Appendix A
Special Functions

Because some special functions are often involved in the fractional calculus research, to fully understand the knowledge of fractional calculus, here we introduce six related special functions. For more details, refer to Refs. [1, 2].

Gamma function

The Gamma function $\Gamma(z)$ is generally defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t \tag{A1.1}$$

which is applied in the right half of the complex plane $Re(z) > 0$ and guarantees the integral convergent at $t = 0$. This definition is also known as the second category Euler integration, which is often used in practical applications and can be further extended to the whole complex plane. Other forms of the definition (e.g. Euler's infinite series expressions, Weierstrass' infinite series, etc.) will be briefly presented in the last part of this section.

The sign of Gamma function $\Gamma(z)$ is used in most cases; in addition, there are two other signs $\Pi(z)$ and $z!$, which are both equal to $\Gamma(z+1)$, $z! = \Pi(z) = \Gamma(z+1)$. The sign $z!$ is normally used only in the case of a positive integer z, but is not restricted in this book. Thus, Eq. (A1.1) can be understood as the promotion of any real number z, non-integer and even complex. Figure A.1 shows the Gamma function graphics. It is easy to observe from Fig. A.1 that there are singularities of Gamma function when $z = 0, -1, -2, \ldots, -n, \ldots$.

1. **Basic Properties**

$\Gamma(z)$ satisfies the following recurrence relations:

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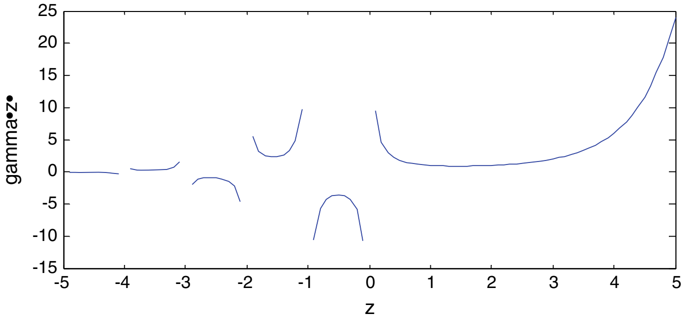

Fig. A.1 Gamma function

$$\Gamma(z+1) = z\Gamma(z) \tag{A1.2}$$

which can be easily proved by integrating by parts:

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z \mathrm{d}t = (-e^{-t}t^z)_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t = z\Gamma(z). \tag{A1.3}$$

If z is assumed to be a positive integer, formula (A1.1) can be generalized as

$$\Gamma(z+n) = (z+n-1)(z+n-2)\ldots(z+1)z\Gamma(z), \tag{A1.4}$$

or

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\ldots(z+n-1)} = \frac{1}{(z)_n}\int_0^\infty e^{-t}t^{z+n-1}\mathrm{d}t, \tag{A1.5}$$

where $(z)_n = z(z+1)\ldots(z+n-1)$.

Obviously, Eq. (A1.5) extends the definition of $\Gamma(z)$ to $Re(z) > -n$, where n is an arbitrary positive integer.

In Equation (A1.1), let $z = 1$, we have

$$\Gamma(1) = 0! = \int_0^\infty e^{-t}\mathrm{d}t = 1. \tag{A1.6}$$

In Eq. (A1.4), let $z = 1$, we have $\Gamma(n+1) = n! = n(n-1)\ldots 2\cdot 1$. This shows when z is a positive integer, $\Gamma(n+1)$ is the factorial $n!$.

2. Euler's Infinite Product Formula

According to the limit relation $e^{-t} = \lim\limits_{n\to\infty}(1-t/n)^n$, the Gamma function $\Gamma(z)$ can be expressed as the limit of the following integration:

$$\mathrm{P}_n(z) = \int\limits_0^n (1-t/n)^n t^{z-1}\mathrm{d}t, \tag{A1.7}$$

where the proof is omitted here. Let $t = n\tau$, integrating $\mathrm{P}_n(z)$ by parts for n times, thus,

$$\begin{aligned}
\mathrm{P}_n(z) &= n^z \int\limits_0^1 (1-\tau)^n \tau^{z-1}\mathrm{d}\tau \\
&= n^z\left[\frac{\tau^z}{z}(1-\tau)^n\right]_0^1 + \frac{n^z\cdot n}{z}\int\limits_0^1 (1-\tau)^{n-1}\tau^z\mathrm{d}\tau \\
&= \cdots = \frac{n^z n(n-1)\cdots 2\cdot 1}{z(z+1)\cdots(z+n-1)}\int\limits_0^1 \tau^{z+n-1}\mathrm{d}\tau \\
&= \frac{1\cdot 2\cdots n}{z(z+1)\cdots(z+n)}n^z,
\end{aligned} \tag{A1.8}$$

that is, $\Gamma(z) = \lim\limits_{n\to\infty}\frac{n!n^z}{z(z+1)\ldots(z+n)}$. Because $\lim\limits_{n\to\infty} n/(z+n) = 1$, this formula can be rewritten as

$$\Gamma(z) = \lim_{n\to\infty}\frac{(n-1)!n^z}{z(z+1)\ldots(z+n-1)}. \tag{A1.9}$$

And because n^z can be written as $n^z = \prod\limits_{m=1}^{n-1}\left(1+\frac{1}{m}\right)^z$, moreover,

$$\frac{(n-1)!}{z(z+1)\ldots(z+n-1)} = \frac{1}{z}\prod_{m=1}^{n-1}\left(1+\frac{z}{m}\right)^{-1}. \tag{A1.10}$$

In the end, we obtain another form of expression of the Gamma function, which is Euler's infinite series formula.

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^z\right\}. \tag{A1.11}$$

The Weierstrass infinite series can be expressed as

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-z/n}\right\}, \tag{A1.12}$$

where γ is the Euler constant

$$\gamma = \lim_{n\to\infty}\left\{\sum_{m=1}^{n}\frac{1}{m} - \ln n\right\} = 0.577\,215\ldots. \tag{A1.13}$$

For the specific derivation, refer to [1].

3. **Important Properties**

(1) When $z \to 0^+$, $\Gamma(z) \to +\infty$.
(2) Euler's reflection formula: $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$.
(3) $\Gamma(n+1/2) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}$.
(4) Multiplication theorem:

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z),$$

$$\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right)$$
$$= (2\pi)^{(m-1)/2}m^{1/2-mz}\Gamma(mz).$$

4. **Special value**

$$\Gamma\left(\frac{-3}{2}\right) = \frac{4\sqrt{\pi}}{3}, \Gamma\left(\frac{-1}{2}\right) = -2\sqrt{\pi},$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma(1) = 0! = 1,$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \Gamma(2) = 1,$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}, \Gamma(3) = 2!,$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}, \Gamma(4) = 3!\,.$$

Beta Function

Beta function, also known as the first-class Euler integration, is another special function defined as

$$B(z, w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1}\mathrm{d}\tau, \tag{A2.1}$$

where the above equation needs to satisfy the condition $Re(z) > 0$, $Re(w) > 0$.

