-1)}$ and let R be an arbitrary graph on $n - 2t + 1$ vertices. Then, put all possible edges between P and R to obtain a $K_{2(t-1),n-2t+1}$, put no edges between Q and R and put an arbitrary $(t-1)$-regular graph between P and Q. This construction can be seen in Figure 8.

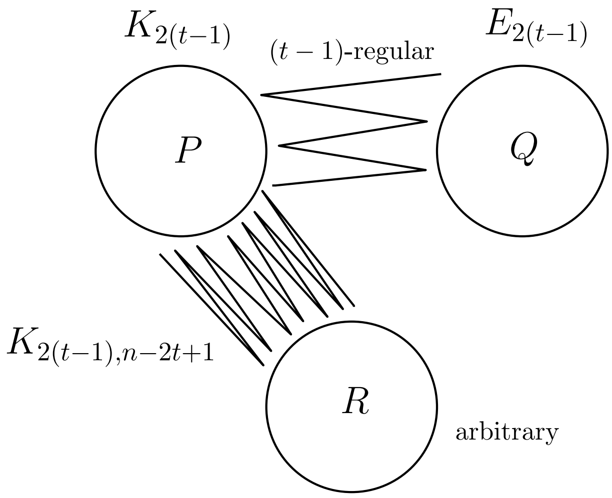


Figure 8: A construction on $n+2t-2$ vertices that shows $\widehat{R}^*_t(n) \geq n+2t-2$ for $t \leq \frac{n+1}{2}$.

Any subgraph H of G of order n must contain a vertex v from Q since $|P|+|R| = n-1$. Since the graph between P and Q is $(t-1)$-regular and Q is empty, we have $d(v) = t-1$. By symmetry, this argument holds true for $\overline{G}$ as well if we swap P and Q.

Finally, we can summarize the bounds in this section.

- For $1 \leq t \leq \frac{n+2}{4}$
$$\widehat{R}^*_t(n) = n + 2t - 2$$
- For $t \leq \frac{1}{2}n - 2\sqrt{(n-1)\log n}$, there exists a constant c such that
$$n + 2t - 2 \leq \widehat{R}^*_t(n) \leq cn \log n.$$
- For $t = \frac{1}{2}n$, there exist constants $c_1, c_2 > 0$ such that
$$c_1 \frac{n \log n}{\log \log n} \leq \widehat{R}^*_t(n) \leq c_2 n \log n.$$
- For $t = \frac{1}{2}n + D\sqrt{(n-1)/\log n}$ with a constant $D > 0$, there exists a constant $c > 0$ such that
$$\widehat{R}^*_t(n) \leq cn \log n.$$

- For $t = \frac{1}{2}n + \nu\sqrt{n \log n}$ with $\nu \geq 0$, there exists a constant $c > 0$ such that

$$cn^{\nu^2+1} \leq \widehat{R}_t^*(n).$$

- For $\frac{1}{2}n < t \leq n-1$

$$e^{(\gamma-\frac{1}{2})^2 n} \leq \widehat{R}_t^*(n) \leq (n-t+3)4^{t-1}.$$

4.4 Full and Co-Full Subgraphs

A similar approach to quasi Rasmey numbers was taken by Falgas-Ravry, Markström and Verstraëte [FRMV18]. After looking at full subgraphs, they introduce the notion of co-full subgraphs. A subgraph H of a graph G is *co-full* if $V(H)$ induces a full subgraph of $\overline{G}$. This means for a graph G with density μ, the subgraph H is *co-full* if it has maximum degree $\mu(|H|-1)$, which implies a minimum degree of $(1-\mu)(|\overline{H}|-1)$ for $\overline{H}$. This leaves a problem that is related to Ramsey numbers in the sense that we look for a substructure in G or $\overline{G}$. We denote the largest integer m such that G has a full or co-full subgraph on m vertices by $g(G) := \max\{f(G), f(\overline{G})\}$ and furthermore define $g(n) := \min\{g(G) \mid |V(G)| = n\}$.

We obtain a lower bound for $g(G)$ if G has density $\frac{1}{2}$ by Theorem 4.4, which states that for every n-vertex graph G, in either G or $\overline{G}$ there exists a subgraph with $m = \Omega(\frac{n}{\log n})$ vertices and minimum degree at least $\frac{1}{2}m$. In the general case, Falgas-Ravry, Markström and Verstraëte prove the following theorem.

Theorem 4.12 (Falgas-Ravry, Markström and Verstraëte [FRMV18])**.** *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 \frac{n}{\log n} \leq g(n) \leq c_2 \frac{n \log \log n}{\log n}.$$

The upper bound is proven by reusing and modifying a unusual weighted graph construction of Erdős and Pach [EP83]. Theorem 4.7 yields a graph G on $\Theta(n \log \log n / \log n)$ vertices with no large full or co-full subgraph. The only problem is that G does not have density $\frac{1}{2}$. This can be solved via the following construction: Let A and B be two disjoint sets of $2n$ vertices, where n is chosen such that $|G| = 2n$. Split each set into n pairs and place a random maximal matching into each of the n^2 sets of pairs, where one pair is from A and one from B. This yields a bipartite graph H between A and B with density $\frac{1}{2}$. By placing G into A and $\overline{G}$ into B the resulting graph G^* has an overall density of $\frac{1}{2}$ as seen in Figure 9. A probabilistic argument then shows that G^* contains no full or co-full subgraph.

They proposed it might be interesting to look deeper into $g(n)$, grouping the considered graphs according to their density.

Problem 4.13. *Define $g(n, \mu) := \min\{g(G) \mid |V(G)| = n, |E(G)| = \mu\binom{n}{2}\}$. Determine the order of magnitude of $g(n, \mu)$.*

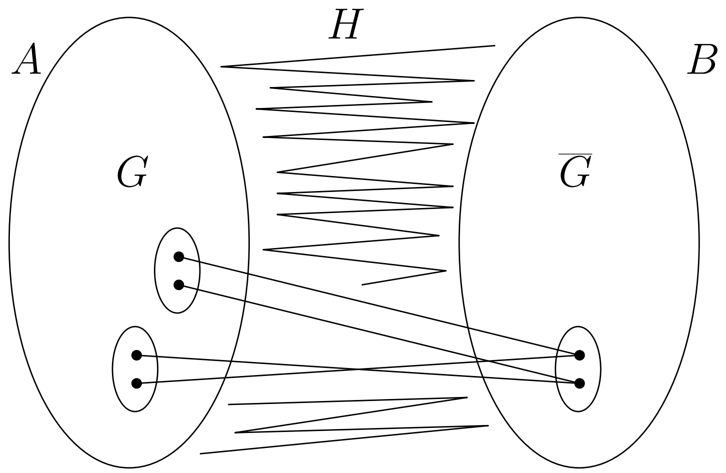


Figure 9: Using Theorem 4.7 to construct a graph of density $\frac{1}{2}$ with no full or co-full subgraph.

4.5 Monochromatic Subgraphs With Fixed Minimum Degree

In this section, we provide insights into work from Caro and Yuster [CY03]. They look into the existence of monochromatic subgraphs that fulfill a fixed minimum degree requirement, i.e. the required minimum degree does not scale with the order of the subgraph. They denote for a graph G and an integer d the function $h_G(d, r)$, defined as the largest integer t such that in every coloring of the edges of G with r colors there is a monochromatic subgraph H with minimum degree at least d and order at least t. They further define for $n > k > d$ the function $h(n, k, d, r)$, that is the minimum over $h_G(d, r)$ where G ranges over all graphs with n vertices and minimum degree at least k. They start with a result for two colors. The function $h_G(d, r)$ is distinct from quasi Ramsey numbers, since the minimum degree of the subgraph H does not depend on the order of H. If we pick the parameters $h(n, n-1, d, 2)$, we look for the minimum order of the largest subgraph with minimum degree d over all edge-two-colorings of K_n, i.e. $h(n, n-1, d, 2) = h_{K_n}(d, 2)$. The first result is for two-colorings.

Theorem 4.14 (Caro and Yuster [CY03]). *(i) For all $d \geq 1$ and $k \geq 4d-3$,*

$$h(n, k, d, 2) \geq \frac{k-4d+4}{2(k-3d+3)}n + \frac{3d(d-1)}{4(k-3d+3)}.$$

(ii) For all $d \geq 1$ and $k \leq 4d-4$, if n is sufficiently large then $h(n, k, d, 2) \leq d^2-d+1$. In particular, $h(n, k, d, 2)$ is independent of n.

It is interesting to note that, when k is at most about four times d, Theorem 4.14 yields an upper bound on $h(n, k, d, 2)$ that is independent of n and only depends on d. Key reason here is that the base graphs considered can be relatively sparse since they only need to fulfill a minimum degree requirement. They also give a general upper bound on $h(n, k, d, r)$, for colorings with r colors.

Theorem 4.15 (Caro and Yuster [CY03]). *For all $d \geq 1$, $r \geq 2$ and $k > 2r(d-1)$, there exists an absolute constant C such that*

$$h(n, k, d, r) \leq n\frac{k-2r(d-1)}{r(k-(r+1)(d-1))} + C.$$

In particular, $h(n, k, d, r) \leq \frac{k-4d+4}{2(k-3d+3)}n + C$.

Finally, they are able to determine $h(n, n-k, d, 2)$ exactly when k is small compared to n.

Theorem 4.16 (Caro and Yuster [CY03]). *Let d and k be positive integers such that n is at least the Ramsey number $R(4d+2k-5, 4d+2k-5)$, then $h(n, n-k, d, 2) = n-2d-k+3$.*

4.6 Highly Connected Monochromatic Subgraphs

In this section, we show some research that does not directly look for subgraphs of high minimum degree, but instead for k-connected subgraphs. Since any k-connected graph has minimum degree at least k (because otherwise one could remove all neighbours of a vertex of minimal degree to disconnect the graph), we mention the results here. Although the bounds might not be tight for our weaker requirements, they present a good start for our own research and contain useful insights. Bollobás and Gyárfás [BG08] conjectured that for $n > 4(k-1)$ every two-coloring of the edges of the complete graph K_n constains a k-connected monochromatic subgraph with at least $n-2(k-1)$ vertices. This conjecture was proven by Łuczak [Łuc16]. In that paper, he looks into highly connected monochromatic subgraphs of two colored complete graphs.

Theorem 4.17 (Łuczak [Łuc16])**.** *Let $k \geq 2$ and $n \geq 4k-3$. Then, in each two-coloring of the edges of K_n there exists either a monochromatic k-connected subgraph on more than $n-2(k-1)$ vertices, or there exist monochromatic k-connected graphs on $n-2(k-1)$ vertices in both colors.*

A construction from Bollobás and Gyárfás [BG08] shows that this is tight. Therefore, we take four sets of vertices $V_1, .., V_4$ of order $k-1$ and a set V_5 of order $n-4(k-1)$ to obtain a graph G on n vertices. We color the edges joining V_1 and V_2 and those joining V_3 and V_4 red. All edges with both ends in $V_2 \cup V_3 \cup V_5$ are colored red as well. All remaining edges are blue, like shown in Figure 10. We see, any largest monochromatic k-connected subgraph that contains a vertex $v \in V_i$ must contain all $w \in V_i$. Furthermore, any largest monochromatic subgraph must contain V_5 to be k-connected. Thus, it follows that any largest monochromatic k-connected subgraph in this graph contains V_5 and two of the other sets.

Using Szemerédis Regularity Lemma, [Sze75] Liu, Morris and Prince [LP09] prove further related results. They weaken the requirement of monochromatic subgraphs to subgraphs with at most s different colors, which is of no particular interest for our case. Still, they show some results for $s = 1$. They define for $n, r, s, k \in \mathbb{N}$ with $s \leq r$ and $k \leq n$, given a graph $G = (V, E)$ with $|V(G)| = n$ and an r-coloring of its edges $c : E(G) \to [r]$:

$$M(c, G, e, s, k) := \max\{|V(H)| : H \subseteq G,\ |c(E(H))| \leq s \text{ and } H \text{ is } k\text{-connected}\}.$$

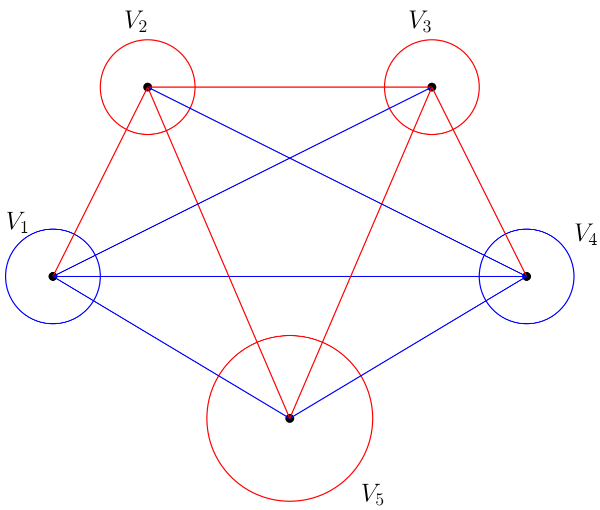


Figure 10: A sketch of an edge coloring of a graph on n vertices with sets $V_1, \dots, V_4$ of order $k-1$ and V_5 of order $n-4(k-1)$. Colored circles imply the color of all edges contained inside, whereas colored lines show the color of all edges in the bipartite graph between two sets.

That means $M(c, G, r, s, k)$ is the order of the largest k-connected subgraph H in G whose edges are colored with at most s different colors in the r-coloring c. They further define

$$m(G, r, s, k) := \min_c \{M(c, G, r, s, k)\}.$$

If $G = K_n$, they write $M(c, n, r, s, k)$ and $m(n, r, s, k)$ respectively. The following theorem yields a lower bound on the minimum order of a largest k-connected monochromatic subgraph over all colorings of K_n.

Theorem 4.18 (Liu and Person [LP09])**.** *For every $\gamma \in (0, \frac{1}{4}), n, r \in \mathbb{N}$ with $r \geq 3$, there exist integers $N_0 = N_0(\gamma, r)$ and $T_0 = T_0(\gamma, r)$ such that, for all $n \geq N_0$,*

$$m\left(n, r, 1, \frac{1-4\gamma}{rT_0}n\right) \geq \frac{n}{r-1} - \frac{6\gamma n}{r-1}.$$

In particular, for fixed r, and $k = o(n), m(n, r, 1, k) \geq \frac{n}{r-1} - o(n)$, with equality if $r-1$ is a prime power.

Finally, we want to mention a result on the order of k-connected monochromatic subgraphs, if we choose a bipartite base graph. Therefore, define for $n, n', r, s, k \in \mathbb{N}$ with $s \leq r$ and $k \leq n \leq n'$ and an r-coloring $c : E(K_{n,n'}) \to [r]$ the functions

$$M_{\text{bip}}(c, n, n', r, s, k) := \max\{|V(H)| : H \subseteq K_{n,n'},\ |c(E(H))| \leq s \text{ and } H \text{ is } k\text{-connected}\},$$

and

$$m_{\mathrm{bip}}(n, n', r, s, k) := \min_{c}\{M_{\mathrm{bip}}(c, n, n', r, s, k)\}.$$

Theorem 4.19 (Liu and Person [LP09]). *For every $\gamma \in (0, \frac{1}{34}), n, n', r \in \mathbb{N}$ with $r \geq 2$ and $n' \geq n$, there exist integers $N_0 = N_0(\gamma, r)$ and $T_0 = T_0(\gamma, r)$ such that, for all $n \geq N_0$,*

$$\frac{n+n'}{r} - \frac{3\gamma(n+n')}{r} \leq m_{bip}\left(n, n', r, 1, \frac{1-4\gamma}{rT_0}n\right) \leq \frac{n+n'}{r} + 2.$$

In particular, for fixed r, and $k = o(n)$ we have $m_{bip}(n, n', r, 1, k) = \frac{n+n'}{r} - o(n)$.

4.7 Ramsey Core Numbers

Bickle [Bic12] defines the *k-core* $C_k(G)$ of a graph G as the maximum induced subgraph $H \subseteq G$ such that $\delta(H) \geq k$, if it exists. This notion was introduced by Seidman [Sei83] and has been studied extensively by Bickle in [Bic10]. It is easy to see that the cores of a graph are nested. i.e. $C_{k+1}(G) \subset C_k(G)$. Bickle furthermore defines the *core number* $C(v)$ of a vertex $v \in V(G)$ as the largest value k such that $v \in C_k(G)$. The maximum core number of a graph $\widehat{C}(G)$ is the maximum of the core numbers of the vertices of G. The k-core of a graph can be determined by iteratively deleting vertices of degree less than k until only the k-core remains. This implies that the maximum core number of a graph G is equal to its degeneracy. Note that, if $k > \widehat{C}(G)$ the k-core is empty.

Theorem 4.20 (Bickle [Bic12])**.** *Every graph with order n, size $m \geq (k-1)n - \binom{k}{2} + 1$, $1 \leq k \leq n-1$ has a k-core.*

Bickle focuses on properties of maximal k-degenerate graphs, i.e. k-degenerate graphs where no edge can be added without violating the k-degeneracy. The basic properties of maximal k-degenerate graphs were established by Lick and White [LW70] and Mitchem [Mit77], a survey type collection of results can be found in [SP76]. Some of them are below.

