Appendices

Undergraduate Structural Analysis course generally focus on the Stiffness Method because it suits the programming, execution and solution of complex redundant structures. The Force Method, on the other hand, is not suitable for computer solutions of complex highly redundant structures because it requires the selection of redundant's and a statically determinant stable load path. The selection of the load path and redundant's must either be made by the user or a clever programming logic that assures a stable and determinate structure. This is unnecessary and impractical given that the Stiffness Method can be programmed using a simple bookkeeping procedure. For determinate structures, however, the Force Method has distinct advantages.

(a) The load path is defined by the determinate, stable structure and there are no redundant's. *b* the forces in the members can be determined directly by executing equilibrium equations.

The example problems this text are solved using the Force Method because every structure, except one, is stable and determinate.

A detailed discussion of the Force Method and the principal of Virtual Work is provided in Appendix A.

The procedure required to solve a redundant problem using the Force Method is shown in Appendix B, where a Truss indeterminate to the second degree is analyzed. The selected redundant's are two members comprising the cross bracing of the truss.

Appendix C contains a simple comparison of the Stiffeners and Force Methods of Analysis for those who are unfamiliar with their difference.

Appendix D is an analysis of a 3-hinge reinforced concrete stairway. The concrete stepped stairway is converted to an 'equivalent beam'. The analysis requires a Transformation Matrix that converts the Global Gravity Loads to Local Loads acting in the direction of the selected degrees of freedom. The analysis includes both a buckling and a vibration solution that includes modes shapes. The stairway is the most complex example included in this text and may be more difficult for the reader to follow.

Appendix

<div style="text-align:right">**A**</div>

Theoretical Development

Matrix Force Method of Analysis

Two fundamental matrices are required in the development

1.0 Equilibrium

   (a) [A]—determinate structure
   (b) [B]—indeterminate (Redundants)

2.0 Influence Coefficients

   (a) $[\alpha] - [L/AE]$ truss member axial flexibility
   (b) $[\alpha] - [2 \times 2]$ beam bending flexibility matrix

$$\alpha(\text{beam}) = \begin{pmatrix} \dfrac{L}{3EI} & \dfrac{L}{6EI} \\ \dfrac{L}{6EI} & \dfrac{L}{3EI} \end{pmatrix}$$

Note: For determinate structure: $\{R\} = 0$ and matrix $[B]$ is not required

The α Matrix

To focus on Geometric Nonlinearity, the problems solved have been carefully selected to be Determinate. For any determinate structure, The equilibrium equations can be written:

Determinate Structure $[A]\{q\} = \{P\}$

   where: $[A]$ is the Equilibrium Matrix

        $\{q\}$ the unknown Force Vector

        $\{P\}$ is the applied Load Vector
   The Flexibility Matrix $[F]$ of a Determinate Structure can be written:

$$F = \left(A^{-1}\right)^T [\alpha] A^{-1}$$

where A^{-1} is the inverse of the Equilibrium Matrix

Truss Structure

$[\alpha]$ matrix is a diagonal Matrix of element flexibility. Listed along the diagonal of the $[\alpha]$ matrix is the element flexibility of every truss member.

$$\text{Truss member} \qquad \alpha = \frac{L}{A \cdot E}$$

Beam or Frame

For a Beam or Frame Structure, the $[\alpha]$ matrix is not diagonal. The element flexibility of a beam member is given by the 2×2 Matrix shown below which is derived on the following pages.

$$\text{Beam member} \quad \alpha = \begin{pmatrix} \dfrac{L}{3 \cdot E \cdot I} & \dfrac{L}{6 \cdot E \cdot I} \\ \dfrac{L}{6 \cdot E \cdot I} & 3 \cdot E \cdot I \end{pmatrix}$$

   A Truss indeterminate to the 2nd degree is worked out in detail in Appendix B.

   A Determinate Beam divided into ten (10) equal segments is worked out in great detail on the following pages. Hopefully these examples will clarify the application of the $[\alpha]$ matrix.

   Virtual Work is a powerful analytical tool in Structural Analysis toolbox that has served Structural Engineers for many years along with other Energy Techniques. Although it has subsequently been replaced by computer software and the Stiffness Method, it remains a powerful analytical tool for those who appreciate and understand its limitations.

Note that the Flexibility Matrix of every statically determinate structural frame analyzed in this book was derived using a combination of Static Analysis (Moment Diagrams) and Virtual Work.

To keep it simple, the technique of Virtual Work works as follows: (a) to find the displacement of any node on the structure you place **a unit load (1) at that structural node in the direction, you want to find the displacement. (Call it Node 1)**

— Using Statics draw a moment diagram for the structure and call that the **Q1 system**

(b) to find the displacement **at any other node** on the structure place a **unit load at that structural node in the direction, you want to find the displacement. (Call it Node 2)**

— Using Statics draw a moment diagram and we call that the **Q2 system**.

To find the displacement at node 1, due to the unit load at node 1 you integrate the Q1 system with itself which gives you the influence coefficient $f11$.

To find the displacement at node 1, due to the unit load at node 2 you integrate the Q1 system with the Q2 system which gives you the influence coefficient $f12$

To find the displacement at node 2, due to the unit load at node 2 you integrate the Q2 system with itself which gives you the influence coefficient $f22$.

SUMMARY   Integrating Q1 with Q1 yields the influence coefficient; $f11$

                  Integrating Q1 with Q2 yields the influence coefficient; $f12$

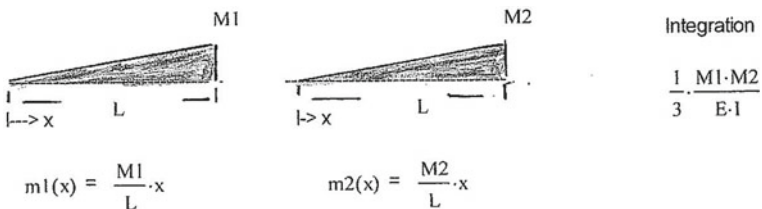   Integrating Q2 with Q2 yields the influence coefficient; $f22$

FLEXIBILITY Matrix $\mathbf{F} = \begin{pmatrix} f11 & f12 \\ f21 & f22 \end{pmatrix}$
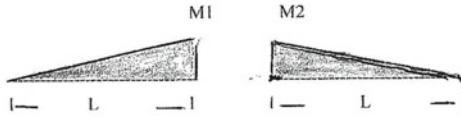
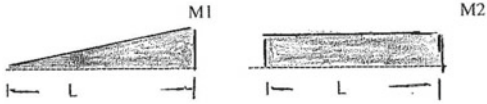## Virtual Work

Integration Diagrams

Member Length $= L$

Constant EI

Virtual Work
Integration
$$\theta = \int_0^L \frac{m1(x) \cdot m2(x)}{E \cdot I} dx = \int_0^L \frac{\frac{M}{L} \cdot \frac{M2}{L} \cdot x^2}{E \cdot I} dx = \frac{1}{3} \cdot \frac{M1 \cdot M2}{E \cdot I}$$

M1   M2

$$\frac{1}{6} \cdot \frac{M1 \cdot M2}{E \cdot I}$$

M1   M2

$$\frac{1}{2} \cdot \frac{M1 \cdot M2}{E \cdot I}$$

MOMENT DIAGRAMS

Appendix

B

A truss indeterminate to the 2nd degree is analyzed. The redundants selected are the 2 diagonal members that form the X bracing in each bay. The removal of the two diagonal members leaves a truss structure that is determinate. The redundants are evaluated and the forces in each member determined. The analysis includes an ASCE Code check which is presented graphically.

The objective of the analysis is to present, for those not familiar with the Force Matrix Method of Analysis (using a simple example), the procedure required to solve an indeterminate structure and the Matrix operations involved.

Force Matrix Method—Theoretical Development Indeterminate Structure

The equilibrium equations can be written in the following form

$$[A]\{q\} + [B]r = \{p\} \tag{T2-1}$$

solving for the member forces $\{q\}$ we have

$$\{q\} = \left[A^{-1}\right]\{P\} - \left[A^{-1} \cdot B\right]\{r\} \tag{T2-2}$$
$$\{r\} = [0]\{P\} + [I]\{R\}$$

which can be written in the form

$$\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1} \cdot B \\ 0 & I \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix} \tag{T2-3}$$

$$\text{Let} \quad Q11 = A^{-1}$$
$$Q12 = -A^{-1} \cdot B$$
$$\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} Q11 & Q12 \\ 0 & I \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix}$$

$$\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} Q11 & Q12 \\ 0 & I \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix} \tag{T2-4}$$

Taking the transpose of equation T2-4 we have:

$$(q \ r) = (P \ R) \begin{pmatrix} Q11^T & 0 \\ Q12^T & I \end{pmatrix} \tag{T2-5}$$

The **Strain Energy**, U, can be written:

$$U = 1/2\{q \ r\} \begin{pmatrix} \delta q \\ \delta r \end{pmatrix} \tag{T2-6}$$

$$\begin{pmatrix} \delta q \\ \delta r \end{pmatrix} = \begin{pmatrix} \alpha q & 0 \\ 0 & \alpha r \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} \tag{T2-7}$$

substituting Eq. T2-7 into Eq. T2-6 gives:

$$U = 1/2\{q \ r\} \begin{pmatrix} \alpha q & 0 \\ 0 & \alpha r \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} \tag{T2-8}$$

substituting Eq. T2-5 and T2-4 into Eq. T2-8 gives:

$$U = 1/2 (PR) \begin{pmatrix} Q11^T & 0 \\ Q12^T & I \end{pmatrix} \begin{pmatrix} \alpha q & 0 \\ 0 & \alpha r \end{pmatrix} \begin{pmatrix} Q11 & Q12 \\ 0 & I \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix} \tag{T2-9}$$

performing the matrix multiplication yields:

$$U = 1/2 (PR) \begin{pmatrix} Q11^T \cdot \alpha q \cdot Q11 & Q11^T \cdot \alpha q \cdot Q12 \\ Q12^T \cdot \alpha q \cdot Q11 & Q12^T \cdot \alpha q \cdot Q12 + \alpha r \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix} \tag{T2-10}$$

Using Castigliano's Theorem we take the partial derivative with respect to the applied load, P, and redundant, R, $\{PR\}$ yields the matrix relation:

$$\begin{pmatrix} \Delta \\ 0 \end{pmatrix} = \begin{pmatrix} Q11^T \cdot \alpha q \cdot Q11 & Q11^T \cdot \alpha q \cdot Q12 \\ Q12^T \cdot \alpha q \cdot Q11 & Q12^T \cdot \alpha q \cdot Q12 + \alpha r \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix} \tag{T2-11}$$

Equation T2-11 is the Flexibility Matrix, F, which may be expressed as:

$$\begin{pmatrix} \Delta \\ 0 \end{pmatrix} = \begin{pmatrix} F11 & F12 \\ F21 & F22 \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix} \tag{T2-12}$$

where

$$F11 = Q11^T \cdot \alpha q \cdot Q11 \qquad\qquad \text{(T2-12.1)}$$

$$F12 = Q11^T \cdot \alpha q \cdot Q12 \qquad\qquad \text{(T2-12.2)}$$

$$F21 = Q12^T \cdot \alpha q \cdot Q11 \qquad\qquad \text{(T2-12.3)}$$

$$F22 = Q12^T \cdot \alpha q \cdot Q12 + \alpha r \qquad\qquad \text{(T2-12.4)}$$

Expanding equation T2-12 gives:

$$\Delta = F11 \cdot P + F12 \cdot R \qquad\qquad \text{(T2-12.5)}$$

$$0 = F21 \cdot P + F22 \cdot R \qquad\qquad \text{(T2-12.6)}$$

solving Eq. T2-12.6 for the redundant, R, gives

$$R = -F22^{-1} \cdot F21 \cdot P \qquad\qquad \text{(T2-12.7)}$$

substituting Eq. T2-12.7 into Eq. T2-12.5 yields:

$$\Delta = F11 \cdot P - F12 \cdot F22^{-1} \cdot F21 \cdot P \qquad\qquad \text{(T2-12.8)}$$

$$\Delta = \left(F11 - F12 \cdot F22^{-1} \cdot F21\right) \cdot P \qquad\qquad \text{(T2-12.9)}$$

Summary

(1) form the A and B Matrices—Eq. T2-1
(2) invert the A matrix to get the matrix $Q11$
(3) multiply the B matrix by the inverse of A to get $Q12 = -Q11^*B$
(4) form the diagonal matrices $[\alpha q]$ and $[\alpha r]$
(5) form the $F11$, $F12$, $F21$ and $F22$ matrices using Eq. T2-12.1 $\rightarrow$ T2-12.4

To show the execution of the equations developed above we select a truss that is indeterminate to the 2nd degree. The truss is X braced and pinned at the base. In each bay, one member of the X brace is selected as redundant which reduces the truss to determinate.