Basic Properties

(1) Beta function is symmetrical, which can be proved by the variable transformation

$$B(z, w) = B(w, z). \tag{A2.2}$$

Beta function has many other forms, including

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{A2.3}$$

$$B(z, w) = 2\int_0^{\pi/2} (\sin\theta)^{2z-1}(\cos\theta)^{2w-1}\mathrm{d}\theta,\ Re(z) > 0,\ Re(w) > 0, \tag{A2.4}$$

$$B(z, w) = 2\int_0^{\infty} \frac{t^{z-1}}{(1+t)^{z+w}}\mathrm{d}t,\ Re(z) > 0,\ Re(w) > 0, \tag{A2.5}$$

$$B(z, w) = \sum_{n=0}^{\infty} \frac{\binom{n-w}{n}}{z+n}, \tag{A2.6}$$

$$B(z, w) = \prod_{n=0}^{\infty}\left(1 + \frac{zw}{n(z+w+n)}\right)^{-1}, \tag{A2.7}$$

$$B(z, w) \cdot B(z+w, 1-w) = \frac{\pi}{z\sin(\pi z)}, \tag{A2.8}$$

$$B(z, w) = \frac{1}{w}\sum_{n=0}^{\infty}(-1)^n \frac{y^{n+1}}{n!(z+n)}. \tag{A2.9}$$

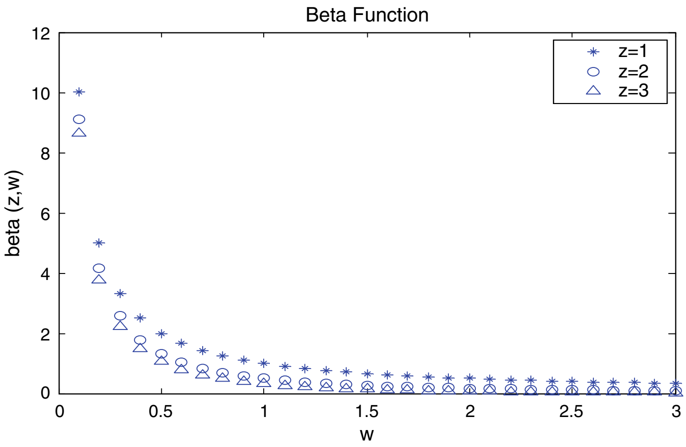

Fig. A.2 Beta function

(2) From the relation between the Beta function and Gamma function, it is easy to obtain the following equation:

$$B(z, 1 - z) = \Gamma(z)\Gamma(1 - z), \tag{A2.10}$$

$$B(z, z) = 2^{1-2z} B(z, 1/2). \tag{A2.11}$$

Figure A.2 shows the Beta function changes with variable w under three selected different constants z

Next, the specific proof of the relationship between the Beta function and Gamma function will be given in the following part.

Considering the following equation:

$$\Gamma(z)\Gamma(w) = \int_0^\infty e^{-u} u^{z-1} \mathrm{d}u \int_0^\infty e^{-v} v^{w-1} \mathrm{d}v, \tag{A2.12}$$

let $u = x^2$, $v = y^2$,

thus,

$$\begin{aligned} \Gamma(z)\Gamma(w) &= 4 \int_0^\infty e^{-x^2} x^{2z-1} \mathrm{d}x \int_0^\infty e^{-y^2} y^{2w-1} \mathrm{d}y, \\ &= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2z-1} y^{2w-1} \mathrm{d}x \mathrm{d}y. \end{aligned} \tag{A2.13}$$

Introducing plane polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, the above equation changes into the following form:

$$\Gamma(z)\Gamma(w) = 4\int_0^\infty e^{-r^2} r^{2(z+w)-1}\mathrm{d}r \int_0^{\pi/2} (\cos\theta)^{2z-1}(\sin\theta)^{2w-1}\mathrm{d}\theta. \tag{A2.14}$$

In the first integration, let $r^2 = t$, then

$$\int_0^\infty e^{-r^2} r^{2(z+w)-1}\mathrm{d}r = \frac{1}{2}\int_0^\infty e^{-t} t^{z+w-1}\mathrm{d}t = \frac{1}{2}\Gamma(z+w); \tag{A2.15}$$

in the second integration, let $\cos^2\theta = x$, then

$$\int_0^{\pi/2} (\cos\theta)^{2z-1}(\sin\theta)^{2w-1}\mathrm{d}\theta = \frac{1}{2}\int_0^1 x^{z-1}(1-x)^{w-1}\mathrm{d}x = \frac{1}{2}B(z,w). \tag{A2.16}$$

Substituting the above two equations into Eq. (A2.14), the relationship between the Gamma function and Beta function is obtained as $\Gamma(z)\Gamma(w) = \Gamma(z+w)B(z,w)$.

Dirac Delta Function

Dirac delta function is a special function widely used in the physical realm. It is a great help for the analysis of physical problems. In particular, it is often used in the analysis of problems, such as diffusion, seepage and wave. A brief description of the function will be given in the following part.

1. The expression of the Dirac delta function

$$\delta(x) = \begin{cases} \infty, x = 0 \\ 0, x \neq 0 \end{cases}. \tag{A3.1}$$

2. The properties of the Dirac delta function:
   (1) Integral property: $\int_{-\infty}^{+\infty}\delta(x)\mathrm{d}x = 1$
   (2) Fourier transform properties: $F(1) = \delta(x)$; $F^{-1}(\delta(x)) = 1$.
   (3) Limit Properties:

$$\lim_{\sigma\to 0}\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{x^2}{2\sigma}\right) = \delta(x), \tag{A3.2}$$

$$\lim_{\alpha\to\infty}\sqrt{\frac{\alpha}{\pi}}\exp\left(-\alpha x^2\right)=\delta(x), \tag{A3.3}$$

$$\lim_{\alpha\to\infty}\sqrt{\frac{\alpha}{\pi}e^{i\pi/4}e^{-i\alpha x^2}}=\delta(x), \tag{A3.4}$$

$$\lim_{\alpha\to\infty}\frac{\sin(\alpha x)}{\pi x}=\delta(x),\quad \lim_{\alpha\to\infty}\frac{\sin^2(\alpha x)}{\pi\alpha x^2}=\delta(x), \tag{A3.5}$$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}e^{-|x|/\varepsilon}=\delta(x), \tag{A3.6}$$

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{x^2+\varepsilon^2}=\pi\delta(x), \tag{A3.7}$$

$$\sqrt{i}=\exp\left(\frac{i\pi}{4}\right), \tag{A3.8}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(ikx)\mathrm{d}k=\delta(x). \tag{A3.9}$$

Proof $\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(ikx)\mathrm{d}k=\frac{\sin(\alpha x)}{\pi x}=\lim_{\alpha\to\infty}\frac{\sin(\alpha x)}{\pi x}=\delta(x).$

(4) The properties of the derivative:

Step function: $\theta(x)=\begin{cases}1 & x>0\\ 0 & x<0\end{cases}\quad \theta'(x)=\delta(x).$

$$\delta'(-x)=-\delta'(x),\quad \delta^{(n)}(-x)=(-1)^n\delta^{(n)}(x),$$

$$\frac{\mathrm{d}(\ln x)}{\mathrm{d}x}=\frac{1}{x}-i\pi\delta(x).$$

(5) Because $\delta(x)$ is an even function, there exists the following result:

$$\int_0^{\infty}\delta(x)\mathrm{d}x=\frac{1}{2}.$$

(6) Convolution Properties:

$$\delta(ax)=\frac{1}{|a|}\delta(x), \tag{A3.10}$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\mathrm{d}x = f(a), \tag{A3.11}$$

$$\delta(x-x') = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}2^n n!} e^{-(x^2+x'2)/2} H_m(x')H_m(x). \tag{A3.12}$$

Mittag-Leffler Function

For a long time, the Mittag-Leffler (M-L) function, especially the generalized (two-parameter) M-L function is not familiar to the public. In fact, Mathematics Subject Classification in 1991 even didn't include the introduction of the M-L function and related content, and the American Mathematical Society (AMS) classification forecast doesn't have its new entry (33E12) until 2000. In recent years, the generalized M-L function has been widely applied in the study of fractal dynamics, fractional anomalous diffusion and fractal random field [1, 3, 4] and coherent states in quantum field theory [5]. The application in these areas, in turn, promotes the development of the study of the function. For example, in the study of the theory of generalized fractional calculus, a recently developed multi-index Mittag-Leffler function has obtained a full use [6].