Theorem 4.21 (Bickle [Bic12])**.** *Let G be a maximal k-degenerate graph of order n, $1 \leq k \leq n-1$. Then*

(i) G contains a $(k+1)$-clique and for $n \geq k+2$, G contains $K_{k+2} - e$ as a subgraph.

(ii) For $n \geq k+2$, G has $\delta(G) = k$, and no two vertices of degree k are adjacent.

(iii) G has connectivity $\kappa(G) = k$.

(iv) For any integer r, $1 \leq r \leq n$, G contains a maximal k-degenerate graph of order r as an induced subgraph. For $n \geq k+2$, if $d(v) = k$, then G is maximal k-degenerate if and only if $G - v$ is maximal k-degenerate.

(v) G is maximal 1-degenerate if and only if G is a tree.

The paper is of special interest for this thesis as they present some results on *Ramsey core numbers*. Given nonnegative integers $t_1, \ldots, t_r$, the *Ramsey core number* is the smallest n such that for all edge colorings of K_n with k colors, there exists an index i

such that the subgraph H_i induced by color i has a t_i-core. In Bickle [Bic12] the Ramsey core number is denoted by $rc(t_1, \ldots, t_r)$, but we extend the definition of the minimum degree Ramsey number for $r \geq 2$ by defining $R_r^{\mathscr{D}}(t_1, \ldots, t_r) := rc(t_1, \ldots, t_r)$. He obtains several basic results immediately.

Proposition 4.22 (Bickle [Bic12]). *(i)* $R_r^{\mathscr{D}}(t_1, \ldots, t_r) \leq R_r^{\mathscr{D}}(t_1 + 1, \ldots, t_r + 1)$.

(ii) $R_r^{\mathscr{D}}(t_1 + 1, \ldots, t_r) \geq R_r^{\mathscr{D}}(t_1, \ldots, t_r) + 1$.

(iii) For any permutation σ of $[r]$, $R_r^{\mathscr{D}}(t_1, \ldots, t_r) = R_r^{\mathscr{D}}(t_{\sigma(1)}, \ldots, t_{\sigma(r)})$.

(iv) $R_r^{\mathscr{D}}(0, t_2, \ldots, t_r) = 1$.

(v) $R_r^{\mathscr{D}}(1, t_2, \ldots, t_r) = R_{r-1}^{\mathscr{D}}(t_2, \ldots, t_r)$.

Proposition 4.23 (Bickle [Bic12]). *Let $t_1 = t_2 = \ldots = t_r = 2$, then $R_r^{\mathscr{D}}(t_1, \ldots, t_r) = 2r + 1$.*

Bickle determines $R_2^{\mathscr{D}}(2, t)$ for infinitely many t. He notices that an upper bound on $R_2^{\mathscr{D}}(2, t)$ can be expressed as a piecewise linear function with each piece having slope one that breaks at the triangular numbers. For the lower bound, he gives an explicit construction based on a *caterpillar graph* as seen in Figure 11. For $t = \binom{k}{2} + 1$, take a $(k, k, k-1, \ldots, 3, 2)$-caterpillar T. Since T is a tree it has no 2-core. For $\overline{T}$ the spine vertices $v_1, \ldots, v_k$ have degrees $\binom{k}{2}, \binom{k}{2}, \binom{k}{2} + 1, \binom{k}{2} + 1, \ldots, \binom{k}{2} + r - 3, \binom{k}{2} + r - 2$. The $(\binom{k}{2} + 1)$-core of $\overline{T}$ can not contain v_1 and v_2. We see one by one, that the spine vertices are not contained in the $(\binom{k}{2} + 1)$-core. Finally, the remaining $\binom{k}{2} + 1$ vertices are too few and thus the $(\binom{k}{2} + 1)$-core is empty.

Theorem 4.24 (Bickle [Bic12]). *Let $t = \binom{k}{2} + q$, $1 \leq q \leq k$. Then $R_2^{\mathscr{D}}(2, t) = t + k + 1$.*

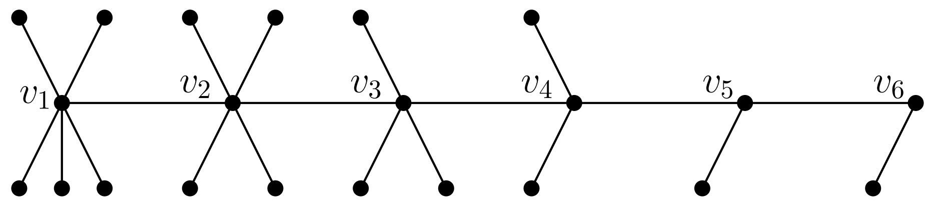


Figure 11: A $(6, 6, 5, 4, 3, 2)$-caterpillar with spine length six and spine vertices of degrees $(6, 6, 5, 4, 3, 2)$ that shows $R_2^{\mathscr{D}}(2, \binom{6}{2} + 1) = \binom{6}{2} + 6 + 2$.

Finally, using results from Klein and Schönheim [KS92], Bickle states

$$R_r^{\mathscr{D}}(t_1, \ldots, t_r) = \left\lceil \frac{1}{2} - r + \sum t_i + \sqrt{\left(\sum t_i\right)^2 - \sum t_i^2 + (2-2r)\sum t_i + r^2 - r + \frac{9}{4}} \right\rceil,$$

where all sums range over $i = 1, \ldots, r$. Parts of this result are further explained in Section 5.2. The upper bound for this result is obtained by the application of the Pigeonhole Principle and Theorem 4.20, we show a similar proof in Section 5.2. The lower bound is obtained from the algorithm of Klein and Schönheim [KS92] that will be shown and explained in Section 5.2 and in the Appendix.

5 New Results and Extensive Explanations

In this chapter, we provide new insights into the existence of subgraphs with high minimum degree and provide extensive explanations on already proven results. We focus primarily on the quasi Ramsey number and the minimum degree Ramsey number.

In Section 5.1, we calculate bounds on $R_\gamma^{\text{bip}}(n)$ for $\gamma \in [0, \frac{1}{2})$ and $\gamma \in (\frac{1}{2}, 1]$. Recall, we define the *bipartite quasi Ramsey number* $R_\gamma^{\text{bip}}(n)$ to be the minimum integer m such that any edge two-coloring of $K_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}$ contains a monochromatic subgraph H of order at least n with $\delta(H) \geq \gamma |V(H)|/2$. Furthermore, we use the same methodology to explicitly calculate the constants in Propositions 4.2 and 4.3.

In Section 5.2, we obtain tight bounds on the *minimum degree Ramsey number* $R_r^{\mathscr{D}}(n)$, which is the smallest integer m such that any edge-coloring of K_m in r colors contains a monochromatic subgraph of minimum degree at least n. In the first part, we focus on the two colored case, where we first give easy estimates for the linearity of the minimum degree Ramsey number. Then, we provide a counting argument based on the Pigeonhole Principle as well as a simple coloring algorithm to precisely calculate $R_2^{\mathscr{D}}(n)$. In the second part, we look at the multicolored case where we use a coloring algorithm from Klein and Schönheim [KS92] to obtain further tight results comparable to Bickle [Bic10].

5.1 Bipartite Quasi Ramsey Number

We start by proving results analogous to the quasi Ramsey number of Erdős and Pach [EP83]. Therefore, we define the *bipartite quasi Ramsey number* $R_\gamma^{\text{bip}}(n)$ as the minimum integer m such that any edge-two-coloring of a balanced bipartite graph on m vertices contains a monochromatic subgraph of order at least n with $\delta(H) = \gamma |V(H)|/2$. In the first Theorem 5.1, we show that $R_\gamma^{\text{bip}}(n)$ grows linear in n if $\gamma \in [0, \frac{1}{2})$, similar to Proposition 4.2. This uses an argument similar to the proof of Theorem 4 in [EP83], where we can bound the maximum degrees of vertices between specific vertex sets.

Theorem 5.1. *Let $\gamma \in [0, \frac{1}{2})$. Then $R_\gamma^{bip}(n) \leq cn$ with $c = \frac{1}{\frac{1}{2} - \gamma}$.*

Proof. Let G be a balanced bipartite graph on $m = cn$ vertices, where c will be determined later. For any two-coloring of the edges of G in red and blue G, let G_r denote the graph containing only the red edges and G_b the blue analogue. Without loss of generality, we assume $\|G_r\| \geq \|G_b\|$. By Pigeonhole Principle, this implies

$$\|G_r\| \geq \max\left(\|G_\mathrm{r}\|, \|G_b\|\right) \geq \frac{m^2}{8}.$$

We define H_1 to be a induced subgraph of G_r on n vertices with maximum size. Let $x = \lfloor \frac{m}{n} \rfloor = \frac{m-k}{n}$ with $k \in \{0, \ldots, n-1\}$. We further define $G_0 = G_r$ and for $i \in \{1, \ldots, x\}$

$$G_i = G_r - \bigcup_{j=1}^{i} H_j$$

and for $i \in \{2, \ldots, x\}$

$$H_i = \max_{\substack{H \subset G_{i-1} \\ |H|=n}} \|H\|.$$

Note that $|G_x| = k < n$. We assume that for every $i \in \{1, \ldots, x\}$, there exists a vertex $v_i \in V(H_i)$ such that $d_{H_i}(v_i) < \gamma|H_i|/2$. Furthermore, we observe that any vertex $v \in V(G_i)$ has less than $\gamma|H_i|/2$ neighbours in $V(H_i)$. If this were not true, then by replacing v_i by v, we would get a graph whose number of edges is greater than $\|H_i\|$, contradicting the definition. Thus, we can bound $\|G_r\|$ from above by

$$\begin{aligned}
\|G_r\| = \|G_0\| &\leq \sum_{i=1}^{x} \|H_i\| + \|G_x\| + \binom{x}{2}\frac{\gamma n^2}{2} \\
&< \frac{xn^2}{4} + \|G_x\| + \frac{\gamma x^2 n^2}{4} \\
&= \frac{xn^2}{4} + \frac{k^2}{4} + \frac{\gamma xkn}{2} + \frac{\gamma x^2 n^2}{4} \\
&= \frac{(m-k)n}{4} + \frac{k^2}{4} + \frac{\gamma k(m-k)}{2} + \frac{\gamma (m-k)^2}{4} \\
&= \frac{\gamma}{4}m^2 + \frac{n}{4}m + \left(\frac{k^2}{4} - \frac{kn}{4} - \frac{\gamma k^2}{4}\right) \\
&\leq \frac{\gamma}{4}m^2 + \frac{n}{4}m.
\end{aligned}$$

This, together with $\|G_r\| \geq m^2/8$, yields a contradiction if $\gamma m^2 + nm < m^2/2$. Thus, we calculate with $m = cn$

$$\gamma m^2 + nm \leq \frac{m^2}{2} \Leftrightarrow \gamma c^2 n^2 + cn^2 \leq \frac{c^2 n^2}{2}$$
$$\Leftrightarrow \gamma c + 1 \leq \frac{c}{2}$$
$$\Leftrightarrow \frac{1}{\frac{1}{2} - \gamma} \leq c.$$

We conclude that $R_\gamma^{\text{bip}}(n) \leq \frac{2}{1-2\gamma} n$.

□

By the same methodology, we can calculate the constant $c(\gamma)$ in Proposition 4.2 to be $(1 - \frac{1}{2n})/(\frac{1}{2} - \gamma)$. The calculations change a little, the definitions stay the same.

Proof of Proposition 4.2. Let G be a complete graph on $m = cn$ vertices, where c will be calculated later. Again, for any two-coloring of G let G_r denote the red graph and G_b the blue graph. Without loss of generality, we assume $\|G_r\| \geq \|G_b\|$. By Pidgeonhole Principle this implies

$$\|G_r\| \geq \max(\|G_r\|, \|G_b\|) \geq \frac{\binom{m}{2}}{2}.$$

Define H_1 to be a induced subgraph of G_r on n vertices with maximum size, $x = \lfloor \frac{m}{n} \rfloor = \frac{m-k}{n}$ with $k \in \{0, \ldots, n-1\}$. Further define $G_0 = G_r$ and for $i \in \{1, \ldots, x\}$

$$G_i = G_r - \bigcup_{j=1}^{i} H_j$$

aswell as for $i \in \{2, \ldots, x\}$

$$H_i = \max_{\substack{H \subset G_{i-1} \\ |H| = n}} \|H\|.$$

Like above, $|G_x| = k < n$. We assume that for every $i \in \{1, \ldots, x\}$, there exists a vertex $v_i \in V(H_i)$ such that $d_{H_i}(v_i) < \gamma(|H_i| - 1)$. Furthermore, we observe that any vertex $v \in V(G_i)$ has less than $\gamma(|H_i| - 1)$ neighbours in $V(H_i)$. If this were not true, then by replacing v_i by v, we would get a graph whose number of edges is greater than $\|H_i\|$, contradicting the definition. Thus, we can bound $\|G_r\|$ from above by

$$\begin{aligned}\|G_r\| = \|G_0\| &\leq \sum_{i=1}^{x} \|H_i\| + \|G_x\| + \binom{x}{2}\gamma n(n-1) \\ &< \frac{xn(n-1)}{2} + \|G_x\| + \frac{\gamma x^2 n^2}{2} \\ &< \frac{xn(n-1)}{2} + \frac{k^2}{2} + \gamma xkn + \frac{\gamma x^2 n^2}{2} \\ &= \frac{(m-k)(n-1)}{2} + \frac{k^2}{2} + \gamma k(m-k) + \frac{\gamma(m-k)^2}{2} \\ &= \frac{\gamma}{2}m^2 + \frac{(n-1)}{2}m + \left(\frac{k^2}{2} - \frac{k(n-1)}{2} - \frac{\gamma k^2}{2}\right) \\ &\leq \frac{\gamma}{2}m^2 + \frac{(n-1)}{2}m.\end{aligned}$$

This, together with $\|G_r\| \geq \binom{m}{2}/2$ yields a contradiction if $\gamma m^2 + (n-1)m \leq \binom{m}{2}$. Thus, we calculate with $m = cn$

$$\begin{aligned}\gamma m^2 + nm - m \leq \binom{m}{2} &\Leftrightarrow \gamma c^2 n^2 + cn^2 - cn \leq \frac{c^2n^2}{2} - \frac{cn}{2} \\ &\Leftrightarrow \gamma cn + n - \frac{1}{2} \leq \frac{cn}{2} \\ &\Leftrightarrow \frac{1-\frac{1}{2n}}{\frac{1}{2}-\gamma} \leq c.\end{aligned}$$

□

For $\gamma \in (\frac{1}{2}, 1]$, Theorem 5.3 proves that $R_\gamma^{\mathrm{bip}}(n)$ grows exponential like the quasi Ramsey number from Proposition 4.3. Theorem 5.3 is proven via a simple probabilistic argument. Therefore we need Hoeffding's inequality [Hoe94].

Theorem 5.2 (Hoeffding [Hoe94])**.** *Let $X \sim Bin(n,p)$ and $k \leq np$ then*

$$\mathbb{P}(X \leq k) = \exp\left(-2\frac{(np-k)^2}{n}\right)$$

Theorem 5.3. *Let $\gamma \in (\frac{1}{2}, 1]$. Then $R_\gamma^{bip}(n) \geq c^n$ with $c = e^{\frac{1}{4}(\gamma-\frac{1}{2})^2}$.*

Proof. We will use a probabilistic argument to show that there exists a coloring of G that does not contain a monochromatic subgraph H of order at least n with minimum degree at least $\gamma|H|/2$.

Let G be a full bipartite graph on $m = e^{\frac{1}{4}n(\gamma-\frac{1}{2})^2}$ vertices with parts A and B of balanced size. An edge between any pair of vertices xy with $x \in A$ and $y \in B$ is colored red with independent probability $p = \frac{1}{2}$, otherwise blue. Without loss of generality, for any $X \subseteq V(G)$ let G_X denote a subgraph induced by X using only red edges from G. Further define A_X as $A \cap X$.