To be consistent with Eq. T2-4, the eight 8 **statically determinate members** are labeled (0, 1, 2, 3, 4, 5, 6, 7) followed by the statically determinate pin reactions (8, 9, 10, 11). The cross bracing members selected as redundant ($r1$, $r2$), become the last two members of the vector: (12, 13).

Equilibrium Equations

Equilibrium equations at each joint are generated using MATHCAD and the following

(a) initially the (x, y) coordinates of each joint are input.
(b) This is followed by defining direction cosines between all joint coordinates. (which allows the input of a member between any two joints.)
(c) At each joint member connectively is now defined by inputting the following:

Joint	Member	Ja node	Jb node
1	5	1	6
1	6	1	9

@Joint1   members 5 and 6 are connected to joint 1

member 5 runs between nodes 1 and 6

member 6 runs between nodes 1 and 9

Using MATHCAD and the member connectively input at each joint, twelve (12) equilibrium equations are generated for the 8 members and 4 reactions of the determinate X truss. Adding a $\{2 \times 2\}$ identity matrix which accounts for the two (2) redundant's to the twelve (12) equilibrium equations forms a (14×14) Matrix in the form of equation T2-4.

$[\alpha]$ Matrix

The $[\alpha]$ matrix for a truss structure is a diagonal matrix. Each value along the diagonal is defined by the flexibility (L/AE) of that member. The **X truss** has 8 members and 4 reactions. Hence the first 8 elements along the diagonal define the flexibility of each member $(0 \rightarrow 7)$. The next 4 diagonals elements define the flexibility of the pin supports. For rigid supports this would consist of the support flexibility $(1/k)$ listed along the diagonal; where (k) is the support stiffness.

The diagonal $[\alpha]$ matrix is 14×14.

(a) 8 values (L/AE) listed along the diagonal representing the flexibility of members (0,1,2,3,4,5,6,7)
(b) 4 diagonal values $(1/k)$ representing the flexibility of each support
(c) 2 diagonal values (L/AE) representing the flexibility of the redundant member $r1$ and $r2$.

The pin supports are assigned a stiffness (k) 100 times greater than the stiffness of each truss member (AE/L). To simplify the programming each support is defined as an 'equivalent' truss member with an area 100 times greater than any truss member. (defined as 'rigid' relative to a truss member stiffness).

Notes

(1) The redundant's are carefully selected such that when they are removed the structure is statically determinate

(2) The numbering of the members, supports, and redundant's is carefully selected such that the numbering is consistent with the ordering of the equilibrium equation (T2-4) where the redundant's are the LAST VALUES listed in the vector member force vector.

By ordering of the equilibrium equation consistent with Eq. (T2-4) we simplify partitioning the matrix and solving the partitioned matrix using MATHCAD.

To clarify the Force Matrix Method of Structural Analysis we select the indeterminate X braced truss structure shown below. The truss is indeterminate to the 2nd degree. The analytical procedure used to solve for the unknown redundants is clearly defined using the Matrix equations previously derived. All matrix operations are performed using the Mathcad program.

The solution for the truss displacements and forces in the truss members follows a typical Linear Analysis. No 2nd order effects are considered or required since the intent of this example is to clarify the Matrix Force Method Analysis procedure.
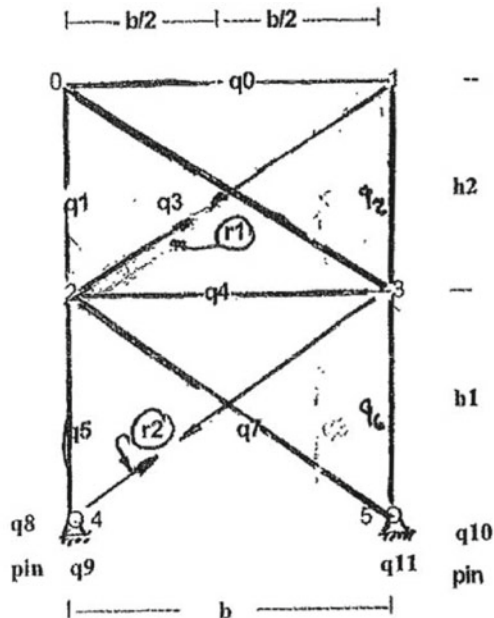
The analytical results are checked against AISC allowable stress design criteria and the results are presented graphically.

Example - X Braced Truss
Indeterminate
to 2nd degree

Geometry $h2 := 15$ ft
              $h1 := 15$ ft
              $b := 15$ ft

Coordinates
$x_0 := 0$ $y_0 := h1 + h2$
$x_1 := b$ $y_1 := h1 + h2$
$x_2 := 0$ $y_2 := h1$
$x_3 := b$ $y_3 := h1$
$x_4 := 0$ $y_4 := 0$
$x_5 := b$ $y_5 := 0$

The *X* braced truss shown above is indeterminate to the 2nd degree. The selected redundant are the cross diagonals $r1$ and $r2$ shown. There are 6 joints. Writing 2 equations per joint yields 12 equations of equilibrium. There are 14 unknowns.

The members are assigned member numbers from 0 to 7. The truss is supported by a pin at each support. The pin reactions become 'rigid' members 8, 9, 10, 11. The cross diagonals are selected as redundant $r1$ and $r2$ and become members 11, 12. Hence we have the 14 members listed below;

#	members
4 reactions	$[q8, q9, q10,$ and $q11]$
8 member forces	$[q0, q1, q2, q3, q4, q5, q6, q7]$.
2 redundant	$[r1, r2] \rightarrow n[q12, q13]$

The equilibrium equations are generated by a MATHCAD program that follows the procedure defined below:

(1) input and store the joint coordinates (x, y)

$$\text{units in ft} \quad x = \begin{pmatrix} 0 \\ 15 \\ 0 \\ 15 \\ 0 \\ 15 \end{pmatrix} \quad y = \begin{pmatrix} 30 \\ 30 \\ 15 \\ 15 \\ 0 \\ 0 \end{pmatrix} \quad \text{ft}$$

(2) Compute the Unit Vectors $[u, v]$ and member lengths

6 joints $i := 0..5 \quad j := 0..5$

Direction Cosines $u_{i,j} := \dfrac{x_j - x_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}} \quad v_{i,j} := \dfrac{y_j - y_i}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}}$

$$L_{i,j} := \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

Note that direction cosines are computed between ALL coordinates. Member connectivity, defined in Step 3, that follows, uses only those required.

Member Connectivity

n—node number

m—member number

ja—from node

jb—to node

n	m	ja	jb
node	**member**	from node	to node
number	number		

$$\text{node}_0 := \begin{pmatrix} n\ m\ ja\ jb \\ 0\ 0\ 0\ 1 \\ 0\ 1\ 0\ 2 \\ 0\ 3\ 0\ 3 \end{pmatrix} \qquad \text{node}_1 := \begin{pmatrix} n\ m\ ja\ jb \\ 1\ 0\ 1\ 0 \\ 1\ 2\ 1\ 3 \\ 1\ 12\ 1\ 2 \end{pmatrix} \qquad \text{node}_2 := \begin{pmatrix} n\ m\ ja\ jb \\ 2\ 1\ 2\ 0 \\ 2\ 12\ 2\ 1 \\ 2\ 4\ 2\ 3 \\ 2\ 7\ 2\ 5 \\ 2\ 5\ 2\ 4 \end{pmatrix}$$

$$\text{node}_3 := \begin{pmatrix} n\ m\ ja\ jb \\ 3\ 2\ 3\ 1 \\ 3\ 3\ 3\ 0 \\ 3\ 4\ 3\ 2 \\ 3\ 13\ 3\ 4 \\ 3\ 6\ 3\ 5 \end{pmatrix} \qquad \text{node}_4 := \begin{pmatrix} n\ m\ ja\ jb \\ 4\ 5\ 4\ 2 \\ 4\ 13\ 4\ 3 \\ 4\ 8\ 4\ 5 \\ 4\ 9\ 4\ 2 \end{pmatrix} \qquad \text{node}_5 := \begin{pmatrix} n\ m\ ja\ jb \\ 5\ 7\ 5\ 2 \\ 5\ 6\ 5\ 3 \\ 5\ 10\ 5\ 4 \\ 5\ 11\ 5\ 3 \end{pmatrix}$$

Form Equilibrium Equations   Row := 12   Col := 14

$n := 0..\text{Row} - 1$ $m := 0..\text{Col} - 1$

Equilibrium Matrix $\Xi\Xi_{n,m} := 0$ i—node number j—number of members connected to node

Joint 0 $i := 0$ $j := 0..2$

equilibrium matrix $\Xi_{2\cdot(\text{node}_i)_{j,0},(\text{node}_i)_{j,1}} := u_{[(\text{node}_i)_{j,2},(\text{node}_i)_{j,3}]}$

$\Xi_{2\cdot(\text{node}_i)_{j,0}+1,(\text{node}_i)_{j,1}} := v_{[(\text{node}_i)_{j,2},(\text{node}_i)_{j,3}]}$

member distance $d_{[(\text{node}_i)_{j,1}]} := L_{[(\text{node}_i)_{(j,2)},(\text{node}_i)_{j,3}]} \cdot 12$

Joint 1 $i := 1$ $j := 0..2$

equilibrium matrix $\Xi_{2\cdot(\text{node}_i)_{j,0},(\text{node}_i)_{j,1}} := u_{[(\text{node}_i)_{j,2},(\text{node}_i)_{j,3}]}$

$\Xi_{2\cdot(\text{node}_i)_{j,0}+1,(\text{node}_i)_{j,1}} := v_{[(\text{node}_i)_{j,2},(\text{node}_i)_{j,3}]}$

$d_{[(\text{node}_i)_{j,1}]} := L_{[(\text{node}_i)_{(j,2)},(\text{node}_i)_{j,3}]} \cdot 12$

Joint 2 $i := 2$ $j := 0..4$

equilibrium matrix $\Xi_{2\cdot(\text{node}_i)_{j,0},(\text{node}_i)_{j,1}} := u_{[(\text{node}_i)_{j,2},(\text{node}_i)_{j,3}]}$

$\Xi_{2\cdot(\text{node}_i)_{j,0}+1,(\text{node}_i)_{j,1}} := v_{[(\text{node}_i)_{j,2},(\text{node}_i)_{j,3}]}$

$d_{[(\text{node}_i)_{j,1}]} := L_{[(\text{node}_i)_{(j,2)},(\text{node}_i)_{j,3}]} \cdot 12$

Joint 3 $i := 3 \quad j := 0..4$

equilibrium
matrix $\Xi_{2 \cdot (\text{node}_i)_{j,0}, (\text{node}_i)_{j,1}} := u_{[(\text{node}_i)_{j,2}, (\text{node}_i)_{j,3}]}$

$\Xi_{2 \cdot (\text{node}_i)_{j,0}+1, (\text{node}_i)_{j,1}} := v_{[(\text{node}_i)_{j,2}, (\text{node}_i)_{j,3}]}$

$d_{[(\text{node}_i)_{j,1}]} := L_{[(\text{node}_i)_{(j,2)}, (\text{node}_i)_{j,3}]} \cdot 12$

Joint 4 $i := 4 \quad j := 0..3$

equilibrium
matrix $\Xi_{2 \cdot (\text{node}_i)_{j,0}, (\text{node}_i)_{j,1}} := u_{[(\text{node}_i)_{j,2}, (\text{node}_i)_{j,3}]}$

$\Xi_{2 \cdot (\text{node}_i)_{j,0}+1, (\text{node}_i)_{j,1}} := v_{[(\text{node}_i)_{j,2}, (\text{node}_i)_{j,3}]}$

$d_{[(\text{node}_i)_{j,1}]} := L_{[(\text{node}_i)_{(j,2)}, (\text{node}_i)_{j,3}]} \cdot 12$

Joint 5 $i := 5 \quad j := 0..3$

equilibrium
matrix $\Xi_{2 \cdot (\text{node}_i)_{j,0}, (\text{node}_i)_{j,1}} := u_{[(\text{node}_i)_{j,2}, (\text{node}_i)_{j,3}]}$

$\Xi_{2 \cdot (\text{node}_i)_{j,0}+1, (\text{node}_i)_{j,1}} := v_{[(\text{node}_i)_{j,2}, (\text{node}_i)_{j,3}]}$