Exponential function e^z plays an important role in the integer-order differential equation; it can be written in the form of a series:$e^z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}$; it is a special case of the single-parameter Mittag-Leffler function. The function $E_\alpha(z)$ was proposed by G. M. Mittag-Leffler, and A. Wiman also did some research on this function. The generalized Mittag-Leffler function has a very important role in the fractional calculus, and it is derived from solving fractional differential equations using the Laplace transform by Humbert and Agarwal.

1. **The definition of the Mittag-Leffler function**

   (1) Single-parameter Mittag-Leffler function:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0. \tag{A4.1}$$

Figure A.3 shows the image of the Mittag-Leffler function of several special cases. The value of the parameter α is 0, 1, 2, 3, 4 and 5.

When $\alpha = 0$, $E_0(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.$x = 1$ is the singular point of the function. When $x \in [-50, 1)$, the function value is a positive number, and the function value increases with x. When $x \to 1^-$, function value tends to $+\infty$. When $x \in (1, 10]$, the function value is negative, and when $x \to 1^+$, the function value tends to $-\infty$. When $\alpha = 1$, M-L function is the exponential function e^x, so we can say that the exponential function is a special case of the M-L function. Taking other different

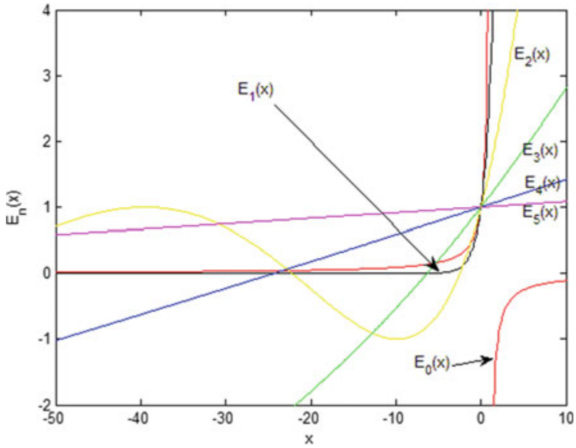

Fig. A.3 Single-parameter Mittag-Leffler function [29]

values of α, the function represented by the M-L function will be listed below:

$$\alpha = 2,\, E_2(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(2k+1)} = \cosh(\sqrt{x}),$$

$$\alpha = 3,\, E_3(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(3k+1)} = \frac{1}{3}\left[e^{x^{1/3}} + 2e^{-x^{13/2}/2}\cos\left(\frac{1}{2}\sqrt{3}z^{1/3}\right)\right],$$

$$\alpha = 4,\, E_4(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(4k+1)} = \frac{1}{2}\left[\cos\left(x^{1/4}\right) + \cosh\left(z^{1/4}\right)\right],$$

For single-parameter Mittag-Leffler function, let $z = -t^{\beta}$, thus,

$$E_{\beta}\left(-t^{\beta}\right) = \sum_{k=0}^{\infty} \frac{\left(-t^{\beta}\right)^k}{\Gamma(\beta k+1)}.$$

The function has the following limiting form:

When $\beta = 1$, the Mittag-Leffler function degrades to an exponential decay function e^{-t}.

When $0 < \beta < 1$, if $t \to 0$, the Mittag-Leffler function can be approximated by the extended exponential decay function: $\exp\left(-t^{\beta}/a\right)$, $a = \Gamma(\beta+1)$; if $t \to \infty$, the Mittag–Leffler function can be approximated by a power function: $bt^{-\beta}$, $b = \Gamma(\beta)\sin(\beta\pi)/\pi$ (see Fig. A.4).

(2) Two-parameter (generalized) Mittag-Leffler function [1]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \beta > 0. \tag{A4.2}$$

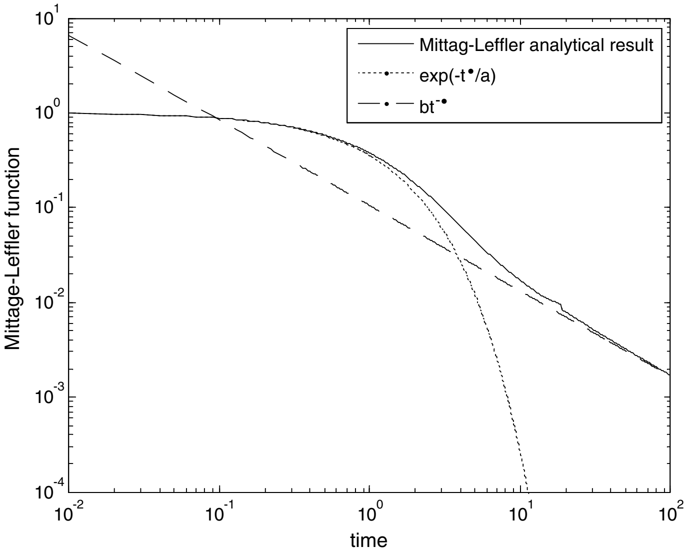

Fig. A.4 The approximation of the Mittag-Leffler function

From the definition of the generalized M-L function, it is not difficult to find that single-parameter M-L function is its special case (when $\beta = 1$); therefore, Fig A.3 can be regarded as a graphic of the generalized M-L function when $\beta = 1$.

Considering several special cases:

(1) $\alpha = 1, \beta = 1,$

$$E_{1,1}(z) = e^z.$$

(2) If $\alpha = \frac{1}{2}, \beta = 1,$

By definition, it can be obtained that

$$E_{1/2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k/2+1)} = e^{z^2} erfc(-z),$$

$$erfc(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2} \mathrm{d}t,$$

where $erfc()$ is the error function.

(3) When $\beta \neq 1$,

$$E_{1,2}(z) = \frac{e^z - 1}{z}.$$

(3) Generalized M-L function with changed parameters

Let $z = 1$ and $\beta = 1$, studying the function changes with parameter α. It is shown in Fig A.5, with the increase of parameter α, the value of the function at the same point reduces, i.e. the smaller α, the greater function value.

Let $z = 1$ and $\alpha = 1$, considering that the function changes with parameter β. It is shown in Fig A.6, for a fixed z, M-L function decreases with the increase in β. The decreasing rate is smaller than that in Fig A.5, and it decreases slowly at first, then increases with the increase in β.

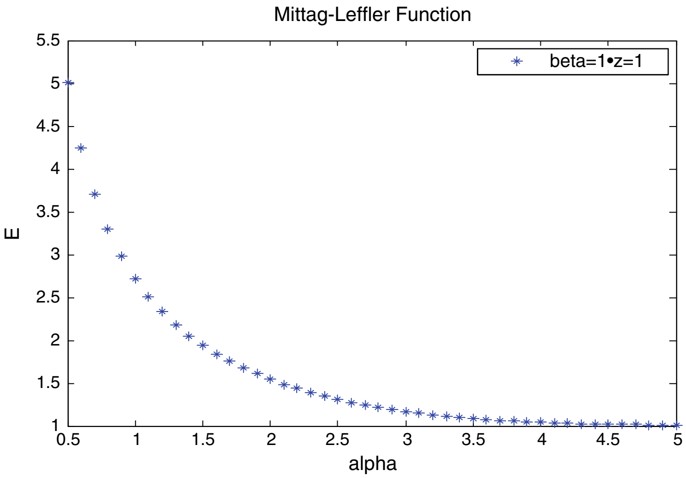

Fig A.5 When $\beta = 1$ and $z = 1$, M-L function changes with α

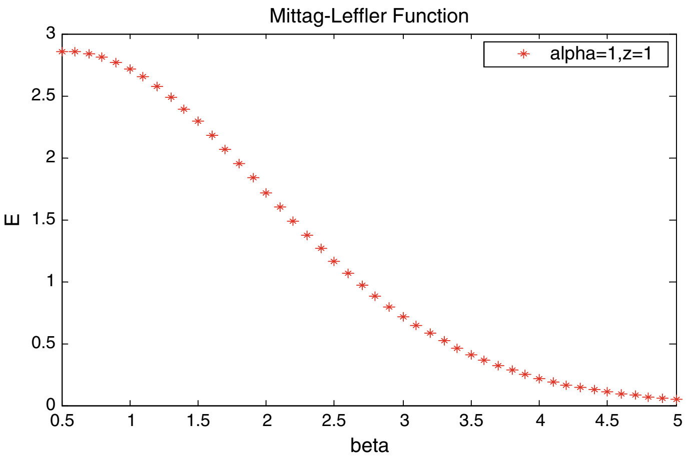

Fig A.6 When $\alpha = 1$ and $z = 1$, M-L function changes with β

(4) Some special functions

Some special functions are introduced below, which can all be expressed as the form containing the M-L function [1].