If $\delta(G_X) \geq \gamma|X|/2$, then G_X has at least $\gamma|X|^2/4$ edges. For $k \in \mathbb{N}$ let Z_k be a independent random variable with $Z_k \sim \mathrm{Bin}(k, \frac{1}{2})$. The probability of a fixed subgraph G_X to fulfill $\delta(G_X) \geq \gamma|X|/2$ can be bounded from above by

$$\begin{aligned}
\mathbb{P}\left(\delta(G_X) \geq \gamma|X|/2\right) &\leq \mathbb{P}\left(\|G_X\| \geq \frac{\gamma|X|^2}{4}\right) \\
&= \mathbb{P}\left(Z_{|A_X|(|X|-|A_X|)} \geq \frac{\gamma|X|^2}{4}\right) \\
&\leq \mathbb{P}\left(Z_{\frac{|X|^2}{4}} \geq \frac{\gamma|X|^2}{4}\right) \\
&= \mathbb{P}\left(Z_{\frac{|X|^2}{4}} \leq (1-\gamma)\frac{|X|^2}{4}\right) \\
&\leq \exp\left(\frac{-2\left(\frac{|X|^2}{8} - (1-\gamma)\frac{|X|^2}{4}\right)^2}{\frac{|X|^2}{4}}\right) \\
&= \exp\left(-\frac{1}{2}\left(\gamma - \frac{1}{2}\right)^2 |X|^2\right).
\end{aligned}$$

We used Hoeffding's inequality 5.2 with $n = |X|^2/4$, $p = 1/2$ and $k = (1-\gamma)|X|^2/4$ to obtain the upper bound for the binomial distribution. Now, we can calculate the probability that G as a whole does not have a monochromatic subgraph of high enough minimum degree. Therefore, we define G as *bad* if there exists a set $X \subseteq V(G)$ with $|X| \geq n$ and $\delta(G_X) \geq \gamma|X|/2$.

$$\begin{aligned}
\mathbb{P}\left(\text{G is bad}\right) &\leq 2\sum_{j=n}^{m}\binom{m}{j}e^{-\frac{1}{2}(\gamma-\frac{1}{2})^2j^2}\\
&\leq 2\sum_{j=n}^{m}\left(\frac{em}{j}\right)^j e^{-\frac{1}{2}(\gamma-\frac{1}{2})^2j^2}\\
&\leq \sum_{j=n}^{m}\frac{2e^j}{j^j}e^{\frac{1}{4}(\gamma-\frac{1}{2})^2nj}e^{-\frac{1}{2}(\gamma-\frac{1}{2})^2j^2}\\
&\leq \sum_{j=n}^{m}e^{-\frac{1}{4}(\gamma-\frac{1}{2})^2j^2}\\
&\leq e^{\frac{1}{4}(\gamma-\frac{1}{2})^2n}e^{-\frac{1}{4}(\gamma-\frac{1}{2})^2n^2}\\
&= e^{-\frac{1}{4}n(n-1)(\gamma-\frac{1}{2})^2}\\
&< 1
\end{aligned}$$

Thus, G is bad with probability less than one. This means that there exists a coloring of G which contains no monochromatic subgraph G_X of order at least n and minimum degree at least $\gamma|X|/2$. This implies $R_\gamma^{\text{bip}}(n) \geq c^n$ with $c = e^{\frac{1}{4}(\gamma-\frac{1}{2})^2}$.

□

Using the methodology above we can calculate the constant $c(\gamma)$ in Proposition 4.3 where we obtain $c(\gamma) = e^{(\gamma-\frac{1}{2})^2}$.

Proof of Proposition 4.3. Let G be a full bipartite graph on m vertices. We independently color an edge between any pair of vertices xy red with probability $p = \frac{1}{2}$, otherwise blue. Without loss of generality, for any $X \subseteq V(G)$ let G_X denote a subgraph induced by X using only red edges from G.

If $\delta(G_X) \geq \gamma(|X|-1)$ then G_X has at more than $\frac{\gamma|X|^2}{2}$ edges. For $k \in \mathbb{N}$, let Z_k be a independent random variable with $Z_k \sim \text{Bin}(k, \frac{1}{2})$. We can bound the probability of a fixed subgraph G_X to fulfill $\delta(G_X) \geq \gamma(|X|-1)$ by

$$\begin{aligned}
\mathbb{P}\left(\delta(G_X) \geq \gamma(|X|-1)\right) &\leq \mathbb{P}\left(\|G_X\| > \frac{\gamma|X|^2}{2}\right) \\
&= \mathbb{P}\left(Z_{\binom{|X|}{2}} > \frac{\gamma|X|^2}{2}\right) \\
&\leq \mathbb{P}\left(Z_{\frac{|X|^2}{2}} > \frac{\gamma|X|^2}{2}\right) \\
&\leq \mathbb{P}\left(Z_{\frac{|X|^2}{2}} \leq (1-\gamma)\frac{|X|^2}{2}\right) \\
&\leq \exp\left(\frac{-2\left(\frac{|X|^2}{4} - (1-\gamma)\frac{|X|^2}{2}\right)^2}{\frac{|X|^2}{4}}\right) \\
&= \exp\left(-2\left(\gamma - \frac{1}{2}\right)^2 |X|^2\right)
\end{aligned}$$

Now, we can calculate the probability that G as a whole does not have a monochromatic subgraph of sufficient size and minimum degree. Again, we define G as *bad* if there exists a set $X \subseteq V(G)$ with $|X| \geq n$ and $\delta(G_X) \geq \gamma|X|/2$. We choose $m = e^{(\gamma-\frac{1}{2})^2 n}$.

$$\begin{aligned}
\mathbb{P}\left(\text{G is bad}\right) &\leq 2\sum_{j=n}^{m} \binom{m}{j} e^{-2(\gamma-\frac{1}{2})^2 j^2} \\
&\leq 2\sum_{j=n}^{m} \left(\frac{em}{j}\right)^j e^{-2(\gamma-\frac{1}{2})^2 j^2} \\
&\leq \sum_{j=n}^{m} \frac{2e^j}{j^j} e^{(\gamma-\frac{1}{2})^2 nj} e^{-2(\gamma-\frac{1}{2})^2 j^2} \\
&\leq \sum_{j=n}^{m} e^{-(\gamma-\frac{1}{2})^2 j^2} \\
&\leq e^{(\gamma-\frac{1}{2})^2 n} e^{-(\gamma-\frac{1}{2})^2 n^2} \\
&= e^{-n(n-1)(\gamma-\frac{1}{2})^2} \\
&< 1
\end{aligned}$$

Again, since G is bad with probability less than one this means there exists a coloring of G which contains no monochromatic subgraph G_X of sufficient order and minimum degree if $|G| \leq e^{n(\gamma-\frac{1}{2})^2}$

$\square$

5.2 Minimum Degree Ramsey Number

In this section, we take a deeper look into results of Bickle [Bic10][Bic12] and Klein and Schönheim [KS92]. We provide results and explanations as well as alternative proofs. Recall the *minimum degree Ramsey number* $R_r^{\mathscr{D}}(n)$ only depends on the minimum degree of the subgraph. We define, for any $n \in \mathbb{N}$, the class $\mathscr{D}_n$ as the class of all graphs with minimum degree at least n. By definition, the classes are nested, i.e. $\mathscr{D}_n \subset \mathscr{D}_{n-1}$. For integers n and r with $r \geq 2$, we define the *minimum degree Ramsey Number* $R_r^{\mathscr{D}}(n)$ as the smallest integer m such that any edge-r-coloring of K_m contains a monochromatic subgraph G with $G \in \mathscr{D}_n$. We may use Bickle's notion of *k-cores*, whereas a k-core of a graph G is defined as the maximal induced subgraph $H \subset G$ such that $\delta(H) \geq k$ if it exists.

5.2.1 Minimum Degree Ramsey Number for Two Colors

In the two colored case, we first obtain simple bounds on $R_2^{\mathscr{D}}(n)$. Then, we improve them to tight bounds for all $n \in \mathbb{N}$ by using a new coloring algorithm for the lower bound and a counting argument for the upper bound. For simplicity, we write $R^{\mathscr{D}}(n) := R_2^{\mathscr{D}}(n)$. First, we introduce a proposition that allows us to only search for subgraphs of minimum degree exactly n since any subgraph of minimum degree greater than n must contain a subgraph with minimum degree exactly n.

Lemma 5.4. *Any graph G in $\mathscr{D}_n \setminus \mathscr{D}_{n+1}$ contains a subgraph $H \subset G$ with $H \in \mathscr{D}_{n-1} \setminus \mathscr{D}_n$.*

Proof. Let $G \in \mathscr{D}_n \setminus \mathscr{D}_{n+1}$. Let $v \in G$ be a vertex of G with $d_G(v) = n$. Let w be a neighbour of v in G. Removing a vertex from G reduces the degree of all remaining vertices in G by at most one. Thus, $\delta(G - w) = n - 1$ since $d_{G-w}(v) = d_G(v) - 1 = n - 1$. □

Obviously $R^{\mathscr{D}}(1) = 2$, since any two vertices connected by an edge induce a subgraph of minimum degree one. A simple argument based on forests is used below to show that $R^{\mathscr{D}}(2) = 5$.

Proposition 5.5. *$R^{\mathscr{D}}(2) = 5$.*

Proof. Take K_4 with the vertices v_1, v_2, v_3 and v_4. Color the edges v_1v_2, v_2v_3 and v_3v_4 red, the rest blue as seen in Figure 12. Then, the red and blue subgraph both form a path, thus there exists no monochromatic subgraph with minimum degree two in this coloring of K_4.

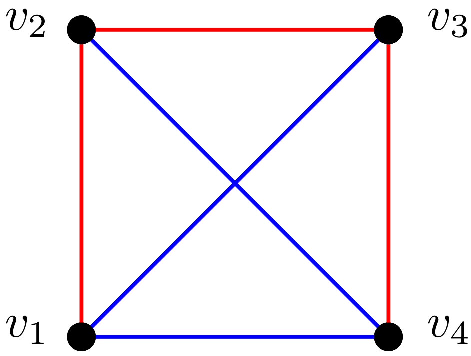


Figure 12: Edge-coloring of K_4 in two colors with no subgraphs of minimum degree greater than one.

Furthermore, any non-empty graph with no subgraph of minimum degree two must be a forest. A forest on five vertices contains at most four edges. Since $2 \cdot 4 < 10 = \binom{5}{2}$, any edge-two-coloring of K_5 must contain a monochromatic subgraph with minimum degree at least 2. $\square$

As a simple general lower bound it is easy to show that for any $n \geq 2$, $R^{\mathscr{D}}(n) \geq 2n+1$. We show that there exists a graph on $2n$ vertices that does not contain a subgraph H with minimum degree $\delta(H) \geq n$. Therefore, we take two red K_n and color all edges between them blue, except for one edge xy which will also be red. Then, there can be no monochromatic red subgraph H_r of sufficient minimum degree, because except for x and y all vertices have red degree $n-1$. There can also be no monochromatic blue subgraph H_b of sufficient minimum degree since x and y have $d_b(x) = d_b(y) = n-1$ and thus can not be in H_b. Since the remaining vertices form a blue $K_{n-1,n-1}$, we are done. An example can be seen in Figure 13.

A simple upper bound for the general case is obtained by using the following lemma about the existence of subgraphs with minimum degree of a fraction of the average degree of the original graph.

Lemma 5.6. *Let G be a graph with average degree ν. Then, there exists a subgraph H with minimum degree at least $\lceil \nu/2 \rceil$.*

Proof. Let G be a n-vertex graph with average degree ν. This implies $\|G\| = \frac{n\nu}{2}$. Define $G = G_0$. Now, in every step we remove a vertex v_i of minimal degree from G_{i-1} if $d_{G_{i-1}}(v) < \frac{\nu}{2}$ to obtain G_i. Suppose the algorithm does not terminate until all vertices are removed. Since this would imply

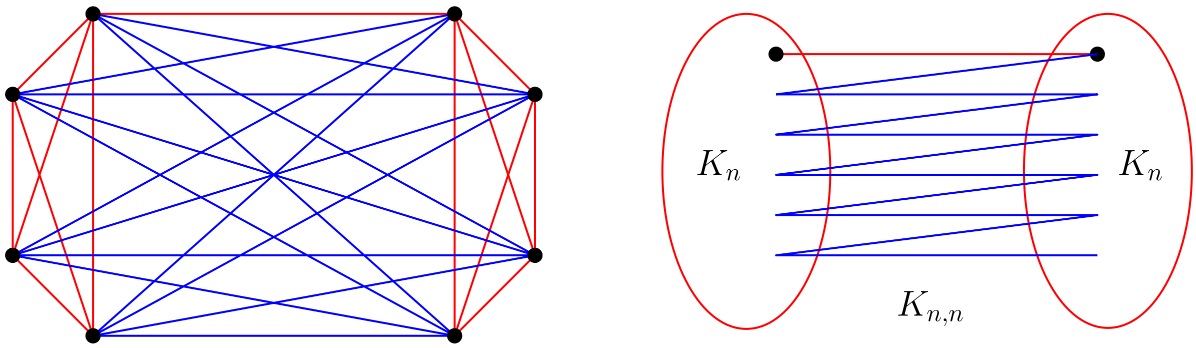


Figure 13: A general construction of a graph on $2n$ vertices with no subgraph of minimum degree n as two red K_n with a blue $K_{n,n}$ with one red edge in between and an example for $n = 4$.

$$\|G\| = \sum_{i=1}^{n} d_{G_{i-1}}(v_i) < \frac{n\nu}{2}$$

we get a contradiction. Thus, the algorithm has to terminate for some G_k with $k < n$. Then, $H = G_k$ is a subgraph with $\delta(H) \geq \lceil \frac{\nu}{2} \rceil$. $\square$

With Lemma 5.6, we prove that for any $n \in \mathbb{N}$ we have $R^{\mathscr{D}}(n) \leq 4n$. Therefore, we let C be any red and blue coloring of K_{4n}. Without loss of generality, we assume that there are at least as many red as blue edges. Let G be the graph consisting of all vertices and only the red edges. We have

$$\|G\| \geq \frac{\binom{4n}{2}}{2} = \frac{4n(4n-1)}{4},$$

thus the average degree of G is $\frac{4n-1}{2}$. Using Lemma 5.6, we get that G contains a subgraph H with minimum degree $\lceil \frac{4n-1}{4} \rceil = n$. We conclude that $R^{\mathscr{D}}(n)$ grows linear in n.

A tight upper bound can be obtained by an inductive approach, where we iteratively remove a vertex of minimal degree to calculate the maximum number of edges in a graph that does not contain a subgraph with minimum degree n. Theorem 5.9 yields an upper bound on the Ramsey number $R^{\mathscr{D}}(n)$ which is approximately $(2 + \sqrt{2})n \approx 3,41n$. We first have to introduce two lemmas. The first yields a vertex with bounded degree, if a graph does not contain a subgraph of high minimum degree.

Lemma 5.7. *Let G be a graph that does not contain a subgraph H with $\delta(H) = n$. Then there exists a vertex v in $V(G)$ with $d(v) \leq n-1$.*

Proof. Suppose there exists no vertex v with $d(v) \leq n-1$. Then $\delta(G) \geq n$ and we choose $H = G$. □

We can use Lemma 5.7 to prove the following lemma via induction. An equivalent statement can be found in [CY03] and [Bol04].

Lemma 5.8. *For fixed natural numbers n and m, any m-vertex graph G that does not contain a subgraph H with $\delta(H) \geq n$ has at most $(m-n)(n-1) + \frac{n(n-1)}{2}$ edges.*

Proof. Define $G_0 = G$. Since G does not contain a subgraph H with $\delta(H) \geq n$, by Lemma 5.7 there exists a vertex $v_1 \in V(G_0)$ with $d_{G_0}(v_1) \leq n-1$. Define $G_i = G_{i-1} - v_i$ with v_i from Lemma 5.7. We have $d_{G_{i-1}}(v_i) \leq n-1$. Stop when $|G_i| = n$. Since only n vertices are left, we may assume $G_i = K_n$ to obtain the maximum number of remaining edges. Now we can bound $|G|$ by

$$|G| \leq \sum_{i=1}^{m-n} d_{G_{i-1}}(v_i) + \binom{n}{2} = (m-n)(n-1) + \frac{n(n-1)}{2}.$$

□

Theorem 5.9. *For $n \in \mathbb{N}$, we have $R^{\mathscr{D}}(n) \leq \left\lfloor \left(2 + \sqrt{2}\sqrt{1 + \frac{1}{8(n-1)^2}}\right)(n-1) + \frac{3}{2} \right\rfloor$.*

Proof. Let m be an integer depending on n that will be determined later. We use Lemma 5.8 and the Pigeonhole Principle to bound the order of K_m based on the number of monochromatic edges in any edge-two-coloring of K_m.

K_m has $\binom{m}{2}$ edges. By Pigeonhole Principle in any edge-two-coloring, one color class has at least $\binom{m}{2}/2 = (m-1)m/4$ edges. Thus, by Lemma 5.8 any edge-two-coloring of K_m contains a monochromatic subgraph H with $\delta(H) \geq n$ if

$$\begin{aligned}
&\frac{(m-1)m}{4} > (m-n)(n-1) + \frac{n(n-1)}{2} \\
\iff\ & m^2 - m > 4mn - 4m - 4n^2 + 4n + 2n^2 - 2n \\
\iff\ & m^2 + (3-4n)m + 2n^2 - 2n > 0.
\end{aligned}$$

For fixed n, the quadratic equation $g(m) := m^2 + (3-4n)m + 2n^2 - 2n = 0$ has the two solutions

$$m_{1,2} = \frac{4n - 3 \pm \sqrt{8n^2 - 16n + 9}}{2}.$$

We take m as the largest integer for which $g(m)$ is negative. That is

$$m = \left\lfloor \frac{4n - 3 + \sqrt{8n^2 - 16n + 9}}{2} \right\rfloor = \left\lfloor \left(2 + \sqrt{2}\sqrt{1 + \frac{1}{8(n-1)^2}}\right)(n-1) + \frac{1}{2} \right\rfloor.$$

Thus, if we pick K_{m+1} in any edge-two-coloring either the red or the blue subgraph will contain a subgraph with minimum degree at least n. This implies the theorem. □

Lemma 5.7 yields an interesting relation between degenerate graphs and graphs with no subgraph of certain minimum degree. Recall, a graph G is called *q-degenerate* if there exists a left to right ordering of the vertices $V(G)$ such that each vertex sends at most q edges to the right.