$d_{[(\text{node}_i)_{j,1}]} := L_{[(\text{node}_i)_{(j,2)}, (\text{node}_i)_{j,3}]} \cdot 12$

Partition the Equilibrium Matrix as follows:

$$i := 0..11 \quad j := 0..11 \quad k := 0..1$$

Form The A and B matrices eq T2-1

$$A_{i,j} := \Xi_{i,j} \qquad B_{i,k} := \Xi_{i,12+k}$$

Invert The [A] matrix to get matrix [Q11] $Q11 := A^{-1}$

Form Q12 $Q12 := -Q11 \cdot B$

Form the diagonal Matrices αq and αr

$i := 0..13 \qquad j := 0..13 \qquad E := 30 \cdot 10^3$

Compute $[\alpha]$ matrix $\qquad \alpha_{i,j} := 0$

Tubular member $6 \times 6 \times 1/4$

$A0 := 6^2 - 5.5^2 \qquad A0 = 5.75$

$I0 := \dfrac{6^4}{12} - \dfrac{5.5^4}{12} \qquad I0 = 31.745$

K Members

$i := 0..13$

$a_i := A0 \qquad\qquad I_i := I0$

Reaction Properties

$a_8 := 100 \cdot A0 \qquad a_{10} := 100 \cdot A0$

$a_9 := 100 \cdot A0 \qquad a_{11} := 100 \cdot A0$

$i := 0..13 \qquad\qquad j := 0..13$

$\alpha_{i,i} := \dfrac{d_i}{a_i \cdot E}$

$I_8 := 1000 \cdot I0 \qquad I_{10} := 1000 \cdot I0$

$I_9 := 1000 \cdot I0 \qquad I_{11} := 1000 \cdot I0$

$d =$

	0
0	180
1	180
2	180
3	254.558
4	180
5	180
6	180
7	254.558
8	180
9	180
10	180
11	180
12	254.558
13	254.558

$a =$

	0
0	5.75
1	5.75
2	5.75
3	5.75
4	5.75
5	5.75
6	5.75
7	5.75
8	575
9	575
10	575
11	575
12	5.75
13	5.75

Partition the $[\alpha\alpha]$ matrix $\qquad i := 0..11 \quad j := 0..11 \quad k := 0..1$

$\alpha q_{i,i} := \alpha_{i,i}$

$\alpha r_{k,k} := \alpha_{12+k,12+k}$

Form the Flexibility Matrix $\quad i := 0..11 \quad j := 0..11 \quad k := 0..1$

Transpose the Q Matrices $\qquad QT11_{j,i} := Q11_{i,j}$

$QT12_{k,i} := Q12_{i,k}$
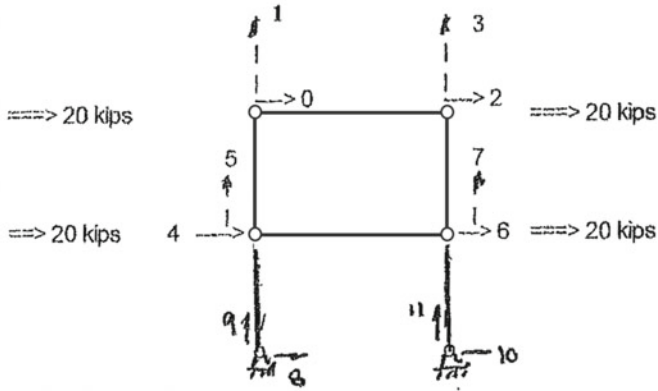
Flexibility Matrices $\qquad F11 := QT11 \cdot \alpha q \cdot Q11$

$F12 := QT11 \cdot \alpha q \cdot Q12$

$F21 := QT12 \cdot \alpha q \cdot Q11$

$F22 := QT12 \cdot \alpha q \cdot Q12 + \alpha r$

Degrees of Freedom

Load Vector

The function of the X truss is to support a structure against lateral wind load which is applied at degree of freedom 0 and 4 as shown above

$W_{\text{roof}} = 40$ kips
$W_{\text{mid}} = 40$ kips

Load Vector $i := 0..11$ $W_i := 0$ solving Eq. T2-12.6 for the redundant, R, gives

$$W_0 := 20 \qquad W_2 := 20$$
$$W_4 := 20 \qquad W_6 := 20$$

$$R := -F22^{-1} \cdot F21 \cdot W \qquad R = \begin{pmatrix} -28.284 \\ -56.569 \end{pmatrix} \text{ kips}$$

The displacements are found from Eq. T2-12.9

$$F := \left(F11 - F12 \cdot F22^{-1} \cdot F21 \right)$$
$$\Delta := F \cdot W \text{ inches}$$

Flexibility Matrix

The first 4 columns of the flexibility matrix are shown below.
Row 0 to 7 correspond to the degrees of freedom of the structure
Rows 8, 9, 10, 11 are the degrees of freedom of the 'rigid' pin supports

Row	$F_{0,i} :=$	$F_{1,i} :=$	$F_{2,i} :=$	$F_{3,i} :=$
0	$8.491 \cdot 10^{-3}$	$2.156 \cdot 10^{-3}$	$8.024 \cdot 10^{-3}$	$-2.059 \cdot 10^{-3}$
1	$2.156 \cdot 10^{-3}$	$1.892 \cdot 10^{-3}$	$2.059 \cdot 10^{-3}$	$-2.059 \cdot 10^{-4}$
2	$8.024 \cdot 10^{-3}$	$2.059 \cdot 10^{-3}$	$8.491 \cdot 10^{-3}$	$-2.156 \cdot 10^{-3}$
3	$-2.059 \cdot 10^{-3}$	$-2.059 \cdot 10^{-4}$	$-2.156 \cdot 10^{-3}$	$1.892 \cdot 10^{-3}$
4	$3.064 \cdot 10^{-3}$	$6.351 \cdot 10^{-4}$	$3.112 \cdot 10^{-3}$	$-4.292 \cdot 10^{-4}$
5	$1.58 \cdot 10^{-3}$	$9.448 \cdot 10^{-4}$	$1.592 \cdot 10^{-3}$	$-1.091 \cdot 10^{-4}$
6	$3.112 \cdot 10^{-3}$	$4.292 \cdot 10^{-4}$	$3.064 \cdot 10^{-3}$	$-6.351 \cdot 10^{-4}$
7	$-1.592 \cdot 10^{-3}$	$-1.091 \cdot 10^{-4}$	$-1.58 \cdot 10^{-3}$	$9.448 \cdot 10^{-4}$
8	$5.28 \cdot 10^{-6}$	$1.091 \cdot 10^{-6}$	$5.154 \cdot 10^{-6}$	$1.091 \cdot 10^{-6}$
9	$2.087 \cdot 10^{-5}$	$1.043 \cdot 10^{-5}$	$2.087 \cdot 10^{-5}$	0
10	$5.154 \cdot 10^{-6}$	$-1.091 \cdot 10^{-6}$	$5.28 \cdot 10^{-6}$	$-1.091 \cdot 10^{-6}$
11	$-2.087 \cdot 10^{-5}$	0	$-2.087 \cdot 10^{-5}$	$1.043 \cdot 10^{-5}$

$$W = \begin{array}{c} \text{Loads} \\ \begin{array}{|c|c|} \hline & 0 \\ \hline 0 & 20 \\ \hline 1 & 0 \\ \hline 2 & 20 \\ \hline 3 & 0 \\ \hline 4 & 20 \\ \hline 5 & 0 \\ \hline 6 & 20 \\ \hline 7 & 0 \\ \hline 8 & 0 \\ \hline 9 & 0 \\ \hline 10 & 0 \\ \hline 11 & 0 \\ \hline \end{array} \end{array} \qquad \Delta = \begin{array}{c} \text{Displacements} \\ \begin{array}{|c|r|} \hline & 0 \\ \hline 0 & 0.454 \\ \hline 1 & 0.106 \\ \hline 2 & 0.454 \\ \hline 3 & -0.106 \\ \hline 4 & 0.204 \\ \hline 5 & 0.085 \\ \hline 6 & 0.204 \\ \hline 7 & -0.085 \\ \hline 8 & 4.174 \cdot 10^{-4} \\ \hline 9 & 1.252 \cdot 10^{-3} \\ \hline 10 & 4.174 \cdot 10^{-4} \\ \hline 11 & -1.252 \cdot 10^{-3} \\ \hline \end{array} \end{array} \text{ inches}$$

$$i := 0..13$$

Stress Analysis $fa_i := \dfrac{Q_i}{a_i}$ $Fy := 36 \text{ ksi}$ $\sigma_{\text{all}} := 0.75 \cdot Fy$

Buckling Analysis $Cc := \sqrt{\dfrac{2 \cdot \pi^2 \cdot E}{Fy}}$ slenderness ratio   stress

radius of gyration $r_i := \sqrt{\dfrac{I_i}{a_i}}$

slenderness ration $\text{ratio}_i := \dfrac{d_i}{r_i}$

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ALLOWABLE STRESS DESIGN

$$Fa_i := \frac{\left[1 - \dfrac{\left(\dfrac{d_i}{r_i}\right)^2}{2 \cdot Cc^2}\right] \cdot Fy}{\dfrac{5}{3} + \dfrac{3}{8} \cdot \dfrac{\left(\dfrac{d_i}{r_i}\right)}{Cc} - \dfrac{\left(\dfrac{d_i}{r_i}\right)^3}{Cc^3} \cdot \dfrac{1}{8}}$$

slenderness ratio

ratio =

	0
0	76.607
1	76.607
2	76.607
3	108.339
4	76.607
5	76.607
6	76.607
7	108.339
8	24.225
9	24.225
10	24.225
11	24.225
12	108.339
13	108.339

member stress

fa =

	0
0	0
1	-3.478
2	3.478
3	4.919
4	$-1.236 \cdot 10^{-15}$
5	-13.913
6	13.913
7	9.838
8	0.07
9	0.209
10	-0.07
11	-0.209
12	-4.919
13	-9.838

allowable stress

Fa =

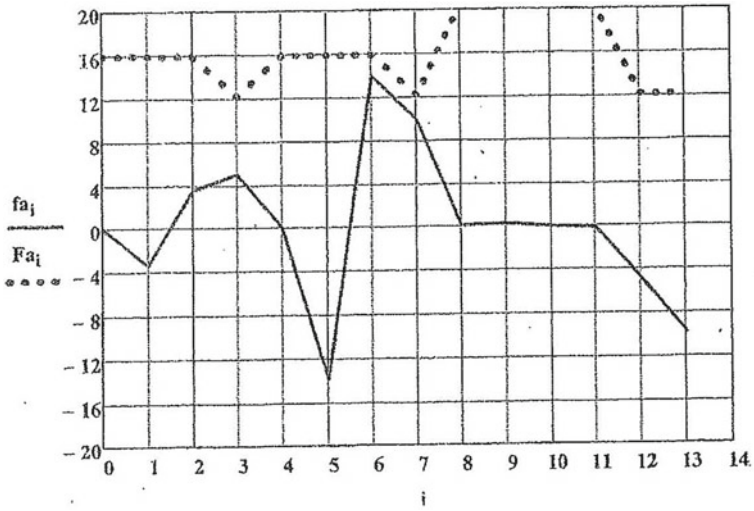|	0
0	15.868
1	15.868
2	15.868
3	12.136
4	15.868
5	15.868
6	15.868
7	12.136
8	20.36
9	20.36
10	20.36
11	20.36
12	12.136
13	12.136

stress ration $sr_i := \text{if}\left[fa_i < 0, \dfrac{(-fa)_i}{Fa_i}, \dfrac{fa_i}{\sigma_{\text{all}}}\right]$

Stress Ratio
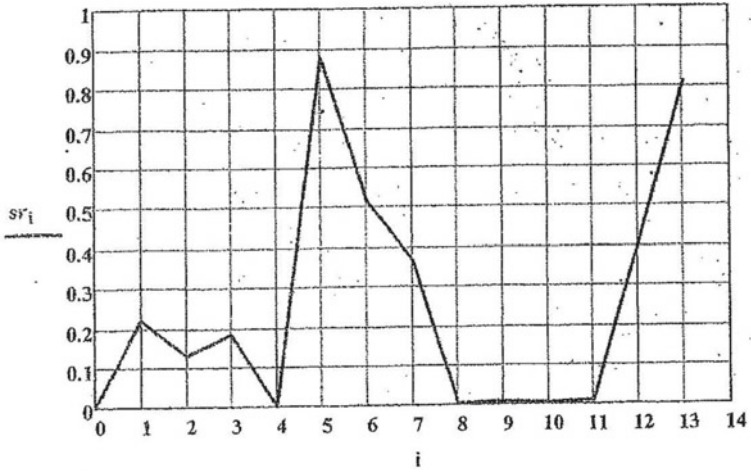
Axial Stress (fa) Allowable Stress (Fa)

Appendix

<div align="right">**C**</div>

Matrix Methods of Structural Analysis

(1) Stiffness Method

  (a) Displacement
  (b) Direct Stiffness

(2) Force Matrix Method

A simple 2 link problem is used to display both the Stiffness and Force Matrix Methods. The Stiffness Method can be approached using either a Displacement technique or a procedure called the Direct Stiffens's Method. The Direct Stiffness Method is easily programmable and is incorporated in most Structural Analysis Software.