(1) The Miller–Ross function

$$\varepsilon_t(v, a) = t^v \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(v + k + 1)} = t^v E_{1,v+1}(at),$$

(2) The Rabotnov function

$$\ni_\alpha(\beta, t) = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((k + 1)(1 + \alpha))} = t^\alpha E_{\alpha+1,\alpha+1}\left(\beta t^{\alpha+1}\right),$$

(3) A class of fractional sine and cosine functions

$$Sc_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-\alpha)n+1}}{\Gamma((2 - \alpha)n + 2)} = zE_{2-\alpha,2}\left(-z^{2-\alpha}\right),$$

$$Cs_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-\alpha)n}}{\Gamma((2 - \alpha)n + 1)} = E_{2-\alpha,1}\left(-z^{2-\alpha}\right),$$

These two functions proposed by Plotnikov and Tseytlin are called "fractional sine and cosine functions".

(4) Another class of fractional sine and cosine functions

$$\sin_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\Gamma(2\mu k + 2\mu - \lambda + 1)} = zE_{2\mu,2\mu-\lambda+1}\left(-z^2\right),$$

$$\cos_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2\mu k + \mu - \lambda + 1)} = E_{2\mu,\mu-\lambda+1}\left(-z^2\right),$$

These two functions are proposed by Luchko and Srivastava; they also can be expressed in the form of M-L function.

(5) Double M-L function

$$\xi^{v,\sigma}_{\alpha,\beta,\lambda,\mu}(x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^{m+\frac{\beta(vn+1)-1}{\alpha}} y^{n+\frac{\mu(\sigma m+1)-1}{\lambda}}}{\Gamma(m\alpha + (vn + 1)\beta)\Gamma(n\lambda + (\sigma m + 1)\mu)},$$

This function is proposed by P. Hubert, P. Delerue and A. M Chak and further expanded by H. M Srivastava.

(5) M-L functions with multi-parameter

$$E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=k \\ l_1>0,\ldots,l_m>0}} \frac{(k;l_1,\ldots,l_m)\prod_{i=1}^{m} z_i^{l_i}}{\Gamma\left(\beta+\sum_{i=1}^{m}\alpha_i l_i\right)},$$

where $(k; l_1, \ldots, l_m)$ denotes polynomial coefficients. This function was originally proposed by Hadid and Luchko for solving fractional differential equations with linear constant coefficient.

2. The Laplace transform of the generalized Mittag-Leffler function

The Laplace transform of the generalized Mittag-Leffler function plays an important role in solving fractional differential equations. And the inverse Laplace transforms of this function are always applied to get the analytical solution of some simple fractional-order equations.

The derivation of the Laplace transforms of the generalized M-L function will be given in the following section.

Firstly, substituting $e^{\pm zt}$ in the integration $\int_0^\infty e^{-t}e^{\pm zt}\mathrm{d}t$ with the Mittag-Leffler function, then performing integral with respect to t, lastly, we have

$$\int_0^\infty e^{-t}e^{\pm zt}\mathrm{d}t = \int_0^\infty e^{-t}\left(\sum_{k=0}^{\infty}\frac{(\pm zt)^k}{k!}\right)\mathrm{d}t,$$
$$= \sum_{k=0}^{\infty}\frac{(\pm z)^k}{k!}\int_0^\infty e^{-t}t^k\mathrm{d}t,$$

According to the definition of Gamma function $\int_0^\infty e^{-t}t^k\mathrm{d}t = \Gamma(k+1) = k!$, thus

$$\int_0^\infty e^{-t}e^{\pm zt}\mathrm{d}t = \sum_{k=0}^{\infty}(\pm z)^k = \frac{1}{1\mp z}. \tag{A4.3}$$

Adding t^k into the above integral term and performing the above integration again, thus

$$\int_0^\infty e^{-t}t^k e^{\pm zt}\mathrm{d}t = \frac{k!}{(1\mp z)^{k+1}},\ (|z|<1). \tag{A4.4}$$

Making appropriate substitution to the equation, the Laplace transform of function $t^k e^{\pm at}$ is obtained as follows:

$$\int_0^\infty e^{-pt} t^k e^{\pm at} \mathrm{d}t = \frac{k!}{(p \mp a)^{k+1}}, (\mathrm{Re}(p) > |a|). \tag{A4.5}$$

According to Eqs. (A4.2) and (A1.1), the Laplace transform of generalized M-L function is considered as follows:

$$\begin{aligned} \int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\beta}(zt^\alpha)\mathrm{d}t &= \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)} \int_0^\infty e^{-t} t^{\alpha k+\beta-1} \mathrm{d}t \\ &= \frac{1}{1-z}, \quad (|z| < 1). \end{aligned} \tag{A4.6}$$

Then, making the same transform to Eq. (A4.5), the Laplace transform of function $t^{\alpha t+\beta-1} E^{(k)}_{\alpha,\beta}(\pm at^\alpha)$ is obtained as follows:

$$\int_0^\infty e^{-pt} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm at^\alpha)\mathrm{d}t = \frac{k!p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}, \left(Re(p) > |a|^{1/\alpha}\right), \tag{A4.7}$$

where $E^{(k)}_{\alpha,\beta}(y) = \frac{\mathrm{d}^k}{\mathrm{d}y^k} E_{\alpha,\beta}(y)$.

Simplified as

$$L\left\{t^{k\alpha+\beta-1} E^{(k)}_{\alpha,\beta}(\mp at^\alpha), p\right\} = \frac{k!p^{\alpha-\beta}}{\left(p^\alpha \pm a^{k+1}\right)}, Re(p) > |a|^{1/a}, \tag{A4.8}$$

where $E^{(k)}_{\alpha,\beta} = \frac{\mathrm{d}^k}{\mathrm{d}z^k} E_{\alpha,\beta}(z)$, the Ref. [7] has given a rigorous and simple proof of this formula.

In particular, if let $\alpha = \beta = \frac{1}{2}$, we can get

$$\int_0^\infty e^{-pt} t^{\frac{k-1}{2}} E^{(k)}_{\frac{1}{2},\frac{1}{2}}(\pm a\sqrt{t})\mathrm{d}t = \frac{k!}{(\sqrt{p} \mp a)^{k+1}}, \quad \left(Re(p) > |a|^2\right). \tag{A4.9}$$

It is noted here that Eq. (A4.9) is extremely useful for solving fractional derivative equation when the order of derivative equals 1/2.

3. **Derivative and integral of the M-L function**

(1) Derivative of the M-L function

For equations given in this section, the derivation process is no longer given, and only the conclusion is given, and the interested reader can try to derive it.