Corollary 5.10. *For any graph G and $n \in \mathbb{N}$, G has no subgraph H with $\delta(H) \geq n$ if and only if G is $(n-1)$-degenerate.*

Proof. "$\Rightarrow$" : Use Lemma 5.7 iteratively to obtain a vertex ordering that shows $(n-1)$-degeneracy.

"$\Leftarrow$" : Let $(v_1, v_2, \ldots, v_n)$ be an ordering of the vertices of G that shows $(n-1)$-degeneracy. Let H be a subgraph of G with $V(H) = (v_{H_1}, v_{H_2}, \ldots, v_{H_k})$ whereas $H_1 < H_2 < \ldots < H_k$. Then $\delta(H) \leq d(v_{H_1}) \leq n - 1$. □

Since Corollary 5.10 leaves quite a lot of flexibility in how to construct a graph such that it contains no subgraph with minimum degree at least n, we use this freedom to distribute edges in a way to obtain a good lower bound on $R^{\mathscr{D}}(n)$. Therefore, we need to construct a graph G that is $(n-1)$–degenerate, where the complement $\overline{G}$ contains no subgraph with minimum degree n.

One way to obtain a lower bound of $R^{\mathscr{D}}(n) \geq 3n - 1$ for $n \in \mathbb{N}$ is to take C_{3n-2}^{n-1} and remove all edges that cross an imaginary line between two vertices that are adjacent on the cycle, as seen in Figure 14. The graph on the left side can not contain a subgraph of minimum degree at least n since it is $(n-1)$–degenerate, which can be seen by numbering the vertices clockwise starting from the cut. The graph on the right can not contain a subgraph of minimum degree at least n, since the vertices without a

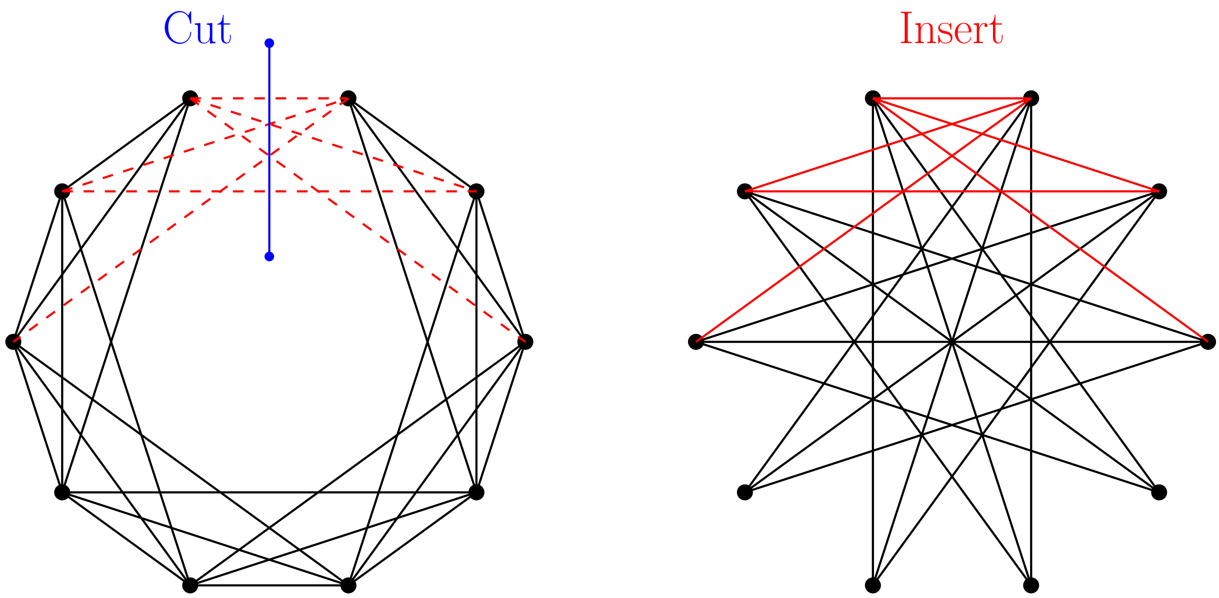


Figure 14: "Coloring" of K_{10} with no subgraph of minimum degree at least 4.

red edge from the cut have degree $n-1$ and thus can not be in such a subgraph. The remaining graph is a $K_{n-1,n-1}$ which can also not contain a subgraph of minimum degree n. This result is tight with the upper bound from Theorem 5.9 until $n=4$, but can be further improved by constructing the red and blue coloring of K_m such that each color class resembles a $(n-1)$–degenerate graph. Therefore, we define an algorithm that constructs a perfect packing of two $(n-1)$-degenerate graphs into a complete graph of order $m(n)$. Since the algorithm presented below manages this for $m(n) = \left\lfloor \left(2+\sqrt{2}\sqrt{1+\frac{1}{8(n-1)^2}}\right)(n-1)+\frac{1}{2} \right\rfloor$, we obtain a tight lower bound on $R^{\mathscr{D}}(n)$ for all $n \in \mathbb{N}$.

Theorem 5.11. *For $n \in \mathbb{N}$, $R^{\mathscr{D}}(n) \geq \left\lfloor \left(2+\sqrt{2}\sqrt{1+\frac{1}{8(n-1)^2}}\right)(n-1)+\frac{3}{2} \right\rfloor$.*

Proof. We provide a coloring algorithm that yields a perfect packing of two $(n-1)$–degenerate graphs R and B into K_m with $m := m(n) = \left\lfloor \left(2+\sqrt{2}\sqrt{1+\frac{1}{8(n-1)^2}}\right)(n-1)+\frac{1}{2} \right\rfloor$. The basic idea is to combine two $(n-1)$-degenerate graphs in opposite directions on a vertex set $v_1, \ldots, v_m$, i.e. one graph sends at most $(n-1)$ edges to the right (with increasing overall degrees to the right) and the other graph at most $(n-1)$ edges to the left (with increasing overall degrees to the left) for each vertex. Then, Corollary 5.10 implies that K_m does not contain a monochromatic subgraph with minimum degree n. An example of the following construction for $n=5$

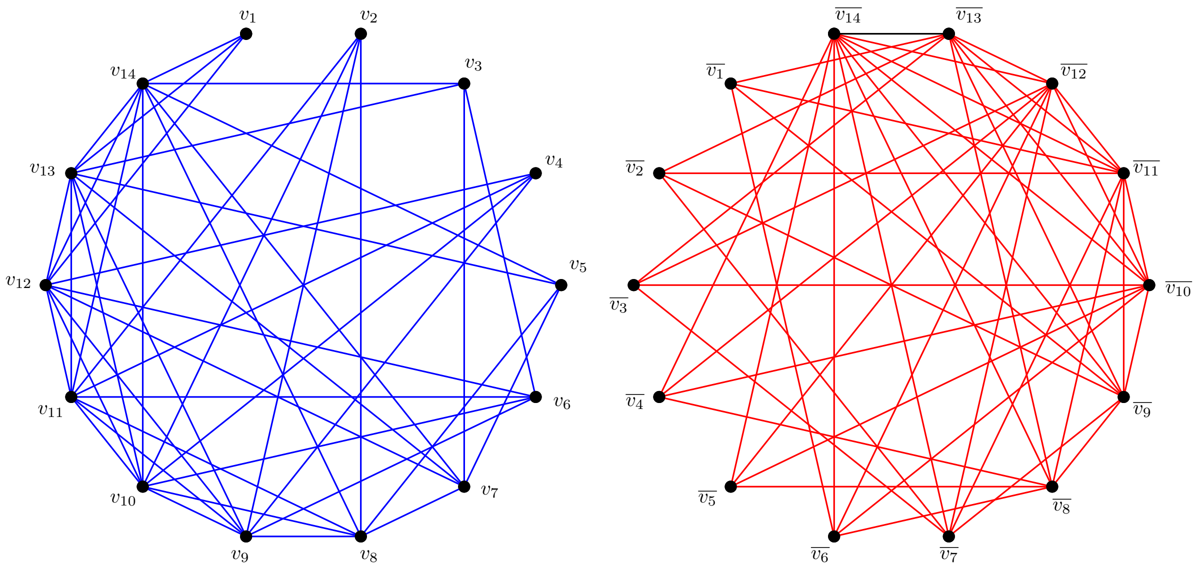


Figure 15: A perfect packing of two 4–degenerate graphs into K_{14} with B in blue and R in red whereas the black edge can be placed in either graph.

can be seen in Figure 15.

Let $v_1, v_2, \dots, v_m$ be m ordered vertices. For convenience, we further label $\overline{v}_i = v_{m+1-i}$ for $i = 1, \dots, m$ as a reverse labeling and define $B_i = B[\overline{v}_1, \dots, \overline{v}_i]$. We construct B by starting with $B_n = K_n$ and iteratively connecting each vertex $\overline{v}_i$ for $i = n+1, \dots, m$ with at most $n-1$ edges to vertices in $\{\overline{v}_1, \overline{v}_2, \dots, \overline{v}_{i-1}\}$, which will be selected in a way that $R_i := \overline{B}_i$ is $(n-1)$–degenerate. Therefore, we calculate a lower bound on $d_B(\overline{v}_i)$ for each $\overline{v}_i$. For $i = 1, \dots, n$, we want $d_B(\overline{v}_i) \geq m-n$. For $i = n+1, \dots, m-n$, we want $d_B(\overline{v}_i) \geq m-i$ and for the last $\overline{v}_i$ with $i = m-n+1, \dots, m$, we want $d_B(\overline{v}_i) \geq n-1$. An overview can be seen in Table 2.

Claim 1: If B is $(n-1)$–degenerate with the vertex ordering $v_1, \dots, v_m$ and each vertex has degree as seen in Table 2, then $R = \overline{B}$ is $(n-1)$–degenerate with the vertex ordering $\overline{v}_1, \dots, \overline{v}_m$.

Proof: For $i = 1, \dots, n$, $d_B(\overline{v}_i) \geq m-n$ implies $d_R(\overline{v}_i) \leq (m-1)-(m-n) = n-1$. Since B is $(n-1)$–degenerate, we have for $i = n+1, \dots, m-n$ that $d_{B_{i-1}}(\overline{v}_i) \leq n-1$, which implies $d_{R_{i-1}}(\overline{v}_i) \geq (i-1)-(n-1) = i-n$. Thus $d_{R[\overline{v}_{i+1},\dots,\overline{v}_m]}(\overline{v}_i) = (m-1)-d_R(\overline{v}_i)-d_{R_{i-1}}(\overline{v}_i) \leq (m-1)-(m-i)-(i-n) = n-1$. The last n vertices $\overline{v}_{m-n+1}, \dots, \overline{v}_m$ send at most $n-1$ edges to the right by definition of the vertex ordering. ■

vertex	v_m	$\dots$	v_{m-n+1}	v_{m-n}	$\dots$	v_{n+1}	v_n	$\dots$	v_1
	$\overline{v}_1$	$\dots$	$\overline{v}_n$	$\overline{v}_{n+1}$	$\dots$	$\overline{v}_{m-n}$	$\overline{v}_{m-n+1}$	$\dots$	$\overline{v}_m$
$d_B(\cdot) \geq$	$m-n$	$\dots$	$m-n$	$m-n-1$	$\dots$	n	$n-1$	$\dots$	$n-1$
$d_R(\cdot) \leq$	$n-1$	$\dots$	$n-1$	n	$\dots$	$m-n-1$	$m-n$	$\dots$	$m-n$

Table 2: Minimum degrees in $(n-1)$-degenerate B such that R is also $(n-1)$-degenerate.

All that is left to show is that there exists a construction of a $(n-1)$–degenerate B such that the degree requirements for Claim 1 are fulfilled. Therefore, define for $i = 2, \dots, m+1$ and $j = 1, \dots, i-1$

$$f_i(\overline{v}_j) = \begin{cases} (m-n) - d_{R_{i-1}}(\overline{v}_j) & \text{if } j \leq n, \\ (m-j) - d_{R_{i-1}}(\overline{v}_j) & \text{if } n < j < m-n+1, \\ (n-1) - d_{R_{i-1}}(\overline{v}_j) & \text{if } j \geq m-n+1, \end{cases}$$

with

$$f(\overline{v}_j) = f_{m+1}(\overline{v}_j) = \begin{cases} (m-n) - d_R(\overline{v}_j) & \text{if } j \leq n, \\ (m-j) - d_R(\overline{v}_j) & \text{if } n < j < m-n+1, \\ (n-1) - d_R(\overline{v}_j) & \text{if } j \geq m-n+1. \end{cases}$$

We start constructing B on $V = \{v_m, \dots, v_1\} = \{\overline{v}_1, \dots, \overline{v}_m\}$ by setting $B_n = K_n$. Now, we iterate for each step $i = n+1, \dots, m$:

1. Calculate $f_i(\overline{v}_1), \dots, f_i(\overline{v}_{i-1})$.

2. Define A_i as a set of $n-1$ vertices from $V(B_{i-1})$ with the largest values of $f_i(\cdot)$. If there are more than $n-1$ possible choices, choose the ones with the smallest indices in $\overline{v}_i$ notation.

3. Connect $\overline{v}_i$ to all vertices $v \in A_i$ with $f_i(v) > 0$.

An example on how the process distributes new edges can be seen in Table 3. We need the following claims to ensure that the construction above always yields a valid graph that fulfills Claim 1. To achieve that, we need to show that the algorithm puts $n-1$ edges in each step until all degree requirements are achieved. Observe that, by definition of the algorithm $f_i(\overline{v}_k) \geq f_i(\overline{v}_j) \geq 0$ for $i = n+1, \ldots, m+1$ and $n+1 \leq k < j \leq i-1$.

$f_i(\overline{v}_j)$	1	2	3	4	5	6	7	8	9	10	11	12	13
6	5	5	5	5	5								
7	5	4	4	4	4	4							
8	4	4	4	3	3	3	3						
9	3	3	3	3	3	3	2	2					
10	3	3	2	2	2	2	2	2	1				
11	2	2	2	2	2	2	1	1	1	0			
12	2	2	1	1	1	1	1	1	1	0	0		
13	1	1	1	1	1	1	1	0	0	0	0	0	
14	1	1	1	0	0	0	0	0	0	0	0	0	0

Table 3: Values of $f_i(\overline{v}_j)$ in each step of the algorithm for $n = 5$, whereas i is on the vertical axis and j on the horizontal axis.

Claim 2: In any step $i = n+1, \ldots, m$, if $\max\{f_i(\overline{v}_1), \ldots, f_i(\overline{v}_k)\} \neq \min\{f_i(\overline{v}_1), \ldots, f_i(\overline{v}_k)\}$ for any $k \in \{1, \ldots, i-1\}$, then

$$\begin{aligned} &\min\{f_{i+1}(\overline{v}_1), \ldots, f_{i+1}(\overline{v}_k)\} = \min\{f_i(\overline{v}_1), \ldots, f_i(\overline{v}_k)\} - 1 \\ \Rightarrow &\max\{f_{i+1}(\overline{v}_1), \ldots, f_{i+1}(\overline{v}_k)\} = \max\{f_i(\overline{v}_1), \ldots, f_i(\overline{v}_k)\} - 1. \end{aligned}$$

Proof: We know the minimum and maximum can change at most by one. Suppose there exists a step $i \in \{n+1, \ldots, m\}$, where only the minimum changes. Then, there exists $w \in V(B_{i-1})$ with $f_{i+1}(w) < \min\{f_i(\overline{v}_1), \ldots, f_i(\overline{v}_{i-1})\}$ and $v \in V(B_{i-1})$ with $f_{i+1}(v) = \max\{f_i(\overline{v}_1), \ldots, f_i(\overline{v}_{i-1})\}$. This implies $w \in A_i$ and $v \notin A_i$ which is a contradiction since $f_i(w) < f_i(v)$. ■

Claim 3: The algorithm distributes $n-1$ edges in every step $i \leq m$ that does not result in $f_{i+1}(\overline{v}_1) = \ldots = f_{i+1}(\overline{v}_i) = 0$.

Proof: Suppose there is a first step $i > n+1$ where the algorithm places less than $n-1$ edges. Since $f_i(\overline{v}_j) - f_{i+1}(\overline{v}_j) \leq 1$ for any $j \in \{1, \ldots, i-1\}$, Claim 2 together with $f_{n+1}(\overline{v}_1) = f_{n+1}(\overline{v}_2) = \ldots = f_{n+1}(\overline{v}_n)$ implies that there exists a $k < n-1$ such that $f_i(\overline{v}_1) = \ldots = f_i(\overline{v}_k) = 1$ and $f_i(\overline{v}_{k+1}) = f_i(\overline{v}_m) = 0$. ■

Suppose after the algorithm finishes, there exists a vertex v with $f(v) > 0$. This implies the number of edges needed to be placed by the algorithm E_{need}

$$\begin{aligned}2E_{\text{need}} &= n(m-n) + n(n-1) + \sum_{i=n+1}^{m-n} (m-i) \\ &= n(m-n) + n(n-1) + (m-2n)m + \frac{n^2+n}{2} - \frac{(m-n)^2-(m-n)}{2} \\ &= \frac{m^2-m}{2} = \binom{m}{2}\end{aligned}$$

exceeds the amount of edges that can be placed by the algorithm E_{poss}, which are by Claim 3 and choice of m

$$E_{\text{poss}} = (m-n)(n-1) + \binom{n}{2} \geq \frac{\binom{m}{2}}{2}.$$

That is a contradiction. Since B is $(n-1)$–degenerate by construction and $f(\overline{v}_1) = \ldots = f(\overline{v}_m) = 0$, Claim 1 implies that R is $(n-1)$–degenerate as well. Finally, Corollary 5.10 implies the theorem.