The Force Matrix Method is consistent with how Structural engineers have historically approached Structural analysis. In general the procedure requires selecting a statically determinate structure (for an indeterminate structure), selecting redundant's or unknowns and writing equilibrium equations. The method is cumbersome and not easily programmable and therefore not preferable for computer analysis.

Matrix Methods of Structural Analysis

Structural analysis is concerned with the determination of forces and displacements. There are basically two approaches that can be used:

(a) The Force Method
(b) The Displacement Method

The Force Method treats the forces as the unknowns. Solve for the forces and from the forces you find the displacements.

The Displacement Method treats the displacements as the unknowns. Find the displacements and from the displacements find the forces.

Nearly all of the finite element programs available today use the displacement method, which will be explained later in the text. The displacement method is very effective for highly redundant (indeterminate structures) because programming algorithms are very effective and efficient. One could say that the displacement method is more computer friendly than the force method.

The primary problem with the displacement method is that it is inconsistent with the thinking of structural engineers who are primarily interested in forces and their effect on the structure.

To fix ideas and to appreciate the difference between the Force and the Displacement Method we solve example problem **T3** using:

 (a) Displacement Method
 (b) Force Matrix Method

   **Example T3**

   Displacement Method

                                                          $+$ Positive (up)

   **Example T3**
        Displacement Method
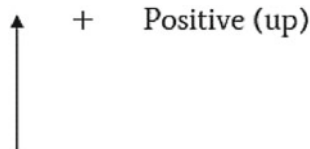
Sign Convention—the sign convention is extremely important when using a displacement method. The programming algorithms are based on a consistent right hand Cartesian coordinate system. For this problem all force acting up are positive. Hence, the force q is shown below are positive acting up. The loads PO and $P1$ are positive acting up
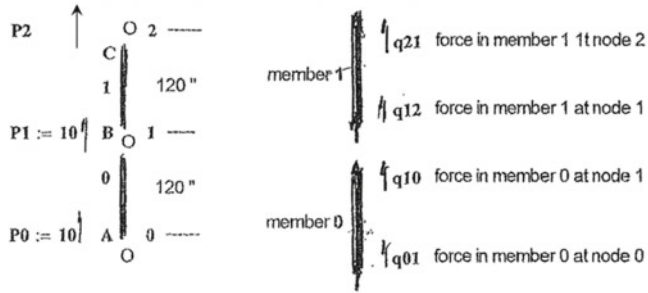
Equilibrium at joint 2—Consistent with the adopted sign convention the force $q12$ acts down and the load $P2$ acts up. Hence equilibrium at joint 2 becomes:

$$q21 = P2$$
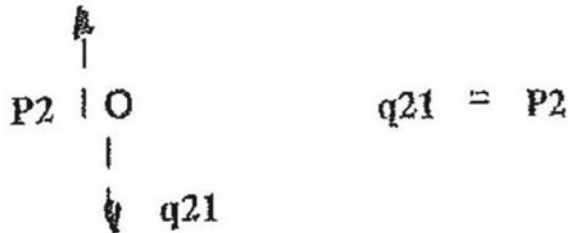
Equilibrium at joint 1—Consistent with the adopted sign convention the forces $q21$ and $q11$ act down and the load $P1$ acts up. Hence equilibrium at joint 1 reduces to:

$$q10 + q12 = P1$$
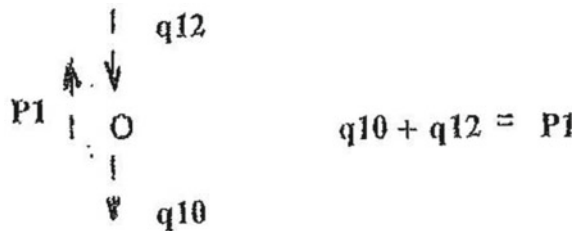
Equilibrium at joint 0—consistent with the sign convention adopted, the force $q00$ shown in the free body diagram of joint 0 acts down (equal and opposite) and the applied load, $P1$, acts up. Hence the equilibrium equation at joint 0 reduces to:

$$q01$$

$$PO \quad\quad O$$

$$q01 \;=\; PO$$
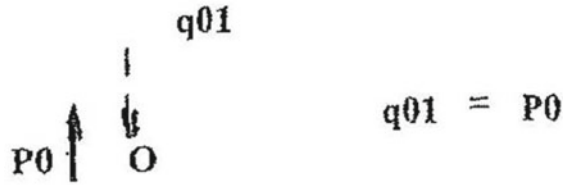
in Matrix form, the equilibrium equations become:

$$
\begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix} \cdot
\begin{pmatrix} q01 \\ q01 \\ q11 \\ q12 \end{pmatrix} =
\begin{pmatrix} P0 \\ P1 \\ P2 \end{pmatrix}
\tag{1.0}
$$

We now need a relationship between the internal forces and displacements in member 0 and member 1. The coordinate system for displacements is consistent with the coordinate system selected for forces. **Up is positive.**

If we fix joint 1 and impose a displacement, $\delta0$, at joint 0, we find that the forces in member 0 are:

$$q00 = \frac{A0 \cdot E0}{L0} \cdot \delta0$$

$$q01 = \frac{-A0 \cdot E0}{L0} \cdot \delta0$$

If we fix joint 0 and impose a displacement, $\delta1$, at joint 1, we find that the forces in member 0 are:

$$q00 = \frac{-A0 \cdot E0}{L0} \cdot \delta1$$

$$q01 = \frac{A0 \cdot E0}{L0} \cdot \delta1$$

Expressing the force displacement equations for member 0 in Matrix form we have:

$$
\begin{pmatrix} q00 \\ q01 \end{pmatrix} = \frac{A0 \cdot E0}{L0}
\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot
\begin{pmatrix} \delta00 \\ \delta01 \end{pmatrix}
$$

repeating the procedure for member 1 we have:

$$
\begin{pmatrix} q11 \\ q12 \end{pmatrix} = \frac{A1 \cdot E1}{L1}
\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot
\begin{pmatrix} \delta11 \\ \delta12 \end{pmatrix}
$$

The Matrix relations for the forces in members 0 and 1 can be expressed as:

$$\begin{pmatrix} q00 \\ q01 \\ q11 \\ q21 \end{pmatrix} = \begin{pmatrix} \kappa 0 & -\kappa 0 & 0 & 0 \\ -\kappa 0 & \kappa 0 & 0 & 0 \\ 0 & 0 & \kappa 1 & -\kappa 1 \\ 0 & 0 & -\kappa 1 & \kappa 1 \end{pmatrix} \begin{pmatrix} \delta 00 \\ \delta 01 \\ \delta 11 \\ \delta 12 \end{pmatrix} \qquad (2.0)$$

$E := 30000.\ \text{ksi}$

where: $\quad \kappa 0 = \dfrac{A0 \cdot E0}{L0} \qquad \kappa 0 := \dfrac{\frac{\pi}{4} \cdot 1^2 \cdot E}{120} = 196.35$

$\quad \kappa 1 = \dfrac{A1 \cdot E1}{L1} \qquad \kappa 1 := \kappa 0 = 196.35$

The vector of member displacements $\{\delta\}$ can be related to the vector of structural displacements $\{\Delta\}$ by observing that:

$$\delta 00 = \Delta 0$$
$$\delta 01 = \Delta 1$$
$$\delta 11 = \Delta 1$$
$$\delta 12 = \Delta 2$$

in matrix form we have:

$$\begin{pmatrix} \delta 00 \\ \delta 01 \\ \delta 11 \\ \delta 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \Delta 0 \\ \Delta 1 \\ \Delta 2 \end{pmatrix}$$

Using Matrix notation we can write equations 1.0, 2.0 and 3.0 as:

Eq. 1.0 $\quad T^T \cdot q = P$

Eq. 2.0 $\quad q = \kappa \cdot \delta$

Eq. 3.0 $\quad \delta = T \cdot \Delta$

Inspection of equations 1.0 and 3.0 show a clear relationship between the matrices. One is the transpose of the other, and this will always be the case. In other words the relationship between the local displacement vector, δ, and the structural displacement, Δ, is the transpose of the relationship between the local forces, q, and the applied loads, P.

Substituting equations 2 and 3 into Eq. 1 gives;

substituting equations 2 and 3 into Eq. 1 gives: $\quad P = T^T \cdot \kappa \cdot T \cdot \Delta$

where the Global Stiffness matrix of the structure: $\quad K = T^T \cdot \kappa \cdot T$

and Transpose: $\quad P = K \cdot \Delta$

$$T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad Tt := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Pt := \begin{pmatrix} -10 \\ -10 \end{pmatrix}$$

Numerical Solution

Element Stiffness Matrix $\quad \kappa := \begin{pmatrix} \kappa 0 & -\kappa 0 & 0 & 0 \\ -\kappa 0 & \kappa 0 & 0 & 0 \\ 0 & 0 & \kappa 1 & -\kappa 1 \\ 0 & 0 & -\kappa 1 & \kappa 1 \end{pmatrix}$

Stiffness Matrix $\qquad\qquad K := Tt \cdot \kappa \cdot T$

$$K = \begin{pmatrix} 196.35 & -196.35 & 0 \\ -196.35 & 392.699 & -196.35 \\ 0 & -196.35 & 196.35 \end{pmatrix}$$

$$\begin{pmatrix} P0 \\ P1 \\ P2 \end{pmatrix} = \begin{pmatrix} 196.35 & -196.35 & 0 \\ -196.35 & 392.699 & -196.35 \\ 0 & -196.35 & 196.35 \end{pmatrix} \cdot \begin{pmatrix} \Delta0 \\ \Delta1 \\ \Delta2 \end{pmatrix}$$

In this form the Global stiffness matrix, K, is singular because the structure **has not been grounded**. We know that the displacement at joint 2 is zero $[\Delta2 = 0]$ because the structure is connected to the ground at this point.

To ground the structure in the matrix formulation we must set, $\Delta2 = 0$ and the force, $P2$, become the reaction, R, as shown in the matrix below.

$$\begin{pmatrix} P0 \\ P1 \\ R \end{pmatrix} = \begin{pmatrix} 196.35 & -196.35 & 0 \\ -196.35 & 392.699 & -196.35 \\ 0 & -196.35 & 196.35 \end{pmatrix} \cdot \begin{pmatrix} \Delta0 \\ \Delta1 \\ \Delta2 \end{pmatrix}$$

To solve the problem, partition the Global Stiffness matrix as follows:

$$\begin{pmatrix} P \\ R \end{pmatrix} = \begin{pmatrix} K11 & K12 \\ K21 & K22 \end{pmatrix} \cdot \begin{pmatrix} \Delta \\ 0 \end{pmatrix}$$

Expansion of the matrix relation gives:

$$P = K11 \cdot \Delta$$
$$R = K21 \cdot \Delta$$

where: $K11 := \begin{pmatrix} 196.35 & -196.35 \\ -196.35 & 392.699 \end{pmatrix}$

$\qquad\quad K21 := \begin{pmatrix} 0 & -196.35 \end{pmatrix}$

solving for the displacement vector, Δ, gives:

$$\Delta := K11^{-1} \cdot Pt \qquad \Delta = \begin{pmatrix} -0.153 \\ -0.102 \end{pmatrix}$$

and the reaction is: $R := K21 \cdot \Delta \qquad R = 20$

Now that the displacements are known we can determine the force in each member by back substituting as follows:

Compute the members deflections:

member deflections

$$\begin{pmatrix} \delta00 \\ \delta01 \\ \delta11 \\ \delta12 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \Delta_0 \\ \Delta_1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \delta00 \\ \delta01 \\ \delta11 \\ \delta12 \end{pmatrix} = \begin{pmatrix} -0.153 \\ -0.102 \\ -0.102 \\ 0 \end{pmatrix}$$

Compute the member forces

member forces

$$\begin{pmatrix} q00 \\ q01 \\ q11 \\ q21 \end{pmatrix} := \begin{pmatrix} \kappa0 & -\kappa0 & 0 & 0 \\ -\kappa0 & \kappa0 & 0 & 0 \\ 0 & 0 & \kappa1 & -\kappa1 \\ 0 & 0 & -\kappa1 & \kappa1 \end{pmatrix} \cdot \begin{pmatrix} \delta00 \\ \delta01 \\ \delta11 \\ \delta12 \end{pmatrix} \qquad \begin{pmatrix} q00 \\ q01 \\ q11 \\ q21 \end{pmatrix} = \begin{pmatrix} -10 \\ 10 \\ -20 \\ 20 \end{pmatrix}$$

Comments

The analysis looks tedious and time consuming. It is and it was prior to the computer. But the computer is an incredible tool. Once you program it, the process is repeatable which can be both good and bad. It is good because a better engineered structure can theoretically be completed in less time. But it could be bad if the analysis is transparent, the matrix operations are pre-programmed and the software is a 'black box'. The displacement analysis described above can be considerably condensed and shortened using a technique called the Direct Stiffness Method.

