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Yuliya S. Mishura

Stochastic Calculus for Fractional Brownian Motion and Related Processes

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Yuliya S. Mishura
Department of Mechanics and Mathematics
Kyiv National Taras Shevchenko University
64 Volodymyrska
01033 Kyiv
Ukraine
myus@mail.univ.kiev.ua

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Preface

For several decades the semimartingale processes were the best model in order to implement many ideas. The stochastic calculus for semimartingales and the general theory of stochastic processes, which are closely connected to the theory of stochastic integration and stochastic differential equations, were originated by N. Wiener (Wie23), P. Lévy (Le48), K. Itô (Itô42), (Itô44), (Itô51), A.N. Kolmogorov (Kol31), W. Feller (Fel36), J.L. Doob, M. Loéve, I. Gikhman and A. Skorohod (the list of related papers and books is very long and we do not mention it here in full). Those ideas were developed further by several authors, among them there are K. Bichteler (Bi81), C.S. Chou, P.A. Meyer and C. Stricker (CMS80), K.L. Chung and R.J. Williams (ChW83), C. Dellacherie (Del72), C. Dellacherie and P.A. Meyer (DM82), C. Doléans-Dade and P.A. Meyer (DDM70), H. Föllmer (Fol81a), P.A. Meyer (Me76) and M. Yor (Yor76). These theoretical data were fruitfully discussed and summarized in the monographs of J. Jacod (Jac79), R. Elliott (Ell82), P.E. Kopp (Kop84), M. Métivier and J. Pellaumail (MP80), B. Øksendal (Oks03), P. Protter (Pro90). Limit theorems in the most general semimartingale framework were proved by J. Jacod and A.N. Shiryaev (JS87). A very convenient way to consider financial markets is to insert them into semimartingale models, as perfectly demonstrated by I. Karatzas and S. Shreve (KS98), A.N. Shiryaev (Shi99), F. Delbaen and W. Schachermayer (DS06). The Malliavin calculus for the Wiener process was presented in the books of P. Malliavin (Mal97) and D. Nualart (Nua95). However, in recent years the well-studied theory of semimartingales turns out to be insufficient in order to describe many phenomena. On one hand, telecommunication connections, asset prices and other objects have "long memory". This effect cannot be described with the help of such processes as the Wiener process, which has independent increments and has no memory. On the other hand, the concept of turbulence in hydrodynamics can be described by self-similar fields with stationary (dependent) increments (A.M. Yaglom (Yag57), A. Monin and A.M. Yaglom (MY67) and A.M. Yaglom (Yag87)).

A.N. Kolmogorov (Kol40) was the first to consider continuous Gaussian processes with stationary increments and with the self-similarity property; it means that for any $a > 0$ there exists $b > 0$ such that

$$Law(X(at);\ t \geq 0) = Law(bX(t);\ t \geq 0).$$

It turns out that such processes with zero mean have a special correlation function:

$$EX(t)X(s) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right),$$

where $0 < H < 1$. A.N. Kolmogorov called such Gaussian processes "Wiener Spirals" ("Wiener screw-lines"). Later, when the papers of H.E. Hurst (Hur51) and H.E. Hurst, R.P. Black and Y.M. Simaika (HBS65), devoted to long-term storage capacity in reservoirs, were published, the parameter H got the name "Hurst parameter". The stochastic calculus of the processes mentioned above originated with the pioneering work of B.B. Mandelbrot and J.W. van Ness (MvN68) who considered the integral moving average representation of X via the Wiener process on an infinite interval and called this process fractional Brownian motion (fBm). Note that B.B. Mandelbrot worked with fractional processes during a long period and his later results concerned the fractals and scaling were summarized in the book (Man97). Note also that it was proved in the paper (GK05) that the moving average representation of fBm is unique in the class of the right-continuous, nondecreasing concave functions on $\mathbb{R}_+$. The first result where fBm appeared as the limit in the Skorohod topology of stationary sums of random variables was obtained by M. Taqqu (Taq75); another scheme of convergence to fBm in the uniform topology was considered in (Gor77). Spectral properties of fBm were studied by G. Molchan (Mol69), G. Molchan and J. Golosov (MG69), G. Molchan (Mol03), and later by K. Dzhaparidze and H. van Zanten (DvZ05), (RLT95), (SL95).

The next intensive wave of interest in fBm arose in the 1990s. It can be explained by various applications of fBm and other long-memory processes in teletraffic, finances, climate and weather derivatives. The paper (DU95) was one of the first paper devoted to stochastic analysis for fBm. Note that fBm is neither a semimartingale (except the case $H = 1/2$ when it is a Brownian motion) nor a Markov process. However, it is closely connected with fractional calculus and can be represented as a "fractional integral" (with the help of a comparatively complicated hypergeometric kernel) via the Wiener process not only on infinite, but also on finite intervals. This was stated by I. Norros, E. Valkeila and J. Virtamo (NVV99) and C. Bender (Ben03a). Such a representation, together with the Gaussian property of fBm and the Hölder property of its trajectories (fBm with Hurst index H is Hölder up to order H) permits us to create an interesting and specific stochastic calculus for fBm. The development of the theory of long-memory processes moved in several directions: stochastic integration, stochastic differential equations, optimal filtering, financial applications, statistical inference, from one side (these topics create the main points of this book) and a lot of other theoretical problems and applications, from the other side. In our Preface we mention for the most part the papers that are not mentioned and used in the text of the book but play a very important role in the development of the theory of long-memory processes. For example, series, spectral and wavelet analysis for fBm was considered in (AS96), (ALP02a), (Mas93), (Mas96), (RZ91), (DvZ05), (DF02), (DvZ04), (SL95), (Mac81b); local times, the Tanaka formula, the law of the iterated logarithm, maximal properties and the Kallianpur–Robbins law for fBm and related processes were studied in (Ber69), (CNT01), (HO02), (HP04b), (Sin97), (HOS05), (GRV03), (KK97), (KM96), (KO99), (Ros87), (KM96), (Kono96), (Sh96), (ElN93), (Tal96) and (Taq77). Furthermore, stochastic evolution equations driven by fBm were investigated in the papers (AG03), (CD01), (MN03), (TTV03) and some methods of construction of fBm were proposed in (Yor88) and (Sai92).