□

Theorem 5.11 together with Theorem 5.9 yields a tight bound on $R^{\mathscr{D}}(n)$ for all $n \in \mathbb{N}$. Thus, we conclude that

$$R^{\mathscr{D}}(n) = \left\lfloor \left(2 + \sqrt{2}\sqrt{1 + \frac{1}{8(n-1)^2}}\right)(n-1) + \frac{3}{2} \right\rfloor.$$

5.2.2 Multicolored Minimum Degree Ramsey Number

Since we obtained tight bounds in the previous section, we focus our attention on the usage of more colors for the multicolored minimum degree Ramsey number. Remember, the *minimum degree Ramsey number* for $r \geq 2$ colors $R^{\mathscr{D}}_r(n)$ is defined as the smallest integer m such that in any edge-r-coloring of K_m, there exists a monochromatic subgraph H with $\delta(H) \geq n$. An upper bound on $R^{\mathscr{D}}_r(n)$ can be obtained by using the same methodology as in the two colored case.

Theorem 5.12. *For integers $n \in \mathbb{N}$ and $r \geq 2$ we have*

$$R^{\mathscr{D}}_r(n) \leq \left\lfloor \left(r + \sqrt{r(r-1)}\sqrt{1 + \frac{1}{4r(r-1)(n-1)^2}}\right)(n-1) + \frac{3}{2} \right\rfloor$$

Proof. We call any graph G *good*, if it contains no subgraph H with $\delta(H) \geq n$ and we call G *bad*, if it does contain a subgraph H with $\delta(H) \geq n$. In the following proof, let G be a good graph on $m(n)$ vertices, whereas m will be determined later as a function of n. For simplicity, we may simply write m instead of $m(n)$. By Lemma 5.8, we get that G can contain at most $(m-n)(n-1)+\frac{n(n-1)}{2}$ edges. We compare this number to the smallest number of edges of the maximum size monochromatic subgraph over all edge-r-colorings of K_m. By Pigeonhole Principle, in any edge-r-coloring exists a color class with at least $\binom{m}{2}/r = m(m-1)/2r$ edges. We call the graph induced by edges of this color G. G is bad, if it has more than $(m-n)(n-1)+\frac{n(n-1)}{2}$ edges. We calculate

$$\begin{aligned} &\frac{m(m-1)}{2r} > (m-n)(n-1)+\frac{n(n-1)}{2} \\ \iff\ & m^2+(2r-2rn-1)m+rn(n-1) > 0. \end{aligned}$$

For fixed n, r, the quadratic equation $g(m) := m^2+(2r-2rn-1)m+rn(n-1)=0$ has the two solutions

$$m_{1,2} = \frac{-2r+2rn+1 \pm \sqrt{(2r-2rn-1)^2-4rn^2+4rn}}{2}.$$

Since K_m can only be good if $g(m)$ is negative which is between m_1 and m_2, we take m as the largest integer for which $g(m)$ is negative, that is

$$\begin{aligned} m(n) &= \left\lfloor \frac{-2r+2rn+1 \pm \sqrt{(2r-2rn-1)^2-4rn^2+4rn}}{2} \right\rfloor \\ &= \left\lfloor r(n-1)+\frac{1}{2}\sqrt{1+4r(r-1)(n-1)^2}+\frac{1}{2} \right\rfloor \\ &= \left\lfloor \left(r+\sqrt{r(r-1)}\sqrt{1+\frac{1}{4r(r-1)(n-1)^2}}\right)(n-1)+\frac{1}{2} \right\rfloor \end{aligned}$$

Thus, if we pick K_{m+1} any edge-r-coloring will contain a bad subgraph. □

By Corollary 5.10, we see a close relation between the multicolored minimum degree Ramsey number $R_r^{\mathscr{D}}(k)$ and packing and decomposition results. If it is possible, for integers r and m, to pack r $(n-1)$-degenerate graphs into K_m then $R_r^{\mathscr{D}}(n) \geq m+1$. Most packing results start with fixed graphs $H_1, \ldots, H_r$ that are to be packed in a graph G, in our case it is sufficient to decompose K_n into r unspecified $(n-1)$-degenerate graphs. A *decomposition* of a graph G is a collection $\mathscr{A}$ of edge disjoint subgraphs

$H_1, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. Our goal is to maximize n such that K_n can be decomposed into subgraphs $H_1, \ldots, H_r$ that are $(k-1)$-degenerate. To get a general lower bound on $R_r^{\mathscr{D}}(2)$, Proposition 5.13 shows it is sufficient to look into cycle free decompositions.

Proposition 5.13. *Any graph G with $\delta(G) \geq 2$ contains a cycle.*

Proof. Let $P = [v_0, v_1, \ldots, v_k]$ be a longest path in G. Since $\delta(G) = 2$, we have $d(v_k) = 2$. Suppose there exists a vertex v adjacent to v_k besides v_{k-1} that is not in P, then P is no longest path. Thus, there exists a vertex $w = v_i$ in $\{v_0, \ldots, v_{k-2}\}$ that is adjacent to v_k. Then $C = (\{v_i, \ldots, v_k\}, \{v_i v_{i+1}, \ldots, v_{k-1}v_k, v_k v_i\})$ is a cycle. □

There is a conjecture from Gallai (see [Lov68]) that states if G is a connected graph on n vertices then G can be decomposed into $\lceil \frac{n}{2} \rceil$ paths. Progress towards that conjecture was made by Lovász [Lov68] by proving that any (not necessarily connected) graph G can be decomposed into $\lfloor \frac{n}{2} \rfloor$ paths and cycles. Despite further progress made by Harary and Schwenk [HS72], Péroche [Pér11], Stanton, James and Cowan [SJC72] and Arumugam and Suseela [SA98] the conjecture remains unproven, whereas weaker versions are of no use for our case.

A general tight lower bound is obtained by a sophisticated coloring algorithm of Klein and Schönheim [KS92] that we will explain in detail below. They prove a theorem on the existence and order of *composed* graphs. A graph G, whose edge set is the disjoint union of the edge sets of graphs $M_1, \ldots, M_r$, where each M_i is a m_i-degenerate graph, is a $(m_1, \ldots, m_r)$-*composed* graph. Using Lemma 5.8 and Corollary 5.10, it is easy to show that if G is $(m_1, \ldots, m_r)$-composed and has n vertices, then

$$|E(G)| \leq \sum_{i=1}^{r} \left(m_i n - \frac{m_i(m_i+1)}{2} \right).$$

If there is equality in the equation above, we call G $(m_1, \ldots, m_r)$-*saturated.* Otherwise, we call the difference between the right and the left part the *deficiency.* If $m_1 = \ldots = m_r = n-1$, we also write G is $(r * (n-1))$-saturated. They state the following result

Theorem 5.14 ([KS92])**.** *The complete graph K_n is $(m_1, \ldots, m_r)$-composed if and only if*

$$n \leq \nu(m_1, \ldots, m_r) := \sum_{i=1}^{r} m_i + \left\lfloor \frac{1}{2} \left(1 + \sqrt{1 + 8 \sum_{1 \leq i < j \leq r} m_i m_j} \right) \right\rfloor.$$

By setting $m_1 = \ldots = m_r = n-1$, together with Corollary 5.10 we deduct

$$\begin{aligned} R_r^{\mathscr{D}}(n) &\geq r(n-1) + \left\lfloor \frac{1}{2}\left(1+\sqrt{1+8\binom{r}{2}(n-1)^2}\right)\right\rfloor + 1 \\ &= \left\lfloor \left(r+\sqrt{r(r-1)}\sqrt{1+\frac{1}{4r(r-1)(n-1)^2}}\right)(n-1)+\frac{3}{2}\right\rfloor. \end{aligned}$$

Since we already found a simple proof for a tight upper bound of $R_r^{\mathscr{D}}(n)$, we are interested in their approach for the lower bound. Klein and Schönheim [KS92] present a iterative coloring algorithm that produces $(m_1, \ldots, m_r)$-composed colorings by increasing the number of vertices and adding a new color in each step. For fixed integers $m_1, \ldots, m_r$ and $k \leq r$, they define $\nu_k := \nu(m_1, \ldots, m_k)$. In each iteration k, the $(m_1, \ldots, m_{k-1})$-composed coloring of $K_{\nu_{k-1}}$ is expanded to a $(m_1, \ldots, m_k)$-composed coloring of K_{ν_k}. A simple implementation of this algorithm, visualizing every step, can be found in the Appendix. Two small corrections to the original algorithm by Klein and Schönheim [KS92] that would yield infinite loops or open cases are annotated. For a set of integers $m_1, \ldots, m_r$, the algorithm fills a shape that corresponds to the upper right part of the incidence matrix of K_{ν_r}, starting above the main diagonal, as seen exemplary in Table 4. For a graph G with m vertices, the *shape* has m columns of increasing length starting from length 0 up to length $m-1$. By labeling the vertices of G as $v_1, \ldots, v_m$, the color of edge $v_i v_j$ with $i < j$ can be found in row i and column j of the shape. Below, we show the exemplary construction of $K_{\nu(3,3,3)}$, which shows $R_3^{\mathscr{D}}(4) > 16$, where we keep count on the variables and steps used in the algorithm. We use the word "to color" as a synonym for filling an empty square of the shape. The algorithm works on the rows and columns of the shape. We further use notation to show when a variable is set and when it is used, i.e. $3 = t$ to show that the variable t with value 3 is used and $t = 3$ to show that the variable t is set to 3.

The algorithm iterates over the number of colors. In each iteration k there are eight steps that might be executed and be called repeatedly. First, we want to provide an informal overview over the functionality of the steps for each iteration k. Therefore, they can be clustered into groups:

- Step 0 extends the shape from the last iteration by adding empty columns of increasing length until the shape consists of ν_k columns of lengths $0, \ldots, \nu_k - 1$. Then the last m_k rows are completely filled with k's.

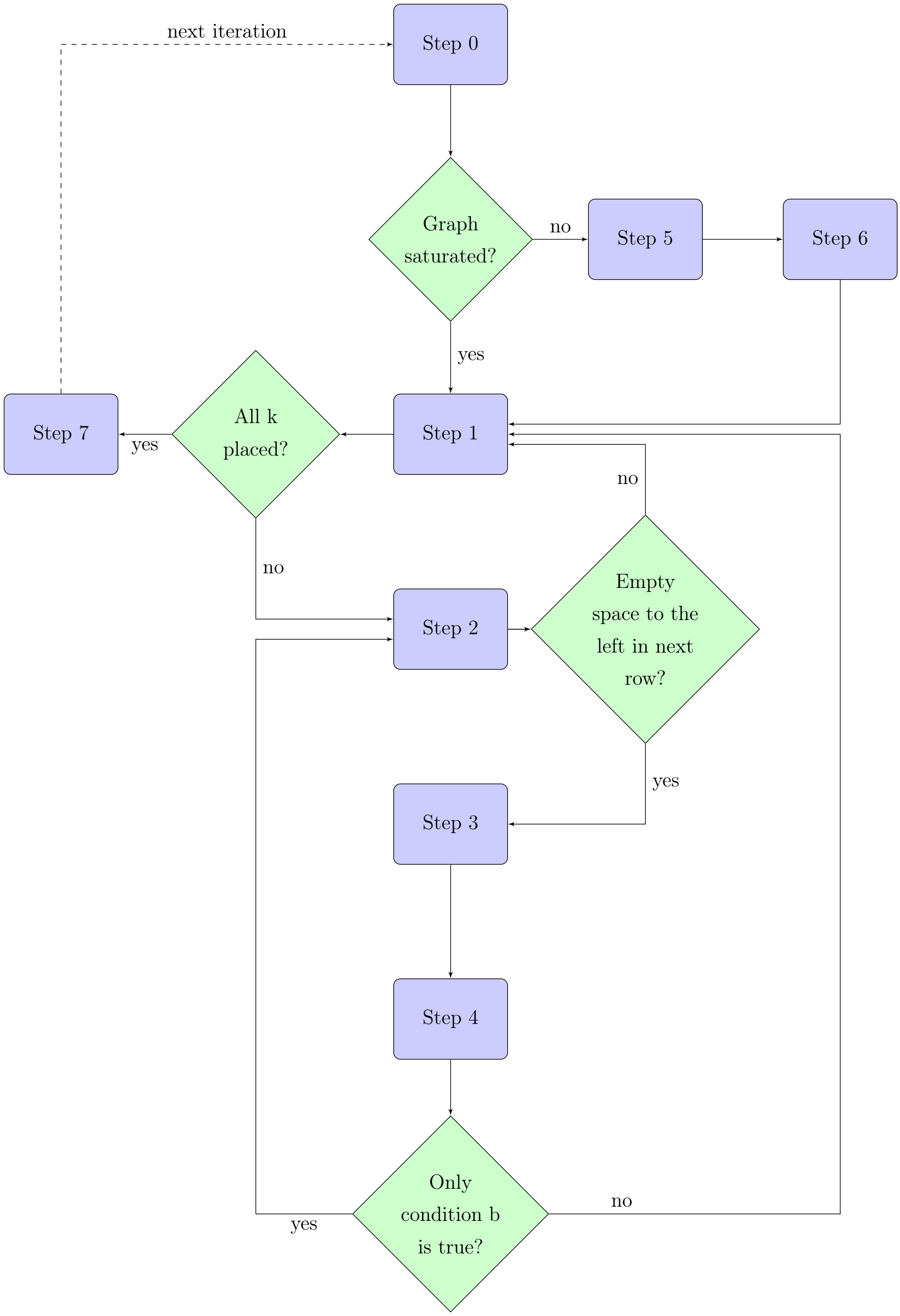


Figure 16: A sketch on the order of steps in the algorithm. Each iteration starts with step 0 and ends with step 7. Steps 5 and 6 are used once at the start of each iteration, if the result of the last iteration is not $(m_1, \dots, m_{k-1})$-saturated.

- Steps 1 to 4 are the only steps that are used several times in each iteration. They are used either after step 0 or after step 6. They fill the uncolored squares with k's and x's firstly from right to left and secondly from top to bottom such that there are m_k k's in each (previously partly uncolored) row and at most $\sum_{i=1}^{k-1}$ squares that are empty or marked with x's in each (previously partly uncolored) column.

- Steps 5 and 6 are used after Step 0, if the result from the last iteration is not $(m_1, \dots, m_{k-1})$-saturated. They recolor some edges from last iterations M_{k-1} such that last iterations $M_1, \dots, M_{k-2}$ are saturated. This recoloring of the result of the last iteration is done in the current iteration as it influences the first added column. After that Steps 1 to 4 are started.

- Step 7 is the final step in each iteration and fills the squares that are empty or marked with x's with colors $1, \dots, k-1$.

An informal overview on how the steps are linked can be found in Figure 16. Now we can start with the exact algorithm.

We iterate over k, the number of colors. For $k = 1$ we start with $K_{\nu(3)} = K_4$. Since there is only one color, which we call 1, we color all edges of K_4 in 1.

K_4	v_1	v_2	v_3	v_4
v_1		1	1	1
v_2	1		1	1
v_3	1	1		1
v_4	1	1	1	

1	1	1
	1	1
		1

Table 4: The incidence matrix of K_4 with non-edges as empty squares and the resulting shape. Note that we count four columns of lengths $0, 1, 2$ and 3 in the shape.

We continue with iteration $k = 2$. The exact definition of each step is annotated in gray:

1) Step 0: Extend the shape corresponding to $K_{\nu_{k-1}}$ to the shape corresponding to K_{ν_k} by appending $\nu_k - \nu_{k-1}$ columns of lengths ν_{k-1} up to length $\nu_k - 1$. The first ν_{k-1} columns retain their coloring. Color the last m_k rows by color k. If $K_{\nu_{k-1}}$ is not $(m_1, \dots, m_{k-1})$-saturated go to step 5, else go to step 1 with $\nu = \nu_{k-1} + 1$ and

$t = 1$.

Increase the number of columns of our triangular shape from $4 = \nu_1$ to $10 = \nu_2$ by adding $6 = v_2 - v_1$ columns starting with length $4 = \nu_1$ and ending with $9 = \nu_2 - 1$. The first $4 = \nu_1$ columns keep their coloring. Color the last $3 = m_k$ rows with $k = 2$. Since $K_{\nu(3)}$ is 3-saturated, go to step 1 with $\nu = 5$ and $t = 1$.

2) Step 1: Put k's into row t from right to left starting from the rightmost square until the number of k's in the row is m_k. Set $l = t + 1$. If $l < \nu_k - m_k$, go to step 2 with $t = l$, else go to step 7.