Direct Stiffness Method

In the Direct Stiffness Method one builds the Structural Stiffness Matrix much as a postman delivers the mail. Consider Example T3. We have 3 degrees of freedom $\{\Delta 0, \Delta 1, \Delta 2\}$ and two (2) members whose element stiffness matrices are:

Member 0 is connected to degrees of freedom 0 and 1.

Member 1 is connected to degrees of freedom 1 and 2.

Imagine that you are a mailman and you are tasked with delivering the elements from the member stiffness matrices $[k0]$ and $[k1]$ to the proper row and column of the **Global Stiffness Matrix** shown below:

For member 0, the $[\kappa 0]$ elements are delivered to locations (0, 0), (0, 1), (1, 0) and (1, 1).

For member 1, the $[\kappa 1]$ elements are delivered to locations (1, 1), (1, 2), (2, 1) and (2, 2).

Note that the (1, 1) element from member 0 is added to the (1, 1) element from member 1 to form the (1, 1) element of the Global Stiffness Matrix.
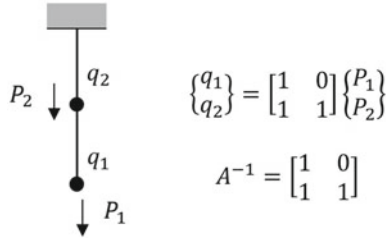
Global Stiffness Matrix

$$
\begin{array}{ccc}
0 & 1 & 2
\end{array}
$$
$$
\begin{pmatrix}
\kappa 0 & -\kappa 0 & 0 \\
-\kappa 0 & \kappa 0 + \kappa 1 & -\kappa 1 \\
0 & -\kappa 1 & \kappa 1
\end{pmatrix} =
\begin{pmatrix}
196.35 & -196.35 & 0 \\
-196.35 & 392.699 & -196.35 \\
0 & -196.35 & 196.35
\end{pmatrix}
\quad \text{Look Familiar?}
$$

The FORCE Matrix Method—Theoretical Development

The equilibrium equations can be expressed in Matrix Form as:

$$[A]\{q\} = \{P\} \tag{1.0}$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

where,

{q}—a vector of unknown forces

{P}—the applied load vector.

[A] is a square ($n \times n$) Equilibrium Matrix.

The number of rows = number of equations.

The number of columns = number of unknown forces.

We can solve for the unknown forces by inverting the [A] matrix.

$$\{q\} = A^{-1}\{P\} \tag{2.0}$$

The strain energy, U, is the dot product of the force vector {q} and the displacement vector {δ}

$$U = q^T \cdot \delta \tag{3.0}$$

where the displacement vector {δ} is given by the Matrix relation:

$$\{\delta\} = [\alpha]\{q\} \tag{4.0}$$

substituting Eqs. 2 and 4 into Eq. 3 yields:

$$U = P^T \cdot \left(A^{-1}\right)^T \cdot \alpha \cdot A^{-1} \cdot P \tag{5.0}$$

Using Castigliano's Theorem we take the partial derivation of the strain energy, U, with respect to the load vector {P} which yields the displacement vector {Δ}.

Taking the partial derivative gives the Matrix relation:

$$\{\Delta\} = [F]\{P\} \tag{6.0}$$

where the matrix [F] is called the flexibility matrix

$$F = \left(A^{-1}\right)^T \cdot \alpha \cdot A^{-1} \tag{7.0}$$

Equation 7.0 provides the matrix operations required to form the flexibility matrix.

The process consists of

(a) writing equilibrium equations to form the $[A]$ matrix.
(b) deriving a 'local' flexibility matrix $[\alpha]$ for each structural element.
(c) performing the matrix operations.

If the structure is determinate, there will be as many equations as there are unknowns and the $[A]$ matrix will be square. If the structure is geometrically stable the $[A]$ matrix can be inverted. If the structure is geometrically unstable the $[A]$ matrix will be singular.

Example T3

The Force Matrix Method of Structural Analysis is applied to the two link example problem
The Strain Energy stored in the two link example problem can be expressed in Matrix form as:

$$U = \{\, q_1 \; q_2 \,\} \begin{pmatrix} \dfrac{L1}{2 \cdot A1 \cdot E} & 0 \\ 0 & \dfrac{L2}{2 \cdot A2 \cdot E} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \qquad (-1.0)$$

The forces in members 1 and 2 can be expressed by the matrix relation:

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} P1 \\ P2 \end{pmatrix} \qquad (-2.0)$$

transposing equation 2.0 gives:

$$\{\, q_1 \; q_2 \,\} = \{\, P1 \; P2 \,\} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad (-3.0)$$

substituting equations 2.0 and 3.0 into Eq. 1.0 yields the matrix relation:

$$U = \{\, P1 \; P2 \,\} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{L1}{2 \cdot A1 \cdot E} & 0 \\ 0 & \dfrac{L2}{2 \cdot A2 \cdot E} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} P1 \\ P2 \end{pmatrix} \qquad (-4.0)$$

performing the matrix multiplication yields:

$$U = \{\, P1 \; P2 \,\} \begin{pmatrix} \dfrac{L1}{2 \cdot A1 \cdot E} + \dfrac{L2}{2 \cdot A2 \cdot E} & \dfrac{L2}{2 \cdot A2 \cdot E} \\ \dfrac{L2}{2 \cdot A2 \cdot E} & \dfrac{L2}{2 \cdot A2 \cdot E} \end{pmatrix} \begin{pmatrix} P1 \\ P2 \end{pmatrix} \qquad (-5.0)$$

Castigliano's Theorem—The partial derivative of the strain energy, U, with respect to the applied load vector $\{P\}$ yields a vector of structural displacements.

Taking the partial derivative of equation 5.0 with respect to the load vector, $\{P\}$, yields:

$$\begin{pmatrix} \Delta 1 \\ \Delta 2 \end{pmatrix} = \begin{pmatrix} \dfrac{L1}{A1 \cdot E} + \dfrac{L2}{A2 \cdot E} & \dfrac{L2}{A2 \cdot E} \\ \\ \dfrac{L2}{A2 \cdot E} & \dfrac{L2}{A2 \cdot E} \end{pmatrix} \begin{pmatrix} P1 \\ P2 \end{pmatrix} \qquad \text{(-6.0)}$$

Equation 6.0 can be written in the more concise form:

$$\begin{pmatrix} \Delta 1 \\ \Delta 2 \end{pmatrix} = \begin{pmatrix} f11 & f12 \\ f21 & f22 \end{pmatrix} \begin{pmatrix} P1 \\ P2 \end{pmatrix} \qquad \text{(-7.0)}$$

The Matrix $\begin{pmatrix} f11 & f12 \\ f21 & f22 \end{pmatrix}$ is the flexibility matrix. The elements f, i, j are called influence coefficients

$$[F] = \begin{pmatrix} f11 & f12 \\ f21 & f22 \end{pmatrix}$$

The Flexibility Matrix

$f11$—the displacement of joint 1 due to a unit load at joint 1

$f12$—the displacement of joint 1 due to a unit load at joint 2

$f21$—the displacement of joint 2 due to a unit load at joint 1

$f22$—the displacement of joint 2 due to a unit load at joint 2

Evaluate [F]

$L1 := 120 \qquad A1 := \dfrac{\pi}{4} \cdot 1^2 = 0.785 \qquad A2 := \dfrac{\pi}{4} \cdot 1^2 = 0.785 \qquad E = 3 \times 10^4$

$L2 := 120$

$f11 := \dfrac{L1}{A1 \cdot E} + \dfrac{L2}{A2 \cdot E} \quad f11 = 0.01$

$f12 := \dfrac{L2}{A2 \cdot E} \qquad\qquad f12 = 5.093 \cdot 10^{-3}$

$f21 := \dfrac{L2}{A2 \cdot E} \qquad\qquad f21 = 5.093 \cdot 10^{-3}$

$f22 := \dfrac{L2}{A2 \cdot E} \qquad\qquad f22 = 5.093 \cdot 10^{-3}$

The Flexibility Matrix is always symmetric. Hence $f12 = f21$ or in general $[fi, j] = fj, i]$ is known as Maxwell's law of reciprocity.

$$F := \begin{pmatrix} f11 & f12 \\ f21 & f22 \end{pmatrix} \qquad F = \begin{pmatrix} 0.01 & 5.093 \times 10^{-3} \\ 5.093 \times 10^{-3} & 5.093 \times 10^{-3} \end{pmatrix}$$

To evaluate the displacements we need to define the load vector: $P1 := 10$ applied
$P2 := 10$ down

$$Po := \begin{pmatrix} P1 \\ P2 \end{pmatrix} \qquad Po = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

The displacements may now be obtained by performing the Matrix operation indicated by Eq. 7.0

$$\Delta o := F \cdot Po \qquad \Delta o = \begin{pmatrix} 0.153 \\ 0.102 \end{pmatrix}$$

Note that the displacements are identical to those previously obtained

Summary

(1.0) Write equilibrium equations that express the internal forces in the structure as a function of the applied loads (see Eq. 2.0)

(2.0) Transpose the equilibrium matrix and perform the Matrix operation indicated by Eq. 4.0

(3.0) The Flexibility Matrix $[F]$ is formed by step 2.0

(4.0) Compute the influence Coefficients fi, j.

(5.0) Define the load vector $\{P\}$

(6.0) Compute the displacements using the Matrix relation defined by Eq. 7.0

The problem that we have completed is very simple and the work required to compute the displacements using the Matrix formulation explained above seems overly cumbersome. But remember that we have a computer and the process can be programmed.

Appendix

D

Appendix D is a rather complicated analysis of a statically determinate reinforced concrete 3 HINGE stairway. The stairway is 'idealized' as composed of equivalent concrete beams. A Transformation Matrix is developed to convert Global Loads and Displacements to local loads and displacements normal to the stairway.

Gravity Loading is applied across the span. The Arch is defined by 5 local degrees of freedom. Because the Arch is statically determinate the Force Matrix Method of Structural Analysis is used to form the 5×5 Flexibility Matrix in the Local Coordinate System. A Transformation Matrix converts the Gravity Loading to Loads in the Local Coordinate System applied normal to the members at the five degrees freedom.

A Vibration and Critical Buckling Analysis of the stairway is evaluated. The Vibration analysis include the frequency and mode shapes of the first two modes. The Buckling analysis includes the Critical Buckling load and corresponding mode shape.