Because individually differentiating the M-L function is relatively complicated, the differential of the product of the M-L function and the power function of t is generally considered. Choosing the definition of the Riemann–Liouville fractional differential, then

$$\frac{\mathrm{d}^{\gamma}}{\mathrm{d}t^{\gamma}}\left(t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\lambda t^{\alpha})\right)=t^{\alpha k+\beta-\gamma-1}E_{\alpha,\beta-\gamma}^{(k)}(\lambda t^{\alpha}), \tag{A4.10}$$

Let $k=0, \lambda=1, \gamma$ is an integer and $m=\gamma$, thus Eq. (A4.10) can be rewritten as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\left(t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\right)=t^{\beta-m-1}E_{\alpha,\beta-m}(t^{\alpha}), (m=1,2,3,\ldots). \tag{A4.11}$$

Considering the following two cases of Eq. (A4.11):

(1) When $\alpha=\frac{m}{n}$, and m, n are natural numbers, thus

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\left(t^{\beta-1}E_{m/n,\beta}\left(t^{m/n}\right)\right)=t^{\beta-1}E_{m/n,\beta}\left(t^{m/n}\right)+t^{\beta-1}\sum_{k=1}^{n}\frac{t^{-\frac{m}{n}k}}{\Gamma\left(\beta-\frac{m}{n}k\right)}. \tag{A4.12}$$

If let $n=1$, then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\left(t^{\beta-1}E_{m,\beta}\left(t^{m}\right)\right)=t^{\beta-1}E_{m,\beta}\left(t^{m}\right)+t^{\beta-1}\frac{t^{-m}}{\Gamma(\beta-m)}. \tag{A4.13}$$

According to the property of Gamma function,

$$\frac{1}{\Gamma(-v)}=0, (v=0,1,2,\ldots).$$

Let $m=1,2,3,\ldots;\quad \beta=0,1,2,\ldots,m$, we can obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\left(t^{\beta-1}E_{m,\beta}\left(t^{m}\right)\right)=t^{\beta-1}E_{m,\beta}\left(t^{m}\right). \tag{A4.14}$$

Substituting $t=z^{n/m}$ into Eq. (A4.11), thus

$$\begin{aligned}&\left(\tfrac{m}{n}z^{1-\frac{n}{m}}\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^{m}\left(z^{(\beta-1)n/m}E_{m/n,\beta}(z)\right)\\&=z^{(\beta-1)n/m}E_{m/n,\beta}(z)+t^{(\beta-1)n/m}\textstyle\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma\left(\beta-\frac{m}{n}k\right)}\end{aligned}, (m,n=1,2,3,\ldots), \tag{A4.15}$$

and let $m=1$, we obtain

$$\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{(\beta-1)n}E_{1/n,\beta}(z)\right) = z^{(\beta n-1}E_{1/n,\beta}(z) + z^{\beta n-1}\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)}, (n = 1, 2, 3, \ldots). \tag{A4.16}$$

(2) The integral of M-L function

Performing integral itemized to the left side of the following equation, thus we obtain

$$\int_0^z E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}\mathrm{d}t = z^\beta E_{\alpha,\beta+1}(\lambda z^\alpha), \beta > 0, \tag{A4.17}$$

then considering the integral

$$\frac{1}{\Gamma(v)}\int_0^z (z-t)^{v-1}E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}\mathrm{d}t,$$

this integration is also relatively easy, and the integral can be solved as follows:

The above integral

$$\begin{aligned}
&\frac{1}{\Gamma(v)}\int_0^z (z-t)^{v-1}E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}\mathrm{d}t \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\frac{1}{\Gamma(v)}\int_0^z (z-t)^{v-1}t^{\alpha k+\beta-1}\mathrm{d}t \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\frac{1}{\Gamma(v)}\int_0^{\frac{\pi}{2}}\left(z - z\sin^2\theta\right)^{v-1}z\sin^2\theta^{\alpha k+\beta-1}\mathrm{d}\left(z\sin^2\theta\right) \\
&= z^{\beta+v-1}E_{\alpha,\beta+v}(\lambda z^\alpha). \quad \beta > 0, v > 0.
\end{aligned} \tag{A4.18}$$

As a result, some special integral equations are obtained:

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}e^{\lambda t}\mathrm{d}t = z^\alpha E_{1,\alpha+1}(\lambda z), \quad (\alpha > 0)). \tag{A4.19}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\cosh(\sqrt{\lambda}t)\mathrm{d}t = z^\alpha E_{2,\alpha+1}\left(\lambda z^2\right), \quad (\alpha > 0). \tag{A4.20}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\frac{\sinh(\sqrt{\lambda}t)}{\sqrt{\lambda}t}\mathrm{d}t = z^{\alpha+1}E_{2,\alpha+2}\left(\lambda z^2\right), \quad (\alpha > 0). \tag{A4.21}$$

The analytical solutions of fractional differential equations involve M-L function many times. Therefore, in order to compare the error relationship between the numerical solution and analytical solution, it is necessary to calculate the function values of M-L function. For this reason, many MATLAB Programs for calculating this function have been written by a lot of scholars until now, and readers can download them on the website [27, 28].

Wright Function

The Wright function plays an important role in solving linear fractional partial differential equations, such as the wave equation. There are some connections between this function and the generalized M-L function. This function was first proposed by British mathematician Wright [8], and a large number of useful equality relations are derived from the Laplace transform of fractional differential equations summarized by Humbert and Agarwal [9].

Series form definition of the Wright function:

$$W(z;\alpha,\beta) = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\alpha k+\beta)}, \tag{A5.1}$$

and Eq. (A5.1) can be written as the following integral form:

$$W(z;\alpha,\beta) = \frac{1}{2\pi i}\int_{Ha}\tau^{-\beta}e^{\tau+z t^{-\alpha}}\mathrm{d}\tau, \tag{A5.2}$$

The Properties of the Wright function [10, 11].

Property 1 *If* $\arg(-z)=\zeta$, $|\zeta|\le\pi$, *and*

$$Z_1 = (\alpha|z|)^{1/(\alpha+1)}e^{i(\zeta+\pi)/(\alpha+1)}, \quad Z_2 = (\alpha|z|)^{1/(\alpha+1)}e^{i(\zeta-\pi)/(\alpha+1)},$$

thus

$$W(z;\alpha,\beta) = H(Z_1) + H(Z_2), \tag{A5.3}$$

in which

$$H(Z) = Z^{1/2-\beta} e^{\{1+(1/\alpha)\}z} \left\{ \sum_{m=0}^{M} \frac{(-1)^m a_m}{Z^m} + O\left(\frac{1}{|Z|^{M+1}}\right) \right\}, Z \to \infty, \quad \text{(A5.4)}$$

where if the value m in Eq. (A5.4) is fixed, the value am can be calculated directly, e.g. $a_0 = (2\pi(\rho + 1))^{-1/2}$.

Property 2 *The relationship between the Wright Function and the Bessel function:*

$$J_v(z) = \left(\frac{z}{2}\right)^v W\left(-\frac{z^2}{4}; 1, v+1\right); \quad \text{(A5.5)}$$

$$I_v(z) = \left(\frac{z}{2}\right)^v W\left(\frac{z^2}{4}; 1, v+1\right). \quad \text{(A5.6)}$$

Property 3 *The relationship between the Wright function and the Mittag–Leffler function [1]:*

$$L\{W(t; \alpha, \beta); s\} = s^{-1} E_{\alpha,\beta}\left(s^{-1}\right). \quad \text{(A5.7)}$$

Property 4 *The relationship between the Wright function and the Meijer G-function.*

When α *is a rational number, and* $\alpha = p/q$*, the Wright function can be expressed with the Meijer G-function as follows:*

$$W(-z; \alpha, \beta) = (2\pi)^{(p-q)/2} q^{1/2} p^{-\beta+1/2} \times G_{0,p+q}^{q,0}\left[\begin{matrix} z^q \\ q^q p^p \end{matrix} \middle| \begin{matrix} - \\ 0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1-\frac{\beta}{p}, 1-\frac{1+\beta}{p}, \ldots, 1-\frac{p-1+\beta}{p} \end{matrix} \right]. \quad \text{(A5.8)}$$

Property 5 *The relationship between the Wright function and the Fox H-function.*

When ρ *is an arbitrary positive number, the Wright function is a special case of the Fox H-function [12–14]:*

$$W(-z; \alpha, \beta) = H_{0,2}^{1,0}\left[z \middle| \begin{matrix} - \\ (0,1), (1-\beta, \alpha) \end{matrix} \right]. \quad \text{(A5.9)}$$

In addition, the Generalized Wright function generally can be expressed as

$$W(z; (\mu, a), (v, b)) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a + \mu k)\Gamma(b + vk)} . \mu, v \in R, a, b \in C. \quad \text{(A5.10)}$$

H-Fox Function

H-Fox function is also called as the Fox function, H-function, the generalized Mellin–Barnes function or generalized Meijer's G-function in different papers. In order to unify and extend the existing results of the symmetric Fourier kernel, Fox has defined the H-function using the general Mellin–Barnes-type integral. It is widely used in the problems of statistics, physics and engineering to get the solution of fractional linear differential equations. It is necessary to note that almost all the special functions applied in the mathematical and statistical area are the special cases of H-Fox function. Even the complex functions such as the Mittag-Leffler function, Meijer's G-function [18], the Maitland generalized hypergeometric function and the Wright generalized Bessel functions are included. This section is compiled based primarily on the literature [2, 19–22].