   Color the last 3 fields in row $t = 1$ with 2. Set $l = 2$ and $t = 2$ and go to step 2.

3) Step 2: Set j as the column of the leftmost k in the rightmost string of $k's$ in row $t - 1$ of the shape. If $j < t + 1$ or the square in row t and column $j - 1$ is already colored, go to step 1, else go to step 3.

   The leftmost $2 = k$ in the rightmost string of k's is in column $j = 8$. Go to step 3.

4) Step 3: Put k's in row l right to left starting from column $j - 1$, until at least one of the following conditions is true:

   a) k is in column ν and the number of empty squares in column ν is $\sum_{i=1}^{k-1} m_i$.
   b) The number of k's in the row is m_k.
   c) The left neighbour of the last k in column $h \leq v_k - m_k$ is x.
   d) The last k was put in the most left empty square of the row.

   Go to step 4.

   In row $2 = l$, after putting $2 = k$ into columns $7, 6, 5$ conditions a and b are true. Go to step 4.

5) Step 4: Continue based on the true cases from step 3:

   - If only b occured, go to step 2. Set $t = l + 1$ and then set $l = t$.
   - If a occured, fill all empty squares in column ν with x and go to step 1 with $t = l + 1$ if b occured, else with $t = l$.
   - If c occured, fill all empty squares in column h with x and go to step 1 with $t = l + 1$ if b occured, else with $t = l$.
   - If d occured, go to step 1 with $t = l + 1$ if b occured, else with $t = l$.

Since a and b occured, fill column $5 = \nu$ with x and go to step 1 with $t = 3$.

6) Step 1: Put $2 = k$ into the $3 = m_k$ rightmost columns in row $3 = t$. Set $l = 4$ and $t = 4$ and go to step 2.

7) Step 2: Set $j = 8$ and go to step 3.

8) Step 3: After placing two $2 = k$ in row $4 = l$, condition c is true with $h = 6$. Go to step 4.

9) Step 4: Since only c occured, we fill column $6 = h$ with x and go to step 1 with $t = 4$. The shape is now filled as seen below in Table 5.

10) Step 1: Put one $2 = k$ in the rightmost column in row $4 = t$. Set $l = 5$ and $t = 5$ and continue with step 2.

11) Step 2: Set $j = 10$ and go to step 3.

12) Step 3: After putting $2 = k$ in columns $9, 8, 7$ in row $5 = t$, conditions b and c are true with $h = 7$. Go to step 4.

13) Step 4: Since b and c occured, fill column $7 = h$ with x and go to step 1 with $t = 6$.

14) Step 1: Put $3 = m_k$, $2 = k$ into the three empty squares of row $6 = t$. Then, go to step 7. The current state of the shape can be seen in Table 6.

15) Step 7: Replace all x's and fill all empty squares with colors $1, \dots, k-1$ such that in each column the number of i's is at most m_i.
    Fill all squares that are empty or marked by x's with color 1.

Finally, we reach iteration $k = 3$.

1) Step 0: Extend the shape by adding $6 = \nu_3 - \nu_2$ columns of length $10 = \nu_2$ up to length $15 = \nu_3 - 1$. Since K_{ν_2} is not $(3,3)$-saturated, go to step 5.

2) Step 5: In each of the last m_{k-1} columns of the shape of $K_{\nu_{k-1}}$ containing less than $\sum_{i=1}^{k-2} m_i$ colors smaller than $k-1$ replace some $k-1$: Replace at most one per row, using only the first $\nu_{k-1} - m_{k-1}$ rows, by smaller colors to have each color i m_i times for $i = 1, \dots, k-2$.
    Replace the $2 = k-1$ in (row, column) $(1, 8), (3, 9)$ and $(4, 10)$ by 1 since there are

1	1	1	x	x		2	2	2
	1	1	2	2	2			
		1	x	x		2	2	2
			x	2	2			
				x				
						2	2	2
							2	2
								2

Table 5: The shape after 9).

1	1	1	x	x	x	2	2	2
	1	1	2	2	2			
		1	x	x	x	2	2	2
			x	2	2			2
				x	2	2	2	
					x	2	2	2
						2	2	2
							2	2
								2

Table 6: The shape after 14).

Table 7: Several states of the shape corresponding to the incidence matrix of $K_{\nu(3,3)}$ during the algorithm.

only $2 = m_1 - 1$ 1's in those columns. Remember the rows where a replacement happened and go to step 6.

3) Step 6: Fill column $\nu_{k-1}+1$ as follows: put $k-1$ in each row where it got replaced in step 5 and in the last m_k rows. Then, fill the remaining empty squares with m_i i's for $i = 1, \dots, k-2$. After that, the remaining empty squares are filled with k. Set $\nu = \nu_{k-1} + 2$, then go to step 1 with $t = 1$. Put $2 = k-1$ in rows $1, 3$ and 4 in column $11 = \nu_{k-1} + 1$ aswell as in the last $3 = m_2$ rows. Fill with $3 = m_1$ 1's. The remaining last empty square is set to $3 = k$. Then, set $\nu = 12$ and go to step 1 with $t = 1$.

4) The state of the shape at this point can be seen in Table 9. Continue like for $2 = k$ …

The final shape for $k = r$ can be seen in Figure 17. Since Theorem 5.14 is defined for all integers $m_1, \dots, m_r$ but we are only interested in the case where $m_1 = \dots = m_r = n-1$, we simplify the proof in Klein an Schönheim [KS92]. Therefore, let r and n be fixed integers for the rest of this section. For $k \leq r$ define

$$\nu_k := \nu(n-1, \dots, n-1) = k(n-1) + \left\lfloor \frac{1}{2}\left(1 + \sqrt{1 + 8\binom{k}{2}(n-1)^2}\right) \right\rfloor$$

with k times $n-1$. We denote the final shape in each iteration $k \leq r$ by $S(k)$ and the

M_1			v_{16}	v_{15}	v_{14}	v_{13}	v_{12}	v_{11}	v_{10}	v_9	v_8	v_7	v_6	v_5	v_4	v_3	v_2	v_1
	M_2		v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_5	v_4	v_3	v_2	v_1
		M_3	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
v_{16}	v_6	v_1		1	1	1	1	1	1	1	2	2	2	2	2	3	3	3
v_{15}	v_7	v_2			1	1	2	2	2	1	1	1	1	3	3	2	2	3
v_{14}	v_8	v_3				1	1	1	1	2	1	2	2	2	3	3	3	2
v_{13}	v_9	v_4					1	2	2	1	1	1	2	3	2	2	3	3
v_{12}	v_{10}	v_5						1	2	2	2	1	1	3	3	3	2	2
v_{11}	v_{11}	v_6							1	2	2	2	1	2	2	3	3	3
v_{10}	v_{12}	v_7								2	2	2	3	3	3	2	2	2
v_9	v_{13}	v_8									2	2	2	1	1	3	3	3
v_8	v_{14}	v_9										2	2	3	3	1	1	3
v_7	v_{15}	v_{10}											2	1	3	3	3	1
v_6	v_{16}	v_{11}												1	1	3	3	3
v_5	v_5	v_{12}													1	3	3	3
v_4	v_4	v_{13}														3	3	3
v_3	v_3	v_{14}															3	3
v_2	v_2	v_{15}																3
v_1	v_1	v_{16}																

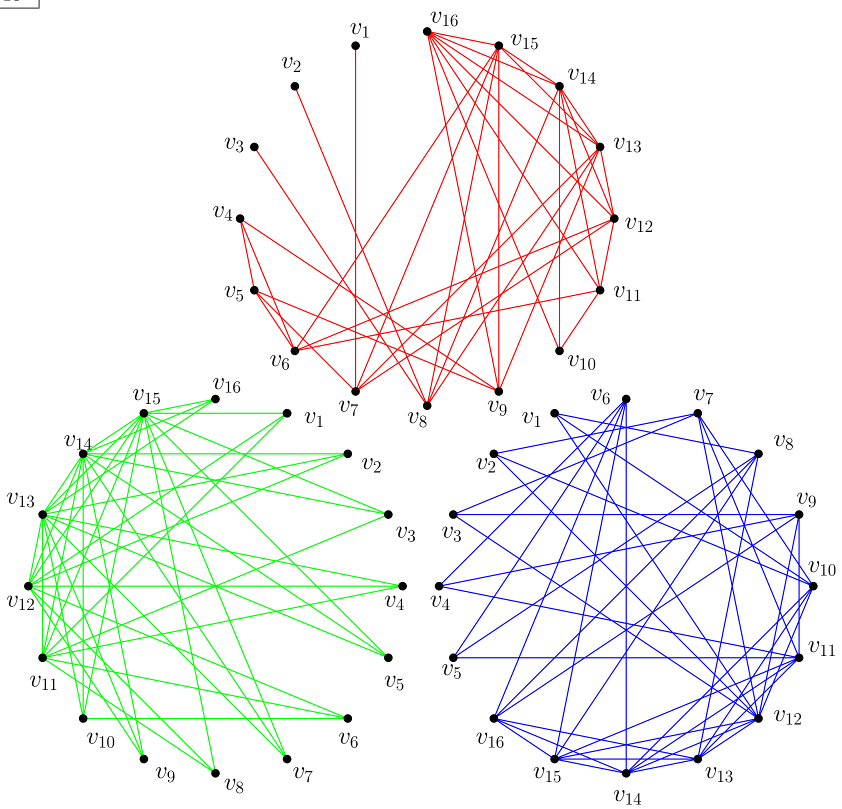


Figure 17: The final shape for $K_{\nu(3,3,3)}$, the labeling of the vertices and the corresponding monochromatic 3-degenerate subgraphs M_1 in red M_2 in blue and M_3 in green.

1	1	1	1	1	1	**2**	2	2	
	1	1	2	2	2	1	1	1	
		1	1	1	1	2	**2**	2	
			1	2	2	1	1	**2**	
				1	2	2	2	1	
					1	2	2	2	
						2	2	2	
							2	2	
								2	

Table 8: State after 1)

1	1	1	1	1	1	1	2	2	**2**
	1	1	2	2	2	1	1	1	1
		1	1	1	1	2	1	2	**2**
			1	2	2	1	1	1	**2**
				1	2	2	2	1	1
					1	2	2	2	1
						2	2	2	3
							2	2	2
								2	2
									2

Table 9: State after 3)

Table 10: Several states of the first $\nu_{k-1}+1$ columns of the shape corresponding to the incidence matrix of $K_{\nu(3,3,3)}$ during the algorithm for $k=3$.

shape before step 7 by $\overline{S}(k)$. The deficiency is denoted by

$$\mu_k := k(n-1)\nu_k - k\binom{n}{2} - \binom{\nu_k}{2}.$$

By choice of ν_k we get $\mu_k \geq 0$ for all $k = 1, \dots, r$. First, we define a labeling function $f_{i,k}^{r,n} = f_{i,k}$ for each monochromatic subgraph M_i with $i \leq k$, that assigns the columns of $S(k)$ to the vertices of M_i. For integers $i \leq k \leq r$ and n with $x \in [1, \dots, \nu_k]$, we define the bijective function into $[v_1, \dots, v_{\nu_k}]$ by

$$f_{i,k}^{r,n}(x) = f_{i,k}(x) = \begin{cases} v_x & \text{if } i = k \\ v_{\nu_k - x + 1} & \text{if } i = 1 \\ v_{\nu_k - x + 1} & \text{if } 2 \leq i < k \text{ and } x \geq \nu_i + 2 \\ v_{\nu_k - \nu_i + 1} & \text{if } 2 \leq i < k \text{ and } x = \nu_i + 1 \text{ and } \mu_i = 0 \\ v_{\nu_k} & \text{if } 2 \leq i < k \text{ and } x = \nu_i + 1 \text{ and } \mu_i > 0 \\ v_{\nu_k - \nu_i + x} & \text{if } 2 \leq i < k \text{ and } x \leq \nu_i \text{ and } \mu_i = 0 \\ v_{\nu_k - \nu_i - 1 + x} & \text{if } 2 \leq i < k \text{ and } x \leq \nu_i \text{ and } \mu_i > 0 \end{cases}$$

We further define the *breaking point* $b_i(k)$ for $i \leq k \leq r$ as $b_i(k) = f_{i,k}^{-1}(v_{\nu_k})$, this is the column of S(k) with the highest label for each M_i. We can use the breaking point to give intuitive labels for each M_i.

- For M_1, $S(k)$ is labeled in decreasing order from left to right.
- For M_k, $S(k)$ is labeled in increasing order from left to right.
- For M_i with $i = 2, \ldots, k-1$, $S(k)$ is labeled first in increasing order from right to left up to column $b_i(k)+1$, then the remaining columns in increasing order from left to right, i.e. $\nu_i+1, \nu_i+2, \ldots, \nu_k, \nu_k-\nu_i, \nu_k-\nu_i-1, \ldots, 1$ or $\nu_i, \nu_i+1, \ldots, \nu_k, \nu_k-\nu_i-1, \nu_k-\nu_i-2, \ldots, 1$, depending on μ_i.

By the chosen labeling, it is easy to see that M_i with $i \leq k$ is $(n-1)$-degenerate after iteration k if and only if each column of $S(k)$ to the right of column $b_i(k)$ contains at most $(n-1)$ i's and each row contains at most $(n-1)$ i's in columns $1, \ldots, b_i(k)$. Furthermore, by definition of $f_{i,k}$, we have $b_1(k) = 1$ for all $k = 1, \ldots, r$ and $b_i(i) = \nu_i$ for $i = 2, \ldots, r$. If $\mu_i > 0$, then $b_i(i+1) = b_i(i) + 1 = \nu_i + 1$, else $b_i(i+1) = b_i(i)$. For $i+1 \leq j \leq r-1$, $b_i(j)$ remains constant, i.e. $b_i(j+1) = b_i(j)$.

We proceed by induction on k. For $k = 1$, there is only M_1 and all edges are color 1. Since $|M_1| = \nu_1 = n$, M_1 is $(n-1)$-degenerate.

For the induction step from $k-1$ to k ($k \leq r$) we need to show:

(i) There are at most $(n-1)$ k's in each row of $S(k)$.

(ii) For $i = 1, \ldots, k-2$, there are at most $(n-1)$ i's in the columns $\nu_{k-1}+1, \ldots, \nu_k$.

(iii) There are at most $(n-1)$ $(k-1)$'s in each row up to column $b_{k-1}(k)$ and in each column $b_{k-1}(k)+1, \ldots, \nu_k$.

(iv) There are at least μ_k k's in rows $1, \ldots, \nu_k - n$ in each of the last $(n-1)$ columns of $S(k)$.

Pats (i),(ii) and (iii) guarantee the $(n-1)$-degeneracy of each M_i and (iv) is needed as a requirement for steps 5 and 6. Part (i) is an immediate consequence of the algorithm.

Proposition 5.15 (Klein and Schönheim [KS92])**.** *The number of k's in row i of $S(k)$ for $i \leq \nu_k - (n-1)$ is exactly $n-1$ and is $n-1-i$ in row $\nu_k + i - (n-1)$ for the remaining rows. In the second case the rows are filled uniquely with k's. Thus, M_k is $(n-1)$-degenerate and $(n-1)$-saturated in $S(k)$.*

Proof. For row $i \le \nu_k - (n-1)$, the algorithm starts to place k's in row i until one of the conditions in step 3 is met. If condition b is one of them, the algorithm continues with the next row, if not it calls step 1 to fill the row with k until there are exactly $n-1$ and then continues with the next row. The last $n-1$ rows are filled completely by k in step 0. Thus M_k is $(n-1)$-saturated. The labeling $f_{k,k}(i) = v_i$ for $i = 1, \ldots, \nu_k$ shows the $(n-1)$-degeneracy. □

For (ii), (iii) and (iv), the following observations shall summarize in an intuitive way how the k's are introduced into the shape, except those introduced by step 0 and step 6. We define the *successor* of a k as the next k that is introduced by the algorithm and the *predecessor* analogously as the k that was previously introduced. The first k does not have a predecessor, the last k does not have a successor.

Observation 1. *The successor of a k is in the column to its left or in the rightmost column and either in the same row (if there are less than $n-1$ k's in that row) or in the next row (if there are $n-1$ k's in that row). The successor of a k is in the rightmost column if and only if one of the following holds*

a) k is in column ν.

b) k is the leftmost square of its row.

c) k's left neighbour is x.

d) k's left neighbour is the leftmost square in the row and the number of k's in the row is $n-1$.

In each column is at most one k in position b, c or d.

Observation 2. *Every k not in the rightmost column has a predecessor in the column to its right, either in the same row or in the row above.*

With Observations 1 and 2, we can define a column wise counting of the k's in all but the last $n-1$ rows of each column of $\overline{S}(k)$. Therefore, we define by $\phi_j(\cdot)$ the number of one of the symbols x, k or 0 in column j of $\overline{S}(k)$ in all but the last $n-1$ rows. The symbol 0 represents empty squares. For the evaluation of ϕ, we distinguish left L, middle M and right part R of the columns $b_{k-1}(k)+1, \ldots, \nu_k$ of $\overline{S}(k)$. Therefore, let t be the number of columns containing x. The left part L consists of columns $b_{k-1}(k)+1, \ldots, b_{k-1}(k)+t$.