Force Matrix Method of Structural Analysis—3 Hinge Archstairway

The ensuing analysis is intended to display the power and versatility of the Force Matrix Method. The analysis focuses on a statically determinate stairway (analyzed as a three hinged arch). **To represent displacements normal to each ramp, a transformation matrix is required to switch from a global system to a local coordinate system. Because the stairway is ramped the development of the geometric stiffness matrix** $[\mathbf{Kg}]$ **becomes somewhat more complicated.**

The 4 ft wide stairway is reinforced concrete. Reinforcement is provided using #4 re bar spaced 6 inch on center. The stairway is analyzed as a series of beams 1 ft wide and 7 inches thick.
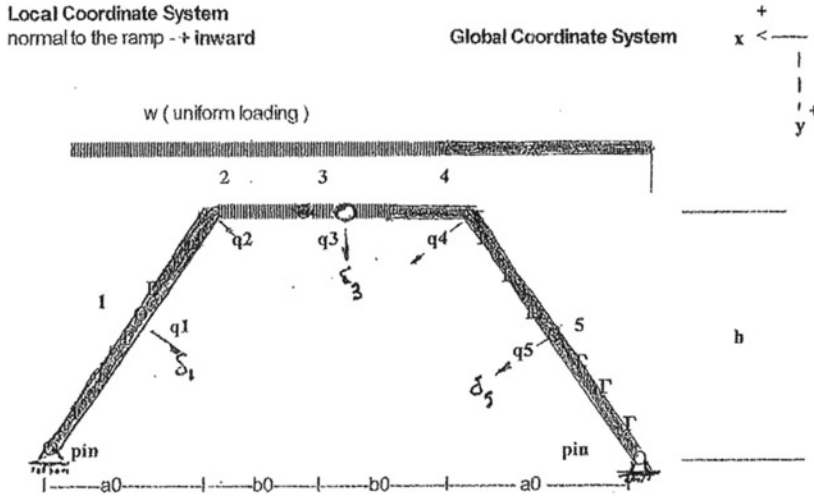
The concrete stairs are idealized as the rectangular beam shown below: The beam is 12" wide 7" deep and reinforced by #4 bars spaced 6 inches on center.

Concrete Beam
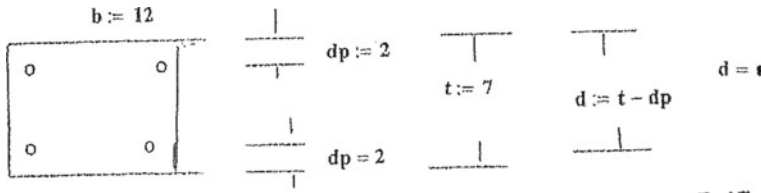
| Compression bars - (2) #4 bars | $n = Es/Ec$ |
| Tension bars - (2) #4 bars | $n := 10$ |

Concrete Section Properties $b = 12$ $t = 7$

Cracked Location of Neutral axis $b \cdot x \dfrac{x}{2} = As \cdot n \cdot (d - x)$

Try $x := 15$ $b \cdot \dfrac{x^2}{2} = 13.5$

$As := \dfrac{\pi}{4} \cdot \left(\dfrac{4}{8}\right)^2 \cdot 2$ $As \cdot n \cdot (d - x) = 13.744$ $x = 1.5$ O.K

Cracked Moment of Inertia $Icr := \dfrac{b \cdot x^3}{3} + n \cdot As \cdot (d - x)^2$

$Icr = 61.606 \text{ in}^4$

Gross Moment of Inertia $Ig := \dfrac{b \cdot t^3}{12}$ $Ig = 343$

$\dfrac{Ig}{Icr} = 5.568$

Note: Gross to Cracked ratio is 5.5 which means that the stairway will remain very stiff unless the external load exceeds the cracking moment, Mcr. If that were to occur during the life of the structure the stiffness would be reduced by a factor that could range from $(1 > 5)$ depending on the extent of the cracking.

Compute the Cracking Moment

2 numbers 4 bars per 6 foot $\rho := \dfrac{As}{b \cdot d}$ $\rho = 6.545 \times 10^{-3}$

$fpc := 4000$

Cracking Moment $\sigma t := 7.5 \cdot \sqrt{fpc}$ $\sigma t = 474.342 \text{ psi}$

$\sigma t = \dfrac{6 \cdot Mcr}{b \cdot t^2}$

$Mcr := \dfrac{b \cdot t^2 \cdot \sigma t}{6}$ $Mcr = 4.649 \times 10^4 \text{ in} - \text{lbs}$

ACI requires that the minimum steel must satisfy the cracking moment. Using working stress design check the steel stress at the cracking moment.

Working Stress

concrete force $\qquad$ $C := \dfrac{Mcr}{d - \dfrac{x}{3}}$ $\qquad$ $C = 1.033 \times 10^4$ lbs

concrete stress at cracking moment $\qquad$ $fc := \dfrac{2 \cdot C}{b \cdot x}$ $\qquad$ $fc = 1.148 \times 10^3$ psi

concrete stress allowable $\qquad$ $0.45 \cdot fpc = 1.8 \times 10^3$ O.k

$$T := C$$

$fy := 60000$ $\qquad$ $fs := \dfrac{T}{As}$ $\qquad$ $fs = 2.631 \times 10^4$ psi

Note that the stress in the steel is less than $fy/2$ and concrete stress is less than $0.45\,fpc$. Hence the steel will work if and when the concrete cracks.

Essentially the section will probably remain uncracked during the life of the building since the actual moment in the stairway may be appreciably less than the cracking moment. Now we check Ultimate Strength design criteria for minimum steel requirements

Ultimate Strength $\qquad$ $fy = 6 \times 10^4$ $\qquad$ $\Phi := 0.9$

Ultimate Moment $\quad T := fy \cdot As \qquad T = 2.356 \times 10^4$

$$0.85 \cdot fpc \cdot a \cdot b = C = T$$

$$a := \frac{T}{0.85 \cdot fpc \cdot b} \qquad a = 0.577 \text{ inches}$$

$$\frac{\Phi \cdot Mn}{12000} = 8.325 \qquad Mu = 8 \text{ ft kips}$$

$$\frac{\Phi \cdot Mn}{Mcr} = 2.149 \quad \text{O.K.}$$

Note: 2 #4 bars per foot satisfy the cracking moment and minimum steal requirements per AIC strength design with a factor of safety of 2 (**satisfies ACI**)
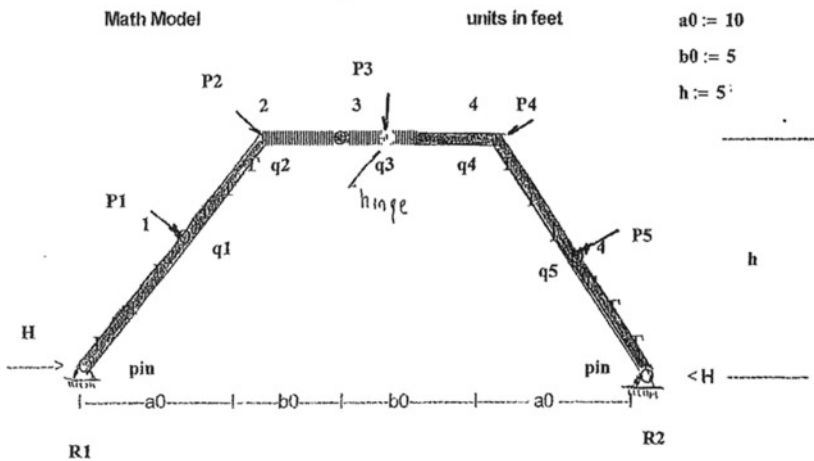
$R1$ and $R2$—vertical reactions—H horizontal reaction at base

External Loads—$[P1, P2, P3, P4]$—applied normal to members (positive as shown in local coordinate system)

$$\text{Geometer} \qquad d := \sqrt{\left(\frac{a0}{2}\right)^2 + \left(\frac{h}{2}\right)^2} \qquad d = 5.59 \text{ ft}$$

$$L := (a0 + b0) \cdot 2 \qquad L = 30 \text{ ft}$$

$$s := \frac{h}{2 \cdot d} \qquad s = 0.447$$

$$c := \frac{a0}{2 \cdot d} \qquad c = 0.894$$

Using the equations of equilibrium, we can express the internal forces and moments in the frame in the form $q = [A]P$. Inverting the equilibrium matrix $[A]$ we have:

$$q = A^{-1} P$$

The ensuing analysis evaluates the $[A^{-1}]$ matrix directly.

First Column of Matrix $[A^{-1}]$

Equilibrium $\qquad P1 := 1$

$$\text{REACTIONS} \qquad R2 := \frac{P1 \cdot d}{L} \qquad\qquad R2 = 0.186$$

$$R1 := P1 \cdot c - R2 \qquad\qquad R1 = 0.708$$

$$H2 := \frac{R2 \cdot (a0 + b0)}{h} \qquad H2 = 0.559$$

$$H1 := H2 - P1 \cdot s \qquad\qquad H1 = 0.112$$

Internal Moments /Forces

$\qquad\qquad i := 0..4 \qquad\qquad j := 0..4$

$$q1 := R1 \cdot \frac{a0}{2} - H1 \cdot \frac{h}{2} \qquad q1 = 3.261 \qquad a11 := q1$$

$$q2 := R1 \cdot (a0) - P1 \cdot d - H1 \cdot h \qquad q2 = 0.932 \qquad a21 := q2$$

$$q3 := R1 - P1 \cdot c \qquad q3 = -0.186 \qquad a31 := q3$$

$$q4 := R2 \cdot a0 - H2 \cdot h \qquad q4 = -0.932 \qquad a41 := q4$$

$$q5 := R2 \cdot \frac{a0}{2} - H2 \cdot \frac{h}{2} \qquad q5 = -0.466 \qquad a51 := q5$$

2nd Column of Matrix $[A^{-1}]$

Equilibrium $\qquad P2 := 1$

REACTIONS $\quad R2 := \dfrac{P2 \cdot d \cdot 2}{L} \qquad\qquad R2 = 0.373$

$\qquad\qquad\qquad R1 := P2 \cdot c - R2 \qquad R1 = 0.522$

$\qquad\qquad\qquad H2 := \dfrac{R2 \cdot (a0 + b0)}{h} \qquad H2 = 1.118$

$\qquad\qquad\qquad H1 := H2 - P2 \cdot s \qquad H1 = 0.671$

Internal Moments /Forces $\qquad i := 0..4 \qquad j := 0..4$

$q1 := R1 \cdot \dfrac{a0}{2} - H1 \cdot \dfrac{h}{2} \qquad q1 = 0.932 \qquad a12 := q1$

$q2 := R1 \cdot (a0) - H1 \cdot h \qquad q2 = 1.863 \qquad a22 := q2$

$q3 := R1 - P2 \cdot c \qquad q3 = -0.373 \qquad a32 := q3$

$q4 := R2 \cdot a0 - H2 \cdot h \qquad q4 = -1.863 \qquad a42 := q4$

$q5 := R2 \cdot \dfrac{a0}{2} - H2 \cdot \dfrac{h}{2} \qquad q5 = -0.932 \qquad a52 := q5$

3rd Column of Matrix $\left[A^{-1} \right]$

Equilibrium $\quad P3 := 1$

REACTIONS $\quad R2 := \dfrac{P3 \cdot (a0 + b0)}{L} \qquad R2 = 0.5$

$\qquad\qquad\qquad R1 := P3 - R2 \qquad R1 = 0.5$

$\qquad\qquad\qquad H2 := \dfrac{R2 \cdot (a0 + b0)}{h} \qquad H2 = 1.5$

$\qquad\qquad\qquad H1 := H2 \qquad\qquad H1 = 1.5$

Internal Moments /Forces $\qquad i := 0..4 \qquad j := 0..4$

$q1 := R1 \cdot \dfrac{a0}{2} - H1 \cdot \dfrac{h}{2} \qquad q1 = -1.25 \qquad a13 := q1$

$q2 := R1 \cdot (a0) - H1 \cdot h \qquad q2 = -2.5 \qquad a23 := q2$

$q3 := R1 \qquad q3 = 0.5 \qquad a33 := q3$

$q4 := R2 \cdot a0 - H2 \cdot h \qquad q4 = -2.5 \qquad a43 := q4$

$q5 := R2 \cdot \dfrac{a0}{2} - H2 \cdot \dfrac{h}{2} \qquad q5 = -1.25 \qquad a53 := q5$