H-Fox function based on the Mellin–Barnes-type integral is [14, 17, 23]

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z\Big|_{(b_q,\beta_q)}^{(a_p,\alpha_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_1,\beta_1),(b_2,\beta_2),\dots,(b_q,\beta_q)}^{(a_1,\alpha_1),(a_2,\alpha_2),\dots,(a_p,\alpha_p)}\right] \\ &= \frac{1}{2\pi i}\int\limits_L \chi(s)z^s \mathrm{d}s,\ z \neq 0, \end{aligned} \tag{A6.1}$$

in which integral density

$$\chi(s) = \frac{A(s)B(s)}{C(s)D(s)},$$

and

$$\begin{aligned} A(s) &= \prod_{j=1}^{m}\Gamma\left(b_j - \beta_j s\right), \quad B(s) = \prod_{j=1}^{n}\Gamma\left(1 - a_j + \alpha_j s\right), \\ C(s) &= \prod_{j=m+1}^{q}\Gamma\left(1 - b_j + \beta_j s\right), D(s) = \prod_{j=n+1}^{p}\Gamma\left(a_j - \alpha_j s\right), \end{aligned} \tag{A6.2}$$

where m, n, p and q are non-negative integers, which satisfy $0 \leq n \leq p$, $1 \leq m \leq q$. When n =0, $B(s) = 1$; $q = m$, $C(s) = 1$; $p = n$, $D(s) = 1$. The parameters $a_j (j = 1, 2, \dots, p)$ and $b_j (j = 1, 2, \dots, q)$ are complex numbers, and $\alpha_j (j = 1, 2, \dots, p)$ and $\beta_j (j = 1, 2, \dots, q)$ are positive numbers. These parameters satisfy the conditions:

$$P(A) \cap P(B) = \varnothing, \tag{A6.3}$$

where

$$P(A) = \left\{ s = \frac{b_j + k}{\beta_j} | j = 1, \dots, m; k = 0, 1, 2, \dots \right\},$$

$$P(B) = \left\{ s = \frac{a_j - 1 - k}{\alpha_j} | j = 1, \dots, n; k = 0, 1, 2, \dots \right\},$$

are the sets of poles $A(s)$ and $B(s)$, respectively. The integral path L is from $s = c - i\infty$ to $s = c + i\infty$ and makes the poles set separate. Then the points in $A(s)$ locate in the right of L, and the points in $B(s)$ locate in the left of L. Equation (A6.3) can also be written as $\alpha_j(b_h + \upsilon) \neq \beta_h(a_j - \lambda - 1)$, $(\upsilon, \lambda = 0, 1, 2, \dots; h = 1, \dots, m; j = 1, \dots, n)$. Note that the path integral (A6.1) is the inverse Mellin transform of $\chi(s)$.

H-Fox function has the following important properties [15, 17, 24]:

Property 1 H-Fox function has the property of permutation symmetry with respect to $(a_1, \alpha_1), \dots, (a_n, \alpha_n)$, $(a_{n+1}, \alpha_{n+1}), \dots, (a_p, \alpha_p)$,$(b_1, \beta_1), \dots, (b_m, \beta_m)$ and $(b_{m+1}, \beta_{m+1}), \dots, (b_q, \beta_q)$.

Property 2 *If an element of the array in $(a_j, \alpha_j)(j = 1, 2, \dots, n)$ equals an element of the array in $(b_j, \beta_j)(j = m + 1, m + 2, \dots, q)$ [or an element of the array in $(b_j, \beta_j)(j = 1, 2, \dots, m)$ equals an element of the array in $(a_j, \alpha_j)(j = n + 1, n+2, \dots, p)$], the H-Fox function can be simplified to a low-level H-Fox function, namely subtract 1 from p, q and n (or m), respectively.*

Therefore, there is the simplification formula:

$$H_{p,q}^{m,n}\left[z|_{(b_1,\beta_1),(b_2,\beta_2),\dots,(b_{q-1},\beta_{q-1}),(a_1,\alpha_1)}^{(a_1,\alpha_1),(a_2,\alpha_2),\dots,(a_p,\alpha_p)} \right] = H_{p-1,q-1}^{m,n-1}\left[z| \begin{matrix} (a_2, \alpha_2), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_{q-1}, \beta_{q-1}) \end{matrix} \right], \quad (A6.4)$$

where $n \geq 1, q > m$.

Property 3

$$H_{p,q}^{m,n}\left[z|_{(b_j,\beta_j)}^{(a_j,\alpha_j)} \right] = H_{q,p}^{n,m}\left[\frac{1}{Z}|_{(1-a_j,\alpha_j)}^{(1-b_j,\beta_j)} \right] \quad (A6.5)$$

According to this nature, we can rewrite the Fox function under the condition of $\mu = \sum_{j=1}^{g} \beta_j - \sum_{j=1}^{p} \alpha_j < 0$ into another Fox function satisfying $\mu > 0$.

Property 4

$$H_{p,q}^{m,n}\left[z|_{(b_q,\beta_q)}^{(a_p,\alpha_p)} \right] = kH_{q,p}^{n,m}\left[Z^k|_{(b_q,k\beta_q)}^{(a_p,k\alpha_p)} \right], \quad (A6.6)$$

where $k > 0$.

Property 5

$$z^{\sigma} H_{p,q}^{m,n}\left[z\Big|_{(b_q,\beta_q)}^{(a_p,\alpha_p)}\right] = H_{q,p}^{n,m}\left[z\Big|_{(b_q+\sigma\beta_q,\beta_q)}^{(a_{p+\sigma\alpha_p},\alpha_p)}\right]. \tag{A6.7}$$

In order to discuss the analytic properties of the H-Fox function and asymptotic expansion, define the following symbols:

$$\mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j; \tag{A6.8}$$

$$\alpha = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j; \tag{A6.9}$$

$$\beta = \prod_{j=1}^{p} \alpha_j^{\alpha_j} \prod_{j=1}^{q} \beta_j^{-\beta_j}; \tag{A6.10}$$

$$\gamma = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}; \tag{A6.11}$$

$$\lambda = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j; \tag{A6.12}$$

$$\delta = \left(\sum_{j=1}^{m} \beta_j - \sum_{j=n+1}^{p} \alpha_j\right)\pi. \tag{A6.13}$$

H-Fox function is the analytic function of z and meaningful if the following existence conditions are met [15, 17, 24, 25]:

Situation 1 If $\mu > 0, \quad z \neq 0$.
Situation 2 If $\mu = 0, \quad 0 < |z| < \beta^{-1}$.