By condition c in steps 3 and 4, we get $t \leq \nu_k - b_{k-1}(k) - (n-1)$. The last $n-1$ columns are the right part R of the shape, the remaining columns in between are M. If $t = \nu_k - b_{k-1}(k) - (n-1)$, the middle part is empty. By conditions a and c in steps 3 and 4 any column in L contains only x and k. All columns in M and R contain only k and empty spaces. In any case, either the number of x's in column $b_{k-1}(k)$ is $(k-1)(n-1)$, or else there the number of empty places is greater than $(k-1)(n-1)$ and there are no x's in any column.

Proposition 5.16 (Klein and Schönheim [KS92]). *With L, M, R and ϕ as defined above, we have:*

1) For $j, j-1$ in $L \cup M \cup R$,

$$\phi_j(k) \geq \phi_{j-1}(k).$$

2) For $j, j-1$ both in M or both in R,

$$\phi_{j-1}(k) + 1 \geq \phi_j(k) \geq \phi_{j-1}(k).$$

3) For $j, j-1$ in L,

$$\phi_j(k) = \phi_{j-1}(k) + 1.$$

Proof. By Observation 2 every k not in the rightmost column has a predecessor in the column to its right, this implies 1). Furthermore, $\phi_{j-1}(k) + 1 \geq \phi_j(k)$ for $j-1, j$ both in M or R follows from Observation 1 since for every k, the successor is in the column to its left except if b or d is true which can only happen once per column. Finally 3) follows from part c of Observation 1 that also happens once per column in L. $\square$

If $\mu_{k-1} = 0$, it is sufficient to show for all columns in $L \cup M \cup R$ that the number of empty or x entries in $\overline{S}(k)$ is at most $(k-1)(n-1)$ to prove (ii). If $\mu_{k-1} > 0$, column $\nu_{k-1} + 1$ is filled with at most $(n-1)$ i's for $i = 1, \ldots, k-2$ in step 6.

Lemma 5.17. *In each column in L the number of x's is $(k-1)(n-1)$, i.e. $\phi_j(x) = (k-1)(n-1)$ for $j \in L$.*

Proof. Remember t is the number of columns containing x and any column in L contains only x's and k's. If $t = 0$, there is nothing to prove. If $t \geq 1$, the first column is $b_{k-1}(k)$ filled with exactly $(k-1)(n-1)$ x's by condition c in steps 3 and 4. By induction on j for $2 \leq j \leq t$, we get by part 3) of Proposition 5.16 that $\phi_j(k) = \phi_{j-1}(k) + 1$. Since there is also exactly one more square in column j than in column $j-1$ it follows that $\phi_j(x) = \phi_{j-1}(x)$. □

We continue with parts M and R.

Proposition 5.18. *For j and $j+1$ in M*

$$\phi_{j+1}(0) \geq \phi_j(0) \geq (k-1)(n-1)$$

Proof. Note that any column in M contains only empty squares and k's. By part 2 of Proposition 5.16, we have $\phi_j(k) + 1 \geq \phi_{j+1}(k)$. Since column j has length $j-1$ we get $\phi_{j+1}(0) = j - \phi_{j+1}(k) \geq j - \phi_j(k) - 1 = \phi_j(0)$. If $L = \emptyset$, note that the leftmost column j in M is $j = b_{k-1} + 1 = \nu$. Since it did not get filled with x's, by condition a of step 3 we have $\phi_j(0) > (k-1)(n-1)$. If $L \neq \emptyset$, part c) of Observation 1 does not occur in the leftmost column of M, thus for every x in the rightmost column of L, there must be a empty space in the leftmost column of M. The claim follows by Lemma 5.17. □

Proposition 5.19. *Let j be the column to the left of R. If $M \neq \emptyset$, there exists an integer s with $0 \leq s \leq (n-1)$ such that the first $(n-1)-s$ columns of R have $\phi_j(0)$ empty squares and the last s columns have $\phi_j(0)$ - 1 empty squares. If $M = \emptyset$, there is a constant $c \leq \phi_j(x)$ such that the claim above remains true with $\phi_j(0)$ replaced by c.*

Proof. Note that rows $\nu_k - (n-1), \ldots, \nu_k - 1$ are filled with k's. The proposition is a direct consequence of Observation 1 and 2. On the one hand, any k not in the rightmost column has a predecessor in the column to its right. On the other hand, every k in R (and $L \cup M$) has a successor in the column to its left, unless it is the last k or the left neighbour is x. Between placing two k's in the rightmost column, the algorithm puts exactly one k in each column in R. The columns of R to the left of the last k have $\phi_j(0)$, respectively c empty squares and the remaining s columns have $\phi_j(0) - 1$ respectively $c - 1$ empty squares. □

Lemma 5.20. *The number of empty squares in each column of $M \cup R$ is at most $(k-1)(n-1)$ and in M exactly $(k-1)(n-1)$.*

Proof. If $t = \nu_k - b_{k-1}(k) - (n-1)$, then $M = \emptyset$ and we only consider R, i.e. columns with index at least $\nu_k - (n-1)$. Column $j = \nu_k - (n-1)$ belongs to L and has by Lemma 5.17 exactly $(k-1)(n-1)$ x's, therefore $\nu_k - n - (k-1)(n-1)$ squares are filled with k's. Observation 2 implies that $\phi_{j+i}(0) \leq \nu_k - n + i - (\nu_k - n - (k-1)(n-1)) = i + (k-1)(n-1)$

with $i = 1, \dots, n-1$. Since the last i columns are filled with k's by step 0, we get $\phi_{j+i}(0) \le (k-1)(n-1)$ for $(i+j) \in R$.

If $M \neq \emptyset$, by Proposition 5.18 for M it is sufficient to show that for $m \in M$, $\phi_m(0) \le (k-1)(n-1)$. Therefore, suppose there is a first column q in M in which there are more than $(k-1)(n-1)$ empty squares. It follows that for all $m \ge q$ the number of empty squares is larger than $(k-1)(n-1)$ and for all $m < q$ the number of empty squares is at least $(k-1)(n-1)$. Using Proposition 5.19 with $\phi_j(0) \ge (k-1)(n-1)+1$, the number of empty squares in each column of R is at least $(k-1)(n-1)$. Furthermore, we know there are $(n-1)\nu_k - \binom{n}{2}$ k's and thus the number of necessary squares N in columns $\nu_{k-1}+1, \dots, \nu_k$ to accommodate all empty and k-squares is

$$N > \nu_k(n-1) - \frac{(n-1)n}{2} + (\nu_k - \nu_{k-1})(k-1)(n-1) + \mu_{k-1}.$$

Comparing with the actual number of squares A in columns $\nu_{k-1}+1, \dots, \nu_k$

$$A = \frac{1}{2}(\nu_k - \nu_{k-1})(\nu_k - 1 + \nu_{k-1}),$$

one can show that by definition of ν_k, there are more necessary squares than actual squares. Thus, there is no column q in M with more than $(k-1)(n-1)$ empty squares. Proposition 5.19 with $\phi_j(x) = \phi_j(0) = (k-1)(n-1)$ implies the number of empty squares in each column of R is at most $(k-1)(n-1)$. □

This completes the requirements of (ii) as well as the second part of (iii). For the first part of (iii) it is sufficient to see that the number of $(k-1)$'s in rows $1, \dots, \nu_{k-1}-(n-1)$ does not change in steps 5 and 6 and is increased by one in rows $\nu_{k-1}-(n-1)+1, \dots, \nu_{k-1}$, which contained at most $(n-2)$ $(k-1)$'s each. All that is left is to show that steps 5 and 6 are possible when used, which is implied by (iv).

Lemma 5.21. *In each of the last $n-1$ columns of $S(k)$, there are at least μ_k k's in rows $1, \dots, \nu_k - n$.*

Proof. Define

$$z_k := \frac{1}{2}\left(1 + \sqrt{1 + 8\binom{k}{2}(n-1)^2}\right)$$

and note that $\nu_k = k(n-1) + \lfloor z_k \rfloor$. Inserting into the definition of μ_k yields

$$\begin{aligned} \mu_k &:= k(n-1)\nu_k - \binom{n}{2}k - \binom{\nu_k}{2} \\ &= k(n-1)(k(n-1) + \lfloor z_k \rfloor) - \frac{n(n-1)k}{2} - \frac{(k(n-1) + \lfloor z_k \rfloor)(k(n-1) + \lfloor z_k \rfloor - 1)}{2} \\ &= \frac{1}{2}(k(k-1)(n-1)^2 - \lfloor z_k \rfloor^2 + \lfloor z_k \rfloor). \end{aligned} \tag{1}$$

Furthermore, with $z_k^2 = z_k + k(k-1)(n-1)^2$

$$\begin{aligned} \lfloor z_k \rfloor^2 + \lfloor z_k \rfloor &> (z_k - 1)^2 + (z_k - 1) \\ &= z_k^2 - z_k \\ &= k(k-1)(n-1)^2. \end{aligned} \tag{2}$$

Inserting (2) into (1) yields $\mu_k < \lfloor z_k \rfloor$. Since $\mu_k \in \mathbb{N}$, we get $\mu_k \leq \lfloor z_k \rfloor - 1$. By Lemma 5.20 the maximum number of empty squares in any column of R is $q \leq (k-1)(n-1)$. Thus the number of k's in the first $\nu_k - n$ rows of any column in R is at least

$$\begin{aligned} \nu_k - 1 - (n-1) - q &= (k-1)(n-1) + \lfloor z_k \rfloor - 1 - q \\ &\geq \lfloor z_k \rfloor - 1 \\ &\geq \mu_k. \end{aligned}$$

□

This proves (iv) and finishes the induction. Finally, we can state the exact multicolored minimum degree Ramsey number as

$$R_r^{\mathscr{D}}(n) = \left\lfloor \left(r + \sqrt{r(r-1)} \sqrt{1 + \frac{1}{4r(r-1)(n-1)^2}} \right)(n-1) + \frac{3}{2} \right\rfloor.$$

An overview on small multicolored minimum degree Ramsey numbers can be found in Table 11.

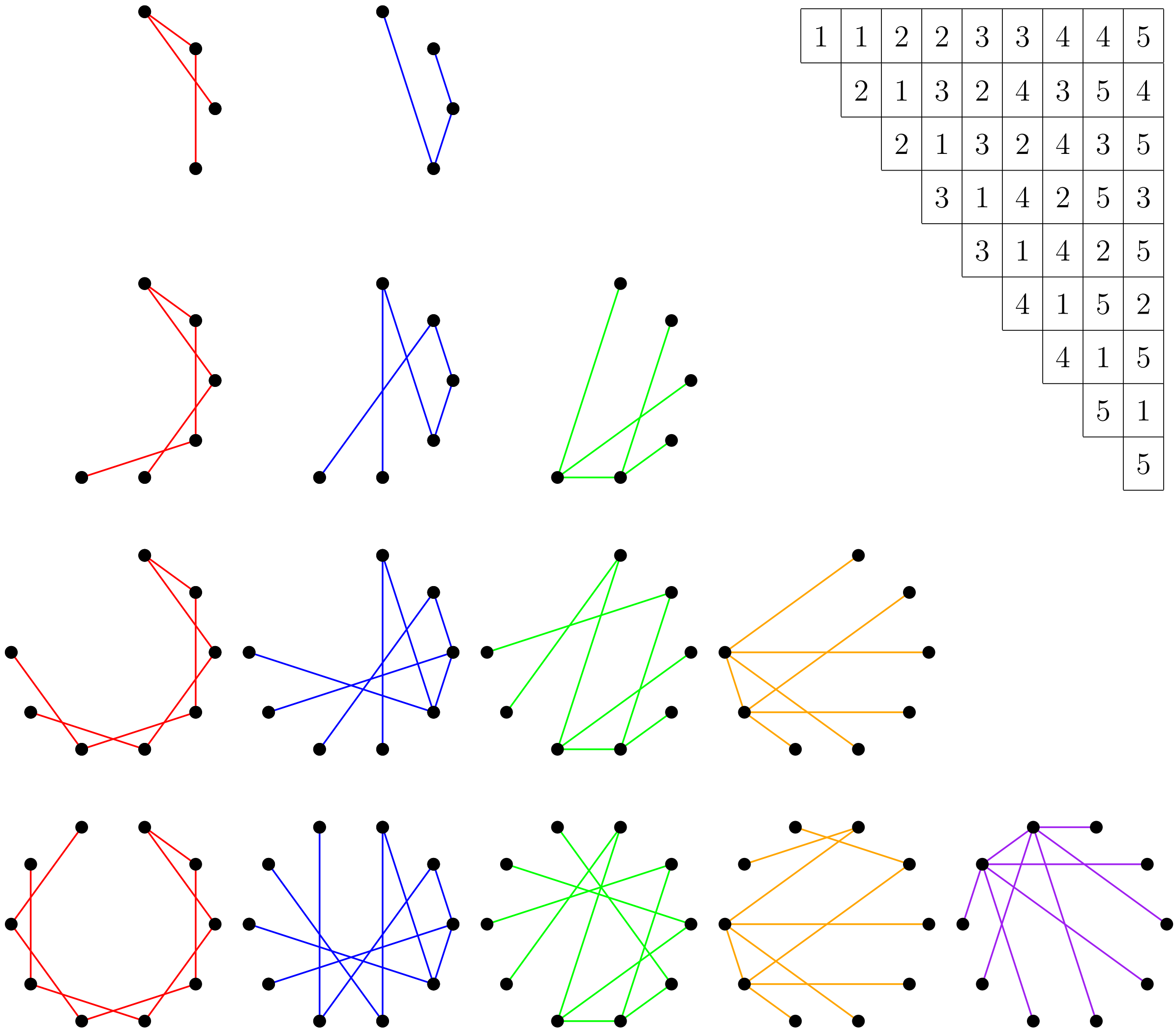
An interesting result can be obtained from Lemma 5.21. By inserting $n = 2$ into the definition of $z_k = \frac{1}{2}\left(1 + \sqrt{1 + 8\binom{k}{2}(n-1)^2}\right)$, we get

$$\begin{aligned} z_k &= \frac{1}{2}\left(1 + \sqrt{1 + 8\binom{k}{2}}\right) \\ &= \frac{1}{2}\left(1 + \sqrt{(2k-1)^2}\right) \\ &= k. \end{aligned}$$

$n =$	1	2	3	4	5	6	7	8	9
$r = 2$	2	5	8	11	15	18	22	25	28
$r = 3$	2	7	12	17	23	28	34	39	45
$r = 4$	2	9	16	23	31	38	46	53	61
$r = 5$	2	11	20	29	39	48	58	67	77

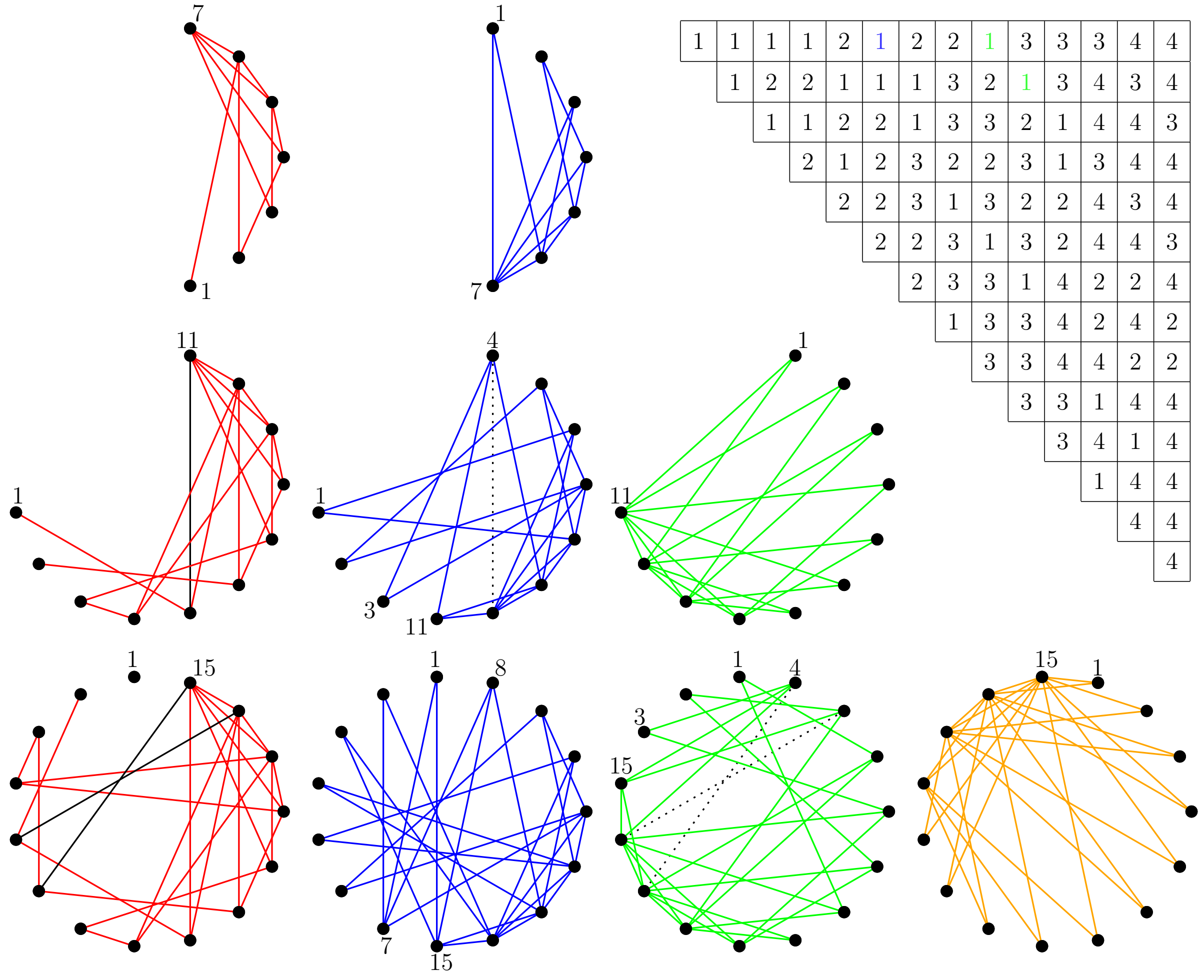
Table 11: Small multicolored minimum degree Ramsey numbers $R_r^{\mathscr{D}}(n)$.