4th Column of Matrix $\left[A^{-1} \right]$

Equilibrium $\quad P4 := 1$

REACTIONS $\quad R1 := \dfrac{P4 \cdot (2 \cdot d)}{L} \qquad R1 = 0.373$

$\qquad\qquad\qquad R2 := P4 \cdot c - R1 \qquad R2 = 0.522$

$\qquad\qquad\qquad H1 := \dfrac{R1 \cdot (a0 + b0)}{h} \qquad H1 = 1.118$

$\qquad\qquad\qquad H2 := H1 - P4 \cdot s \qquad H2 = 0.671$

Internal Moments /Forces $i := 0..4$ $j := 0..4$

$q1 := R1 \cdot \dfrac{a0}{2} - H1 \cdot \dfrac{h}{2}$ $q1 = -0.932$ $a14 := q1$

$q2 := R1 \cdot (a0) - H1 \cdot h$ $q2 = -1.863$ $a24 := q2$

$q3 := R1$ $q3 = 0.373$ $a34 := q3$

$q4 := R2 \cdot a0 - H2 \cdot h$ $q4 = 1.863$ $a44 := q4$

$q5 := R2 \cdot \dfrac{a0}{2} - H2 \cdot \dfrac{h}{2}$ $q5 = 0.932$ $a54 := q5$

5th Column of Matrix $\left[A^{-1}\right]$

Equilibrium $P5 := 1$

REACTIONS $R1 := \dfrac{P5 \cdot (d)}{L}$ $R1 = 0.186$

$R2 := P5 \cdot c - R1$ $R2 = 0.708$

$H1 := \dfrac{R1 \cdot (a0 + b0)}{h}$ $H1 = 0.559$

$H2 := H1 - P5 \cdot s$ $H2 = 0.112$

Internal Moments /Forces $i := 0..4$ $j := 0..4$

$q1 := R1 \cdot \dfrac{a0}{2} - H1 \cdot \dfrac{h}{2}$ $q1 = -0.466$ $a15 := q1$

$q2 := R1 \cdot (a0) - H1 \cdot h$ $q2 = -0.932$ $a25 := q2$

$q3 := R1$ $q3 = 0.186$ $a35 := q3$

$q4 := R2 \cdot a0 - H2 \cdot h - P5 \cdot d$ $q4 = 0.932$ $a45 := q4$

$q5 := R2 \cdot \dfrac{a0}{2} - H2 \cdot \dfrac{h}{2}$ $q5 = 3.261$ $a55 := q5$

The internal forces and moments in the frame $\{q\}$ are related to the applied load vector $\{P\}$ as follows:

$$\{q\} = [\text{Ainv}]\{P\}$$

where:

$$\text{Ainv:} = \begin{pmatrix} a11\ a12\ a13\ a14\ a15 \\ a21\ a22\ a23\ a24\ a25 \\ a31\ a32\ a33\ a34\ a35 \\ a41\ a42\ a43\ a44\ a45 \\ a51\ a52\ a53\ a54\ a55 \end{pmatrix}$$

$$\text{Ainv:} = \begin{pmatrix} 3.261 & 0.932 & -1.25 & -0.932 & -0.466 \\ 0.932 & 1.863 & -2.5 & -1.863 & -0.932 \\ -0.186 & -0.373 & 0.5 & 0.373 & 0.186 \\ -0.932 & -1.863 & -2.5 & 1.863 & 0.932 \\ -0.466 & -0.932 & -1.25 & 0.932 & 3.261 \end{pmatrix}$$

Dead Load

Dead Load—concrete weighs 150 lbs/cubic foot

Stairways weight $\quad W_s := 2 \cdot d \cdot \dfrac{t}{12} \cdot \dfrac{12}{12} \cdot (150) \qquad W_s = 978.28 \text{ lbs}$

Platform Weight $\quad W_p := 2 \cdot b0 \cdot \dfrac{t}{12} \cdot \dfrac{12}{12} \cdot (150) \qquad W_p = 875 \text{ lbs}$

Total weight $\qquad W_{\text{tot}} := 2 \cdot W_s + W_p \qquad\qquad W_{\text{tot}} = 2.832 \times 10^3 \text{ lbs}$

Stairway weight per horizontal foot

$$ws := \frac{W_s}{a0} \qquad ws = 97.828 \frac{\text{lbs}}{\text{ft}}$$

Platform weight per horizontal foot

$$wp := \frac{W_p}{2 \cdot b0} \qquad wp = 87.5 \frac{\text{lbs}}{\text{ft}}$$

Use a distributed dead load of 100 lbs/horizontal ft across the stairway

LIVE LOAD—design for 100 lbs/sq horizontal foot

SERVICE LOAD (dead + live) = 200 lbs/hor ft $\qquad wser := 200$

Global Load Vector $P1x := 0 \quad P2x := 0 \quad P3x := 0 \quad P4x := 0 \quad P5x := 0$

Assume a one foot section

$$P1y := wser \cdot \left(\frac{a0}{2}\right)$$

$$P2y := wser \cdot \left(\frac{a0}{4} + \frac{b0}{2}\right) \quad P4y := P2y$$

$$P3y := wser \cdot (b0) \qquad P5y := P1y$$

Global Load Vector (lbs)

$$Pg := \begin{pmatrix} P1x \\ P1y \\ P2x \\ P2y \\ P3x \\ P3y \\ P4x \\ P4y \\ P5x \\ P5y \end{pmatrix} \qquad Pg =$$

	0
0	0
1	$1 \cdot 10^3$
2	0
3	$1 \cdot 10^3$
4	0
5	$1 \cdot 10^3$
6	0
7	$1 \cdot 10^3$
8	0
9	$1 \cdot 10^3$

Transformation Matrix

$c = 0.894$

$s = 0.447$

Convert- Global Loads to
Local Loads

$$T\eta := \begin{pmatrix} c & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -s & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & s \end{pmatrix}$$

Total Dead Load (1 ft of stairway) $W := P1y + P2y + P3y + P4y + P5y$

$W = 5 \times 10^3$ lbs

Transform Global Loads to Local Loads (normal and tangent)
LOCAL Coordinates

$$P\eta := T\eta \cdot Pg \quad P\eta =$$

	0
0	447.214
1	894.427
2	447.214
3	894.427
4	0
5	$1 \cdot 10^3$
6	894.427
7	447.214
8	894.427
9	447.214

$$Q\eta := \begin{pmatrix} P\eta_1 \\ P\eta_3 \\ P\eta_5 \\ P\eta_6 \\ P\eta_8 \end{pmatrix} \quad Q\eta = \begin{pmatrix} 894.427 \\ 894.427 \\ 1 \times 10^3 \\ 894.427 \\ 894.427 \end{pmatrix}$$

Compute unknown {q} vector/forces and moments in frame

$$q := \text{Ainv} \cdot Q\eta \quad q = \begin{pmatrix} 1.25 \times 10^3 \\ -2.5 \times 10^3 \\ 500 \\ -2.5 \times 10^3 \\ 1.25 \times 10^3 \end{pmatrix} \text{ ft lbs}$$

Maximum positive service moment is: $Mpos := q_0$ $Mpos = 1.25 \times 10^3$ ft $-$ lbs
Maximum negative service moment is: $Mneg := q_1$ $Mneg = -2.5 \times 10^3$ ft $-$ lbs
Compare with cracking moment $\dfrac{Mcr}{12} = 3.874 \times 10^3$ ft lbs

Under full dead + live loading ($wser = 200$ lbs/ft) the section could possibly crack. Cracking would occur at joints 2 and 4 (corner top of the stairway). The stress in the steel and concrete would satisfy the working stress criteria of ACI.

Check Ultimate Strength $Mu := \dfrac{\Phi \cdot Mn}{12000}$ $Mu = 8.325$ ft $-$ kips
Ultimate Factored Loads $wu := 1.4 \cdot 100 + 1.7 \cdot 100$ $wu = 310$ $\frac{\text{lbs}}{\text{ft}}$
                  $\frac{wu}{wser} = 1.55$
                  $Mneg \cdot \frac{wu}{wser} = -3.875 \times 10^3 < Mu$ [O.K]

Although one could perform additional code checks using partial loading conditions we return our attention to the following linear and 2nd order behavior:

(1) The Flexibility Matrix
(2) Linear Displacements

(3) The Mass Matrix
(4) Eigenvalues and Eigenvectors
(5) Geometric Stiffness Matrix
(6) 2nd order deflections, forces and moments

Deflections—The ensuing analysis will develop the following Flexibility Matrices
   $[Fn]$—Flexibility Matrix in Local Coordinates
   where: $F\eta = \text{Ainv}^T \cdot \alpha \cdot \text{Ainv}$
The $[\alpha]$ matrix
   There are 5 degrees of freedom in this local model $\{\delta 1 \to \delta 5\}$. There are 10 degrees of
freedom in the global model $\{(\Delta x1, \Delta y1) \to (\Delta x5, \Delta y5)\}$
   $i := 0..4$ $j := 0..4$ $\alpha_{i,j} := 0$
   $Ec := 3 \cdot 10^6$ psi $Icr = 61.606$ in^4 $Ig = 343$ in^4

converting to consistent units (lb $-$ ft^2) gives $EIcr := \dfrac{Ec \cdot Icr}{144}$ $EIcr = 1.283 \times 10^6$

Connectivity $EIg := \dfrac{Ec \cdot Ig}{144}$ $EIg = 7.146 \times 10^6$

From Node	To Node	Beam Elements————————[α] Matrix		
$i := 0$	$j := 1$	$\alpha_{i,i} := \dfrac{d}{3EIg} + \dfrac{d}{3 \cdot EIcr}$	$\alpha_{i,j} := \dfrac{d}{3 \cdot EIcr}$	$\alpha_{j,i} := \alpha_{i,j}$
$i := 1$	$j := 2$	$\alpha_{i,i} := \dfrac{d}{3EIcr} + \dfrac{d}{3 \cdot EIcr}$	$\alpha_{i,j} := 0$	$\alpha_{j,i} := \alpha_{i,j}$
$i := 2$	$j := 3$	$\alpha_{i,i} := 0$	$\alpha_{i,j} := 0$	$\alpha_{j,i} := \alpha_{i,j}$
$i := 3$	$j := 4$	$\alpha_{i,i} := \dfrac{d}{3EIcr} + \dfrac{d}{3 \cdot EIcr}$	$\alpha_{i,j} := \dfrac{d}{3 \cdot EIcr}$	$\alpha_{j,i} := \alpha_{i,j}$
$i := 4$		$\alpha_{i,i} := \dfrac{d}{3EIcr} + \dfrac{d}{3 \cdot EIg}$		

Ainv Matrix

$$\alpha = \begin{pmatrix} 1.713 \times 10^{-6} & 1.452 \times 10^{-6} & 0 & 0 & 0 \\ 1.452 \times 10^{-6} & 2.904 \times 10^{-6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.904 \times 10^{-6} & 1.452 \times 10^{-6} \\ 0 & 0 & 0 & 1.452 \times 10^{-6} & 1.713 \times 10^{-6} \end{pmatrix}$$

Transposing the Ainv Matrix gives:

$$\text{AinvT} := \text{Ainv}^T = \begin{pmatrix} 3.261 & 0.932 & -0.186 & -0.932 & -0.466 \\ 0.932 & 1.863 & -0.373 & -1.863 & -0.932 \\ -1.25 & -2.5 & 0.5 & -2.5 & -1.25 \\ -0.932 & -1.863 & 0.373 & 1.863 & 0.932 \\ -0.466 & -0.932 & 0.186 & 0.932 & 3.261 \end{pmatrix}$$

Local Flexibility Matrix $F\eta := \text{Ainv}^T \cdot \alpha \cdot \text{Ainv}$

Local Flexibility Matrix

$$F\eta = \begin{pmatrix} 3.371 \times 10^{-5} & 2.863 \times 10^{-5} & -1.613 \times 10^{-5} & -2.863 \times 10^{-5} & -2.033 \times 10^{-5} \\ 2.863 \times 10^{-5} & 3.322 \times 10^{-5} & 0 & -3.322 \times 10^{-5} & -2.863 \times 10^{-5} \\ -1.613 \times 10^{-5} & 0 & 5.98 \times 10^{-5} & 0 & -1.613 \times 10^{-5} \\ -2.863 \times 10^{-5} & -3.322 \times 10^{-5} & 0 & 3.322 \times 10^{-5} & 2.863 \times 10^{-5} \\ -2.033 \times 10^{-5} & -2.863 \times 10^{-5} & -1.613 \times 10^{-5} & 2.863 \times 10^{-5} & 3.371 \times 10^{-5} \end{pmatrix}$$

Local Loading applied normal to frame $\qquad Q\eta = \begin{pmatrix} 894.427 \\ 894.427 \\ 1 \times 10^3 \\ 894.427 \\ 894.427 \end{pmatrix}$

Displacements normal to frame $\qquad \delta\eta := F\eta \cdot Q\eta = \begin{pmatrix} -4.162 \times 10^{-3} \\ 0 \\ 0.031 \\ 0 \\ -4.162 \times 10^{-3} \end{pmatrix}$

Eigenvectors and Eigenvalues
Mode Shape and Frequency
Units- M mass—W(kips)/g (ft/s/s)

$\qquad$ K—kips/ft

$\qquad$ ω—rad/s

$\qquad$ x—eigenvector

$\qquad$ **Eigenvalue Problem** $\qquad\qquad \omega^2 \cdot M \cdot x = K \cdot x$

$\qquad$ multiplying by K^{-1} gives
$\qquad$ the eigenvalue problem: $\qquad\qquad F \cdot M \cdot x = \lambda^2 \cdot x$