Then generally, the H-Fox function is multivalued; however, it is single-valued in the Riemann surface of log z and can be obtained as follows:

$$H_{p,q}^{m,n}(z) = -\sum_{s \in P(A)} \mathrm{Res}\left(\frac{A(s)B(s)}{C(s)D(s)} z^s\right). \tag{A6.14}$$

When the pole of the function $\prod_{j=1}^{m} \Gamma\left(b_j - \beta_j s\right)$ is a single pole, i.e. when $j \neq h$; $j, h = 1, 2, \ldots, m$; $\lambda, \upsilon = 0, 1, 2, \ldots$, $\beta_h\left(b_j + \lambda\right) \neq \beta_j(b_h + \upsilon)$, we get the H-Fox function as follows:

$$H_{p,q}^{m,n}(z) = \sum_{h=1}^{m}\sum_{v=0}^{\infty} \frac{\prod_{j=1,j\neq h}^{m}\Gamma\left(b_j - \beta_j(b_h+v)/\beta_h\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\beta_j(b_h+v)/\beta_h\right)}$$
$$\times \frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j(b_h+v)/\beta_h\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\alpha_j(b_h+v)/\beta_h\right)} \times \frac{(-1)^v z^{(b_h+v)/\beta_i}}{v!\beta_h}. \quad \text{(A6.15)}$$

Braaksma pointed out [17, 24, 26]

$$H_{p,q}^{m,n}(z) = \mathrm{O}\left(|z|^c\right), z \le 1, \quad \text{(A6.16)}$$

where $\mu \ge 0, c = \min \Re\left(b_j/B_j\right)(j = 1, 2, \ldots, m)$;

$$H_{p,q}^{m,n}(z) = \mathrm{O}\left(|z|^d\right), z \ge 1, \quad \text{(A6.17)}$$

in which $\mu \ge 0, \alpha > 0, |\arg z| < \alpha\pi/2, d = \max \Re\left(\frac{a_j-1}{\alpha_j}\right)(j = 1, 2, \ldots, n)$.

Especially, if $\lambda > 0$, $|\arg z| < \lambda\pi/2$ and $\mu > 0$, then when $n = 0$, for the bigger z, H-Fox function tends to 0 exponentially, thus

$$H_{p,q}^{m,0}(z) \to \mathrm{O}\left\{\exp\left(-\mu z^{1/\mu}\beta^{1/\mu}\right)z^{(\gamma+1/2)/\mu}\right\}. \quad \text{(A6.18)}$$

If $n > 0, \delta > \mu\pi/2$, when $|z| \to \infty$, we get the H-Fox function at every closed subspace of $|\arg z| < \delta - \pi\mu/2$ as follows:

$$H_{p,q}^{m,n}(z) \to \sum_{s\in P(-1)} \mathrm{Re}\, s\left(\frac{A(s)B(s)}{C(s)D(s)}z^s\right). \quad \text{(A6.19)}$$

In addition, if ω and η are both complex numbers, $\omega \neq 0$ and $\eta \neq 0$, $\mu > 0$ (μ is defined in Eq. (A6.8)), thus we get

$$H_{p,q}^{m,n}\left(\eta\omega\Big|_{(b_q,\beta_q)}^{a_p,\alpha_p}\right) = \eta^{b_q/\beta_q}\sum_{r=0}^{\infty}\frac{\left(\eta^{1/\beta_q}-1\right)^r}{r!}$$
$$\times H_{p,q}^{m,n}\left(\omega\Big|_{(b_1,\beta_1),\ldots,\left(b_{q-1},\beta_{q-1}\right),\left(r+b_q,\beta_q\right)}^{(a_1,\alpha_1),\ldots,\left(a_p,\alpha_p\right)}\right), \quad \text{(A6.20)}$$

where $q > m$, $\left|\eta^{1/\beta_q-1}\right| < 1$, $\arg(\eta\omega) = \beta_q \arg\left(\eta^{1/\beta_q}\right) + \arg(\omega)$, $\left|\arg\left(\eta^{1/\beta_q}\right)\right| < \pi/2$.

Some special cases of the Fox functions are discussed below. When $\alpha_j = 1(j = 1, 2, \ldots, p)$, $\beta_j = 1(j = 1, 2, \ldots, q)$, H-Fox function reduces to a Meijer G-function [2,18]:

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,1),\ldots,\left(b_q,1\right)}^{(a_1,1),\ldots,\left(a_p,1\right)}\right] = G_{p,q}^{m,n}\left[z\Big|_{b_1,\ldots,b_q}^{a_q,\ldots,a_p}\right]. \quad \text{(A6.21)}$$

If adding other conditions $m = 1$ and $p \leq q$, the Fox function can be expressed as a generalized hypergeometric function ${}_pF_q$ as follows:

$$H_{p,q}^{m,n}\left[Z\Big|_{(b_1,1),\ldots,(b_q,1)}^{(a_1,1),\ldots,(a_p,1)}\right] = \frac{\prod_{j=1}^{n}\Gamma\left(1+b_1-a_j\right)z^{b_1}}{\prod_{j=2}^{q}\Gamma\left(1+b_1-a_j\right)z^{b_1}} \times_{\mathrm{p}} F_{q-1}\left(\begin{matrix}1+b_1-a_1,\ldots,1+b_1-a_p\\1+b_1-b_2,\ldots,1+b_1-b_q\end{matrix};(-1)^{p-n-1}z\right). \tag{A6.22}$$

Many of the so-called special functions, such as the error function, the Bessel functions, the Whittaker functions, the Jacobi polynomials and elliptic integrals, are special cases of the generalized hypergeometric function.

An important H-Fox function not included in the G-function class is shown as follows:

$$H_{p,q+1}^{1,p}\left[Z\Big|_{(0,1),(1-b_1,\beta_1)\ldots,(1-b_q,\beta_q)}^{(1-a_1,a_1),\quad\ldots,(1-a_p,a_p)}\right] = \sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j r\right)}{\prod_{j=2}^{g}\Gamma\left(b_j+\beta_j r\right)}\times\frac{(-z)^r}{r!} =_p \Psi_q\left(\begin{matrix}(a_1,\alpha_1),\ldots,\left(a_p,\alpha_p\right)\\(b_1,\beta_1),\ldots,\left(b_q,\beta_q\right)\end{matrix};-z\right), \tag{A6.23}$$

where ${}_p\Psi_q(z)$ is called the Maitland generalized hypergeometric function. A special case in Eq. (A6.23) gives the relationship between the H-fox function and the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ as follows:

$$H_{1,2}^{1,1}\left(z\Big|_{(0,1),(1-\beta,\alpha)}^{(0,1)}\right) = E_{\alpha,\beta}(-z). \tag{A6.24}$$

References

1. I. Podlubny, *Fractional Differential Equations* [M] (Academic Press, San Diego, 1999)
2. C.X. Wang, D.R. Guo, *Special Functions* [M] (Peking University Press, Beijing, 2000)
3. V.V. Anh, N.N. Leomenko. Scaling laws for fractional diffusion-wave equations with singular data [J]. Stat. Prob. Lett. **48**(3), 239–252 (2000)
4. Y.Q. Liu, *The Solution of Fractional Anomalous Diffusion Equation* [D] (Shandong University, 2006)
5. J.M. Sixdeniers, K.A. Penson, A.I. Solomon, Mittag-Leffler coherent states [J]. J. Phys. A Math. General **32**, 7543–7563 (1999)
6. V.S. Kiryakova, Multiple Mittag-Leffler functions and relations to generalized fractional calculus [J]. J. Comput. Appl. Math. **118**(1–2), 241–259 (2000)