By inserting $z_k = k$ into $\mu_k = \frac{1}{2}(k(k-1) - \lfloor z_k \rfloor^2 + \lfloor z_k \rfloor)$, we get $\mu_k = 0$ for all k. Thus, the algorithm will never use steps 5 and 6. In this case the algorithm will never change the color of already colored edges, as seen in Figure 18. It is easy to see that this holds true in all cases where z_k is an integer. An example of the algorithm recoloring edges can be seen in Figure 19.



1	1	2	2	3	3	4	4	5
	2	1	3	2	4	3	5	4
		2	1	3	2	4	3	5
			3	1	4	2	5	3
				3	1	4	2	5
					4	1	5	2
						4	1	5
							5	1
								5

Figure 18: The colorings obtained from the algorithm for $n = 2$, $r = 2, \ldots, 5$ and the shape $S(5)$. In this special case the algorithm never recolors, i.e. $S(k)$ for $k = 1, \ldots, 4$ consists of the first ν_k columns of $S(5)$.



1	1	1	1	2	1	2	2	1	3	3	3	4	4
	1	2	2	1	1	1	3	2	1	3	4	3	4
		1	1	2	2	1	3	3	2	1	4	4	3
			2	1	2	3	2	2	3	1	3	4	4
				2	2	3	1	3	2	2	4	3	4
					2	2	3	1	3	2	4	4	3
						2	3	3	1	4	2	2	4
							1	3	3	4	2	4	2
								3	3	4	4	2	2
									3	3	1	4	4
										3	4	1	4
											1	4	4
												4	4
													4

Figure 19: The colorings obtained from the algorithm for $n = 3$, $r = 2, \ldots, 4$ and the shape $S(4)$. Recolored edges in each iteration are drawn in black. They are indicated by a dotted line in their former color class. The former colors of recolored edges can be seen as the color of the number in $S(4)$. The vertex orderings obtained by the labeling function that prove 2-degeneracy are indicated by numbers on the vertices. Every labeling is either completely clockwise, completely counterclockwise or changes once from counterclockwise to clockwise.

6 Appendix

Appended is a corrected python implementation of the coloring algorithm from Klein and Schönheim [KS92] that displays each step. Some small mistakes that yield infinite loops in the original algorithm in steps 3 and 4 are corrected and annotated. By setting the values $m_1, \dots, m_r$ in the variable m_array, the algorithm constructs the shape $S(k)$ such that each M_i is m_i-degenerate.

6.1 Coloring Algorithm

```
# -*- coding: utf-8 -*-

import math
import numpy as np
import logging

# Degeneracy of the different subgraphs
m_array=[3,3,6]

"""This class contains a iterative construction algorithm for a
coloring of K_n in len(m_array) colors such that n is maximal while
each monochromatic subgraph has degeneracy as given in m_array. The
result is presented as the upper right triangle of the incidence
matrix of K_n, starting above the main diagonal with
integers filled in representing the color of each edge."""
class coloring:


    """ Constructor of the colorign class."""
    def __init__(self, m_array):

        self.m_array = m_array
        self.n = self.function_v(len(self.m_array))
        self.coloring_array = np.zeros((self.n, self.n))


    """ This method calculates the maximal order n of K_n when using the first k
    color classes with given degeneracy."""
```

```
    def function_v(self, k):

        # Select the k first m_i
        array = self.m_array[:k]

        # Calculate the triangle sum product
        sum_prod = 0
        for j in range(len(array)):
            for i in range(0,j):
                sum_prod += array[j]*array[i]

        # Set v_k
        v_k = sum(array) + math.floor(1/2*(1+math.sqrt(1+8*sum_prod)))

        return v_k


    """ This method calculates the maximum number of edges a (m_1,...,m_k)
    graph on n vertices can contain."""
    def function_saturation(self, k):

        n = self.function_v(k)

        saturation = 0
        for i in self.m_array[:k]:
            saturation += i*n - i*(i+1)/2

        return int(saturation)

    """ This method calculates the deficiency."""
    def function_deficiency(self, k):

        n = self.function_v(k)

        deficiency = self.function_saturation(k) - int(n*(n-1)/2)

        return deficiency


    """ This is the main method that iteratively constructs the coloring,
```

```
    starting with one color up to len(m_array) colors."""
    def create_coloring(self):

        # Step for k = 1
        k=1

        # Set v_1
        current_v_i = self.function_v(k)

        # Create coloring array
        coloring_array = np.zeros((current_v_i,current_v_i))

        for i in range(1, current_v_i):
            for j in range(i, current_v_i):
                    coloring_array[i,j] = k

        self.v_k = current_v_i
        self.coloring_array = coloring_array
        k += 1

        # Steps for other k
        while k <= len(self.m_array):
            coloring_array = self.next_color(coloring_array, k)
            k += 1

        return coloring_array


    """ This method takes a valid coloring on k colors and creates a valid
    coloring for k+1 colors while increasing the order of the colored
       K_n."""
    def next_color(self, old_coloring_array, k):

        # Calculate current v_i
        current_v_i = self.function_v(k)

        # Create new coloring array and insert the previous one
        coloring_array = np.zeros((current_v_i, current_v_i))
        coloring_array[:old_coloring_array.shape[0],
            :old_coloring_array.shape[1]] = old_coloring_array
```

```
        self.coloring_array = coloring_array

        # Parameters used in the Algorithm
        self.k = k
        self.m_k = self.m_array[k-1]
        self.t = None
        self.v = None
        self.l = None
        self.j = None
        self.h = None
        self.v_k_old = self.v_k
        self.v_k = current_v_i

        # Start Algorithm for k
        self.step_0()

        return self.coloring_array


    """ Step 0 is the start for each new k. If the graph obtained in step
    k-1 is saturated we continue with Step 1, else we recolor in Step
        5."""
    def step_0(self):

        logging.debug("step_0")
        self.show_array("step_0:")

        # Color the last m_k rows as k
        for i in range(self.m_k):
            self.coloring_array[self.v_k - (i+1), -(i+1):] = self.k

        # If the coloring from last k is not saturated go to step 5
        if self.function_deficiency(self.k-1) > 0:
            self.step_5()

        # If it is saturated continue with step 1, set t = 1 and v =
            v_{k-1}
        else:
            self.v = self.v_k_old
            self.t = 1
```

```python
            self.step_1()


    """ Step 1 fills row t with missing colors k so that we can step
    to row t+1 in Step 2 or go to the final Step 7."""
    def step_1(self):

        logging.debug("step_1")
        self.show_array("step_1:")

        # Fill row t until m_k k are in it
        i = 1
        while len([k for k in self.coloring_array[self.t,self.t:] if k ==
            self.k]) < self.m_k:
            self.coloring_array[self.t, -i] = self.k
            i += 1

        # Set l = t+1 (since t rows are colored)
        self.l = self.t + 1

        # If l = v_k - m_k goto step 7
        if self.l >= self.v_k - self.m_k:
            self.step_7()

        # Else set t=l and go to step 2
        else:
            self.t = self.l
            self.step_2()


    """Step 2 chooses the column index j of the leftmost k in the
    rightmost string of k's in row t-1. Then it decides if we are in a new
    row to go to Step 3, or in a old row to finish it in Step 1."""
    def step_2(self):

        logging.debug("step_2")
        self.show_array("step_2:")

        # Calculate the column index of the leftmost k in the rightmost
            k-string
```

```
        # in row t-1, set that index as j
        row = self.coloring_array[self.t-1, self.t-1:]
        in_k = False
        i = 1
        while row[-i] == self.k or in_k == False:
            if row[-i] == self.k:
                in_k = True
            i += 1
        self.j = self.v_k - i  + 1

        # If j < t+1 or [t, j-1] is already colored go to step 1
        if self.j < self.t+1 or self.coloring_array[self.t,self.j-1] != 0:
            self.step_1()

        # else go to step 3
        else:
            self.step_3()


    """Step 3 fills row l starrting from column j-1 from right to left
    until at least one of four conditions is met. Then it goes to Step 4
    where the met conditions decide how to continue."""
    def step_3(self):
        logging.debug("step_3")
        self.show_array("step_3:")

        # These conditions get checked after every new color placed until at
        # least one is True
        a = False
        b = False
        c = False
        d = False

        # First checks before starting to insert
        if self.coloring_array[self.l, self.v] == self.k:
            number_zeros = 0
            for cand in self.coloring_array[1:self.v+1, self.v]:
                if cand == 0:
                    number_zeros += 1
```

```
            if number_zeros == np.sum(self.m_array[:self.k-1]):
                a = True
        if len([i for i in self.coloring_array[self.l, self.l:] if i == self.k]) == self.m_k:
            b = True
        if self.coloring_array[self.l, self.j - 1] == -1:
            c = True
            self.h = self.j
        if self.j == self.l or self.coloring_array[self.l, self.j - 1] != 0:
            d = True

        # Stop if one condition is True
        sub = 0
        while not (a or b or c or d):

            sub += 1
            self.coloring_array[self.l, self.j - sub] = self.k

            # Set True if k in column v in row l and number of zero squares in
            #column v is m_1+...+m_{k-1}
            if  self.coloring_array[self.l, self.v] == self.k:
                number_zeros = 0
                for cand in self.coloring_array[1:self.v+1, self.v]:
                    if cand == 0:
                        number_zeros += 1
                if number_zeros == np.sum(self.m_array[:self.k-1]):
                    a = True

            # Set True if k is in the row m_k times
            if len([i for i in self.coloring_array[self.l, self.l:] if i == self.k]) == self.m_k:
                b = True

            # Set True if the left neighbor of the last k is -1 (which represents x)
            # and there are enough columns to the right
            if self.coloring_array[self.l, self.j - sub - 1] == -1:
                self.h = self.j - sub
```

```
                if self.h < self.v_k - self.m_k: # CORRECTION: THIS NEEDS TO BE CHECKED HERE, ELSE OPEN CASES
                    c = True

            # Set True if the k got filled in the most right empty square of the shape
            if self.j - sub == self.l or self.coloring_array[self.l, self.j-sub-1] != 0:
                d = True

        # Go to step 4 with conditions a,b,c,d
        self.step_4(a,b,c,d)


    """Step 4 decides based on the conditions given by Step 3 how to
    continue."""
    def step_4(self,a,b,c,d):
        logging.debug("step_4")
        print("\n",a,b,c,d)
        self.show_array("step_4:")

        # If only b occured go to step 2 with t = l+1
        if b and not (a or c or d):
            self.t = self.l+1
            self.l = self.t # CORRECTION: THIS WAS NOT IN THE PAPER BUT IS NEEDED
            self.step_2()

        # If a occured replace all 0 in column v by -1
        # go to step 1 with either t = l+1 if b was met or t = l if not
        elif a:
            for i in range(1, self.v+1):
                if self.coloring_array[i, self.v] == 0:
                    self.coloring_array[i, self.v] = -1
            if b:
                self.t = self.l+1
                self.step_1()
            if not b:
                self.t = self.l
                self.step_1()
```

```
            return

        # If c occured in column h < v_k - m_k replace all 0 in column h by -1
        # go to step 1 with either t = l+1 if b was met or t = l if not
        elif c:
            for i in range(1, self.h+1):
                if self.coloring_array[i, self.h] == 0:
                    self.coloring_array[i, self.h] = -1
            if b:
                self.t = self.l+1
                self.step_1()
            if not b:
                self.t = self.l
                self.step_1()
            return

        # If d occured go to step 1 with either t = l+1 if b was met or t = l if not
        elif d:
            if b:
                self.t = self.l+1
                self.step_1()
            if not b:
                self.t = self.l
                self.step_1()
            return

        # If none of the cases fits return error (this should not happen)
        else:
            raise ValueError("None of the cases in step 4 true, something went wrong.")


    """"Step 5 is activated if the graph recieved in step k-1 is not
    saturated. Together with step 6 it performs some changes on the
    coloring. We replace color k-1 by suitable smaller colors and remember
    the rows where it happened."""
    def step_5(self):

```

```
        logging.debug("step_5")
        self.show_array("step 5:")

        # Each row can only be used once thus we keep the indices here
        unused_rows = list(range(1, self.v_k_old))

        # We collect the changes we do
        changed_fields = []

        # Iterate through the last m_{k-1} columns
        for i in range(self.v_k_old - self.m_array[self.k-2],
            self.v_k_old):
            column = self.coloring_array[ :i+1, i]

            # Check each color class
            for color in range(1, self.k-1):

                # If we could have more edges of that color in that column
                while list(column).count(color) < self.m_array[color - 1]:

                    # Replace color k-1 by suitable smaller color
                    for r_index in unused_rows:
                        if r_index <= i:
                            if column[r_index] == self.k-1:
                                column[r_index] = color
                                unused_rows.remove(r_index)

                                # Remember the replacements for step 6
                                changed_fields.append((r_index, i, color))
                                break

        # Go to step 6 while remembering the changes
        self.step_6(changed_fields)


    """In Step 6 we color column v_{k-1} by filling k-1 in the rows where
    it was replaced in Step 5. The rest is filled with colors 1,...,k-2
    until the classes are saturated. The rest is filled with k. Then
    v_{k-1}+1 is set to be v and we continue normally with Step 1."""
    def step_6(self, changed_fields):
```

```python

        logging.debug("step_6")
        self.show_array("step 6:")

        # Name the column for convenience
        column = self.coloring_array[ : self.v_k_old + 1, self.v_k_old]

        # Fill with k-1 in the rows where it was replaced in step 5
        for i in changed_fields:
            column[i[0]] = self.k-1
            column[-self.m_array[self.k-2]:] = self.k-1

        # Iterate through the colors, fill column until color is saturated
        # Then go to next color
        color = 1
        placed = len([i for i in column if column[i] == color])
        i = 1
        while i < len(column):
            if column[i] == 0:
                if placed < self.m_array[color - 1] or color == self.k:
                    column[i] = color
                    placed += 1
                    i += 1
                else:
                    color += 1
                    placed = len([i for i in column if column[i] ==
                        color])
            else:
                    i += 1

        # Set v = v_{k-1}+1, t=1 and start step 1
        self.v = self.v_k_old + 1
        self.t = 1
        self.step_1()


    """Step 7 is the last step for each k. The yet uncolored edges are
    colored with colors up to k-1 that are not saturated."""
    def step_7(self):

```

```
        logging.debug("step_7")
        self.show_array("step 7:")

        # Fill the remaining -1 and 0 in the array with colors are not
            saturated
        # left
        for c_index in range(1, self.v_k):
            for r_index in range(1, c_index + 1):
                if (self.coloring_array[r_index, c_index] == 0 or
                    self.coloring_array[r_index, c_index] == -1):

                    for i in range(1, self.k):
                        if (list(self.coloring_array[1:c_index +1,
                            c_index]).count(i)
                        < self.m_array[i-1]):

                            self.coloring_array[r_index, c_index] = i

        # If the last k is reached print the finished result
        if self.k == len(self.m_array):
            self.show_array("finished:")


    """This metod prints the relevant part of the incidience matrix. It is
    the upper right part starting above the main diagonal."""
    def show_array(self, text = ""):

        if text != "":
            print ("\n", "\n" , text)
        else:
            print()

        self.coloring_array = self.coloring_array.astype(int)

        # -1 are replaced by x to fit the description in paper
        space = len(str(np.max(self.coloring_array))) + 1
        for i in range(1,self.v_k):
            print("\n", end ="")
            for j in range(1, self.v_k):
                if self.coloring_array[i,j] == 0:
```

```
                    print("{0:{1}}".format("", space), end ="")
                elif self.coloring_array[i,j] == -1:
                    print("{0:>{1}}".format("x", space), end ="")
                else:
                    print("{0:{1}}".format(self.coloring_array[i,j],
                        space), end = "")


# Create a object from the coloring class with m_array and start the
    coloring algorithm
new = coloring(m_array)
new.create_coloring()
```

List of Figures

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Erklärung

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