$\qquad$ where $\qquad\qquad\qquad\qquad\qquad \omega^2 = \dfrac{1}{\lambda^2}$

$\qquad$ The frequency of the ith mode is: $\qquad f_i = \dfrac{1}{2\cdot\pi} \cdot \sqrt{\dfrac{1}{\lambda_i}}$

5 Degree of Freedom Frame Model—5 Translations

Displacement Vector: $\delta 0$ $\delta 1$ $\delta 2$ $\delta 3$ $\delta 4$

Mass Matrix $m0$ $m1$ $m2$ $m3$ $m4$

$g := 32.2$ Mass at Node 1—dead load $\Rightarrow$ 100 lbs/f

$$W1 := 100 \cdot \frac{a0}{2} \qquad\qquad m1 := \frac{W1}{g}$$

$$W2 := 100 \cdot \left(\frac{a0}{2} + \frac{b0}{2} \right) m2 := \frac{W2}{g}$$

$$W3 := 100 \cdot (b0) \qquad\qquad m3 := \frac{W3}{g}$$

$$W4 := 100 \cdot \left(\frac{a0}{2} + \frac{b0}{2} \right) m4 := \frac{W4}{g}$$

$$W5 := 100 \cdot \frac{a0}{2} \qquad\qquad m5 := \frac{W5}{g}$$

MASS MATRIX

$$M\eta := \begin{pmatrix} m1 & 0 & 0 & 0 & 0 \\ 0 & m2 & 0 & 0 & 0 \\ 0 & 0 & m3 & 0 & 0 \\ 0 & 0 & 0 & m4 & 0 \\ 0 & 0 & 0 & 0 & m5 \end{pmatrix}$$

$$M\eta = \begin{pmatrix} 15.528 & 0 & 0 & 0 & 0 \\ 0 & 23.292 & 0 & 0 & 0 \\ 0 & 0 & 15.528 & 0 & 0 \\ 0 & 0 & 0 & 23.292 & 0 \\ 0 & 0 & 0 & 0 & 15.528 \end{pmatrix}$$

Form the FM Matrix $FM := F\eta \cdot M\eta$

Eigenvectors and Eigenvalues of the FM Matrix $c := \text{eigenvals}(FM)$

<div align="center">

Eigenvalues   Frequency-cps

</div>

$$\omega_0 := \frac{1}{\sqrt{c_{00}}} \quad \omega_0 = 20.679 \quad f_0 := \frac{\omega_0}{2 \cdot \pi} \quad f_0 = 3.291$$

$$\omega_1 := \frac{1}{\sqrt{c_{01}}} \quad \omega_1 = 30.522 \quad f_1 := \frac{\omega_1}{2 \cdot \pi} \quad f_1 = 4.858$$

$$\omega_2 := \frac{1}{\sqrt{c_{02}}} \quad \omega_2 = 126.141 \quad f_2 := \frac{\omega_2}{2 \cdot \pi} \quad f_2 = 20.076$$

Eigenvectors: $v := \text{eigenvecs}(FM)$

$i := 0..4 \quad j := 0..4$

Modes $0 \Rightarrow 2$

$v_{i,0} =$

0.47
0.528
0
−0.528
−0.47

$v_{i,1} =$

0.268
0
−0.926
0
0.268

$v_{i,2} =$

0.654
0
0.379
$1.197 \cdot 10^{-15}$
0.654

Normalize Modes to unity

$$\Phi_{i,0} := \frac{v_{i,0}}{v_{1,0}} \quad \Phi_{i,1} := \frac{v_{i,1}}{v_{2,1}} \quad \Phi_{i,2} := \frac{v_{i,2}}{v_{0,2}}$$

Mode 1   Mode 2   Mode 3

$f_0 = 3.291$ cps $f_1 = 4.858$ cps $f_2 = 20.076$ cps

$\Phi_{i,0} =$

0.89
1
0
−1
−0.89

$\Phi_{i,1} =$

−0.289
0
1
0
−0.289

$\Phi_{i,2} =$

1
0
0.579
$1.83 \cdot 10^{-15}$
1

Mode Shape is shown below

MODE 1
3.3 hz

MODE 2
4.9 hz
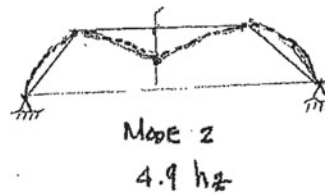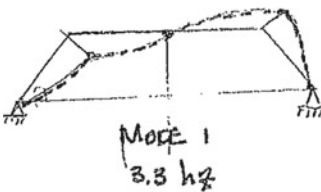
Buckling
The Elastic Buckling Load (Euler Load, Pe) can be computed from the eigenvalue problem

$$[K] \cdot \delta = \lambda \cdot [Kg] \cdot \delta$$

where the linear displacement ($\Delta = 0$) since ($Q = 0$) multiplying by $K^{-1} = F_\eta$ we have:

$\delta = \lambda \cdot F\eta \cdot [Kg] \cdot \delta$ solve the eigenvalue problem $\dfrac{1}{\lambda} \cdot \delta = F\eta \cdot Kg \cdot \delta$

Derivation of the Geometric Stiffness Matrix

Let the bucked shape of the Frame be described in local coordinates by a vector of displacements, $\{\delta\}$, and let the displacement at the ith node be δ_i equilibrium at the ith node can be written in terms of the applied axial load, N, and the buckled shape of the beam connecting node $(i-1)$ and node (i) **equilibrium @ the ith Node is given by the inner product**

$$W g_i = -\left[\frac{-1}{d_{i-1}}\left(\frac{1}{d_{i-1}} + \frac{1}{d_i}\right)\frac{-1}{d_{i+1}}\right] \cdot \begin{pmatrix} \delta_{i-1} \\ \delta_i \\ \delta_{i+1} \end{pmatrix}$$

The Geometric Stiffness Matrix (5×5—Local coordinates) is obtained by applying the above equation to the frame shown below:
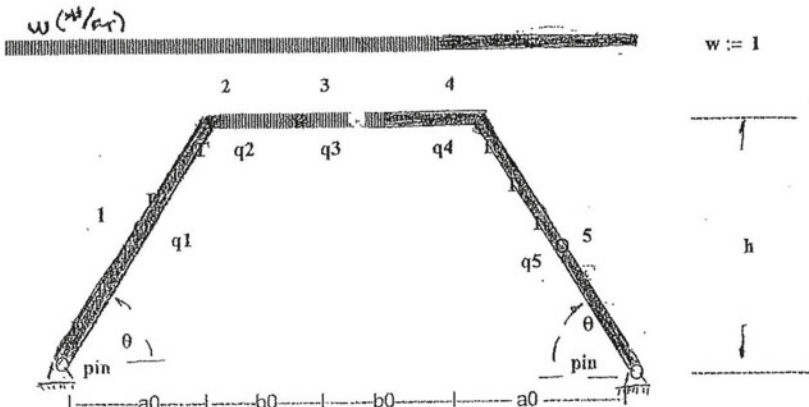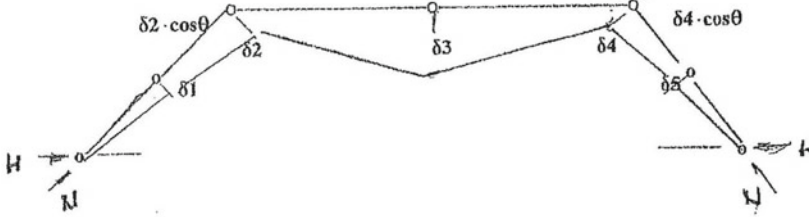

Frame fully Loaded with service load (deal + live = 200 lbs/ft). Compute the axial load in each member of the frame $wser = 200$

$$R2 := \frac{wser \cdot L \cdot \dfrac{L}{2}}{L} \qquad\qquad R2 = 3 \times 10^3 \text{ lbs}$$

$$R1 := R2 \qquad\qquad R1 = 3 \times 10^3 \text{ lbs}$$

Horizontal force $\quad\quad\quad H := \dfrac{wser \cdot \dfrac{L}{2} \cdot \dfrac{L}{4}}{h}$ $\quad\quad H = 4.5 \times 10^3$ lbs

Axial Load in stairway $\quad N := H \cdot \dfrac{a0}{2 \cdot d} + R1 \cdot \dfrac{h}{2 \cdot d}$ $\quad N = 5.367 \times 10^3$

Local Coordinate System

$$[Kg] := \begin{bmatrix} \dfrac{2}{d} \cdot N & \dfrac{-1}{d} \cdot N & 0 & 0 & 0 \\ \dfrac{-1}{d} \cdot N & \left[\dfrac{1}{d} \cdot N + \dfrac{1}{b0} \cdot H \cdot \left(\dfrac{a0}{2 \cdot d}\right)^2\right] & \dfrac{-1}{b0} \cdot H \cdot \left(\dfrac{a0}{2 \cdot d}\right) & 0 & 0 \\ 0 & \dfrac{-1}{b0} \cdot H \cdot \left(\dfrac{a0}{2 \cdot d}\right) & \dfrac{2}{b0} \cdot H & \dfrac{-1}{b0} \cdot H \cdot \left(\dfrac{a0}{2 \cdot d}\right) & 0 \\ 0 & 0 & \dfrac{-1}{b0} \cdot H \cdot \left(\dfrac{a0}{2 \cdot d}\right) & \left[\dfrac{1}{d} \cdot N + \dfrac{H}{b0} \cdot \left(\dfrac{a0}{2 \cdot d}\right)^2\right] & \dfrac{-1}{d} \cdot N \\ 0 & 0 & 0 & \dfrac{-1}{d} \cdot N & \dfrac{2}{d} \cdot N \end{bmatrix}$$

Eigenvalue Problem $\quad \dfrac{1}{\lambda} \cdot \delta = F\eta \cdot [Kg] \cdot \delta$

Note ($\lambda = 1$) since member axial loads, N and H, are included in the matrix $[Kg]$ a

Let: $\quad\quad\quad\quad\quad PE := F\eta \cdot [Kg]$

Eigenvalues $\quad\quad\quad c := \text{eigenvals}(PE)$

Eigenvectors $\quad\quad\quad v := \text{eigenvecs}(PE)$

Critical $\quad\quad\quad c = \begin{pmatrix} 0.126 \\ 0.098 \\ 7.697 \times 10^{-3} \\ 7.3 \times 10^{-3} \\ 0 \end{pmatrix}$ $\quad \lambda cr := \dfrac{1}{0.126} = 7.937$
Buckling
Load, λcr

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ factor of 8

Assuming a single stairway ramp buckles as a pinned column—the axial load in the ramp due to applied service load ($wser = 200$ lbs/ft) $N = 5367$ lbs

Estimate Euler $\quad \lambda 0 := \dfrac{\pi^2 \cdot EIcr}{(3 \cdot d)^2}$ $\quad\quad \dfrac{\lambda 0}{N} = 8.392$
Buckling Load $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ factor agrees $\rightarrow \lambda cr$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \lambda cr = 7.937$
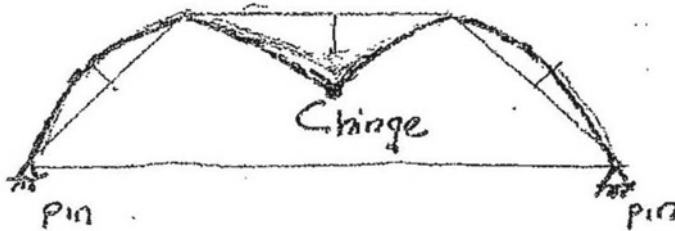
Symmetrical Buckling—Mode 1

Buckled Modes

$$v = \begin{pmatrix} 0.269 & 0.473 & 0.648 & 0.516 & 0.275 \\ 0 & 0.526 & -1.063 \times 10^{-14} & -0.483 & 0.55 \\ -0.925 & 0 & 0.401 & -8.474 \times 10^{-15} & 0.492 \\ 0 & -0.526 & 9.311 \times 10^{-15} & 0.483 & 0.55 \\ 0.269 & -0.473 & 0.648 & -0.516 & 0.275 \end{pmatrix}$$

Critical Buckling Mode—Mode 1

$$\frac{v_{i,0}}{v_{2,0}} =$$

symmetric mode		$\delta 1$—stairway leg buckles out
maximum node 3	-0.291	$\delta 2$
	0	$\delta 3$—platform buckles in
nodes 2 and 5 move out as	1	$\delta 4$
node 3 (hinge) moves down	0	$\delta 5$—stairway leg buckles out
	-0.291	

Buckled Mode Shape

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