7. M.Y. Xu, W.C Tan, Theoretical analysis of the generalized second-order fluid fractional anomalous diffusion velocity field, stress field and vortex layer [J]. Sci. China (Ser. A) **31**(7), 626–638 (2001)
8. E.M. Wright, On the coefficients of power series having exponential singularities [J]. J. Lond. Math. Soc. **8**, 71–79 (1933)
9. P. Humbert, R.P. Agarwal, Sur la function de Mittag-Leffler et quelques-unes de ses generalizations [J]. Bulletin des Sciences Mathématiques **77**(10), 180–185 (1953)
10. R. Gorenflo, Y. Luchko, F. Mainardi, Analytical properties and applications of the Wright function [J]. arXiv preprint math-ph/0701069 (2007)
11. R. Gorenflo, Y. Luchko, F. Mainardi, Wright function as scale-invariant solutions of the diffusion-wave equation [J]. J. Comput. Appl. Math. **118**, 175–191 (2000)
12. R. Gorenflo, F. Mainardi, H.M. Srivastava, Special functions in fractional relaxation–oscillation and fractional diffusion-wave phenomena [C], in: Proceedings VIII International Colloquium on Differential Equations, ed. By D. Bainov, Plovdiv 1997, VSP, Utrecht, 1998, pp. 195–202
13. V. Kiryakova, *Generalized Fractional Calculus and Applications* [M] (Longman, Harlow, 1994)
14. H.M. Srivastava, K.C. Gupta, S.P. Goyal, *The H-Functions of One and Two Variables with Applications* [M] (South Asian Publishers, New Delhi, 1982)
15. R. Hilfer, *Applications of Fractional Calculus in Physics* [M] (World Scientific Press, Singapore, 2000)
16. B.D. Carter, M.D. Springer, The distribution of products quotients and powers of independent H-function variates [J]. SIAM J. Appl. Math. **33**(4), 542–558 (1977)
17. A.M. Mathai, R.K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines* [M] (Wiley Eastern Limited, New Delhi, 1978)
18. E.W. Weisstein, *CRC Concise Encyclopedia of Mathematics CD-ROM* [M]. (CRC Press, Boca Raton, FL, 1999)
19. F. Mainardi, G. Pagnini, R.K. Saxena, Fox H functions in fractional diffusion [J]. J. Comput. Appl. Math. **178**, 321–331 (2005)
20. J.G. Liu, *Some Applications of Fractional Calculus to the Constitutive Equations of Viscoelastic Materials* [D] (Shandong University, 2006), pp. 19–24
21. X.Y. Guo, *Some Applications of Fractional Calculus to the Researches on Quantum Mechanics and Non-Newtonian Fluid Mechanics* [D] (Shandong University, 2007) pp. 18–21
22. S.W. Wang, *Some Applications of the Theory of Fractional Calculus in the Viscoelastic Fluid Mechanics and Quantum Mechanics* [D] (Shandong University, 2007), pp. 17–21
23. W.G. GlÖckle, T.F. Nonnenmacher, Fractional integral operators and Fox functions in the theory of viscoelasticity [J]. Macromolecules **24**, 6426–6434 (1991)
24. W. Wyss, The fractional diffusion equation [J]. J. Math. Phys. **27**(11), 2782–2785 (1986)

25. W.R. Schneider, W. Wyss, Fractional diffusion and wave equations [J]. J. Math. Phys. **30**(1), 134–144 (1989)
26. B.L.J. Braaksma, Asymtotic expansions and analytic continuations for a class of Barnes-integrals [J]. Compos. Math. **15**, 239–341 (1964)
27. http://www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-function
28. http://www.mathworks.com/matlabcentral/fileexchange/21454-generalized-generalized-mittag-leffler-function
29. http://mathworld.wolfram.com/Mittag-LefflerFunction.html25.

Appendix B
Related Electronic Resources of Fractional Dynamics

Web Resources

1. Power-law phenomenon and Fractional dynamic system (http://www.ismm.ac.cn/ismmlink/PLFD/index.html).
2. Center for Self-Organizing and Intelligent Systems (http://www.csois.usu.edu).
3. Fractional calculus in Utah State University (http://www.mechatronics.ece.usu.edu/foc).
4. Institute of Soft Matter Mechanics (http://www.ismm.ac.cn/).
5. Group of Robotics and Intelligent Systems (http://www.ave.dee.isep.ipp.pt/~gris/index.htm).
6. Fractional calculus modeling (http://www.fracalmo.org/).
7. Jordan Research Group in Applied Mathematics (JRGAM) (http://www.mutah.edu.jo/jrgam/index.html)
8. Equipe CRONE (http://www.ims-bordeaux.fr/IMS//pages/accueilEquipe.php?guidPage=les_equipes).

Professional Journals

1. Fractional Calculus & Applied Analysis (Fract. Calc. Appl. Anal.), ISSN 1311–0454 Website: http://www.math.bas.bg/~fcaa/

W. Chen et al., *Fractional Derivative Modeling in Mechanics and Engineering*,
https://doi.org/10.1007/978-981-16-8802-7

2. Journal of Fractional Calculus, ISSN 0918-5402.
3. Fractional Dynamic Systems Website: http://www.fds.ele-math.com/.

Open Source Codes

1. Program package on the Adams method and finite difference method by Kai. Diethelm

   (http://www-public.tu-bs.de:8080/~diethelm/software/software.html).
2. Predictor corrector method for solving the relaxation equation

   (http://www.mathworks.com/matlabcentral/fileexchange/26407-predictor-corrector-method-for-variable-order-random-order-fractional-relaxation-equ ation).
3. Matrix method for solving fractional partial differential equations

   (http://www.mathworks.com/matlabcentral/fx_files/22071/14/content/html/Matrix_Approach.html).
4. CRONE Toolbox (http://wWW.ims-bordeaux.fr/IMS//pages/pageAccueilP erso.php?email=alain.outloup)
5. Mittag-Leffler function curve

   (http://www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-function).
   (http://www.mathworks.com/matlabcentral/fileexchange/21454-genera lized-generalized-mittag-leffler-function)
6. The random number generator of Mittag–Leffler distribution

   (http://www.mathworks.com/matlabcentral/fileexchange/19392-mittag-lef fler-random-number-generator).
7. Fractional chaotic system

   (http://www.mathworks.com/matlabcentral/fileexchange/27336-fractional-order-chaotic-systems).
8. Impulse response invariant discretization of distributed-order low-pass filter

   (http://www.mathworks.com/matlabcentral/fileexchange/26868-impulse-res ponse-invariant-discretization-of-distributed-order-low-pass-filter).
9. Digital Fractional-Order Differentiator/integrator—FIR type

   (http://www.mathworks.com/matlabcentral/fileexchange/3673-digital-fracti onal-order-differentiatorintegrator-fir-type).
10. A New IIR-type Digital Fractional-order differentiator

    (http://www.mathworks.com/matlabcentral/fileexchange/3518-a-new-iir-type-digital-fractional-order-differentiator).
11. Variable-order derivative

(http://www.mathworks.com/matlabcentral/fileexchange/24444-variable-order-derivatives).

12. Fractional-order–differential-order equation solver

(http://www.mathworks.com/matlabcentral/fileexchange/13866-fractional-order-differential-order-equation-solver).

13. Fractional-order control

(http://www.mathworks.com/matlabcentral/fileexchange/8312-ninteger).

14. Part of the program code of Professor Mark M. Meerschaert

(http://www.stt.msu.edu/~mcubed/).

Key Words

Fractional calculus

Fractional derivative

Fractional differential equation

Anomalous diffusion

Power law

Frequency-dependent dissipation

Softer matter

Path dependency

Stable distribution

Fractional Brownian motion

Fractal

Fractal derivative

Variable-order derivative

Random-order derivative

Distributed-order derivative

Fractional Fourier transform

Stretched Gaussian distribution

Fractional variational principle

Time-fractional derivative

Spatial/space fractional derivative

Continuous-time random walk

The Relevant Pages of This Book

Owing to the limitation of our knowledge, although the book has been modified several times, surely there are many errors or improprieties. We urge readers of this book if you find any error or irregularity, please tell us your opinion by email, and we will further improve the book.

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References

1. J.T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus [J]. Commun. Nonlinear Sci. Numer. Simul. **16**(3), 1140–1153 (2011).
2. http://www.ismm.ac.cn/ismmlink/PLFD/index.html.