R.J. Adler and G. Samorodnitsky (AS95) considered super processes connected to fBm. The Clark–Ocone theorem for fBm was established in (BE03) and (AOPU00); forward and symmetric integrals for fBm were constructed in (BO04), (CN02), (Zah02b) (note that the general theory of forward, backward and symmetric integrals was created by F. Russo and P. Vallois in (RV93), (RV95a), (RV95b), (RV98) and (RV00)).

Detection and prediction problems were discussed in the papers (BP88), (GN96), (Dun06); the stochastic maximum principle for a controlled process governed by an SDE involving fBm was proved in (BHOS02); stochastic Fubini theorem for fBm was studied in (KM00); time rescaling for fBm was investigated in (Mac81a); Hausdorff measure and packing dimension connected to fBm were considered in (Tal95), (TX96), (Xiao91), (Xiao96), (Xiao97a), (Xiao97b); estimation of the parameters of long-memory processes, in particular, the estimates of the Hurst parameter are presented in (Ber94), (BGK06), (BG96), (BG98), (GR03a). Markov properties of some functionals connected with an fBm were considered in (CC98).

Rough path analysis for fBm was studied in (CQ02) and some of its applications were considered in the manuscript (HN06); the properties of the Gaussian spaces generated by an fBm were established in (PT01); distribution of functionals connected with fBm was obtained in (CM96), (LN03), (ElN99) (Sin97), (Zha96), (Zha97); the Skorohod–Stratonovich integral for fBm was studied in (Dec01), (ALN01), (AMN01), (AN02); the properties of spectral exponent of fBm were established in (LP95); multi-parameter fractional Brownian fields were studied in (ENO02), (Kam96), (ALP02b), (Lind93), (Gol84), (KK99), (OZ01), (PT02a), (Tal95), (TV03), (TT03), (Tud03), (MisIl03), (MisIl04), (MisIl06), (Mur92); set-parametrized fractional Brownian fields have been studied in the papers (HM06a), (HM06b); asymptotic properties of two-dimensional fractional Brownian fields were considered in (BaNu06). The Malliavin calculus for fBm was developed in (Hu05), (Pri98), (Nua03), (Nua06); fBm in Hilbert space was constructed and investigated in (DPM02). The papers (HN04), (KLeB02), (AHL01), (ALN01), (CKM03) are devoted to stochastic fractional Ornstein–Uhlenbeck, Riesz–Bessel and Lévy type processes. An interesting formula of transformation of fBm with Hurst index H into fBm with index $1 - H$ was obtained in (Jost06). Mention also the papers (DU98), (Daye03) and forthcoming book (BHOZ07).

Note that fBm has a long-memory property only for $H \in (1/2, 1)$. In the case $H \in (0, 1/2)$ it is a process with short memory. The theory of such processes is quite different. FBm with $H \in (0, 1/2)$ was studied in (ALN01), (AMN00), (AI04) and (CN05); simulation of fBm and various applications of fBm were considered in (CM95), (CM96), (Nor95), (Yin96), (Dun00), (Dun01), (DF02), (Seb95) and (Sin94).

Fractional Brownian motion as a model of financial markets was proposed in a large number of papers. (See, for example, (AM06), (BE04), (BSV06), (BO02), (BH05), (Che01b), (Dun04), (EvH01), (EvH03), (Gap04), (HO03), (HOS03), (HOS05), (Rog97), (Sch99), (Shi01), (Sot01), (SV03), (WRL03), (WTT99), (Wyss00) and (Zah02a).) Financial markets with memory were considered in (AI05a), (AI05b), (INA07) and (IN07). Moreover, filtering and prediction problems were considered in (CD99), (INA06), (KKA98b), (LeB98), (KLeBR99), (KLeB99), (KLeBR00), (Dun06) and (GN96). In addition, some related applied problems were studied, e.g., in (MS99), (Nar98), (Nor95), (Nor97), (Nor99). An estimate of ruin probabilities for the models with the long-range dependence was studied in (Mis05), (HP04b). Statistical inferences for the processes related to fBm are a very extended area. The major contributions to this theory were made, among other authors, by M. Taqqu and P.M. Robinson. We mention here also the papers of P. Doukhan, A. Khezour and G. Lang (DKL03), L. Giraitis and P.M. Robinson (GR03b), and the papers (DH03), (HH03), (KS03), (MS03), (BLOPST03), (WTT99). Of course, our list of the papers devoted to the theory of fBm is not exhaustive. The book of P. Doukhan, G. Oppenheim, M. Taqqu (editors): *Theory and Applications of Long-range Dependence* (Birkhäuser, Boston 2003) contains papers devoted to different aspects of stochastic calculus for fractional Brownian motion and related processes. We mention, in particular, the papers of D. Surgailis (Sur03a), (Sur03b) and M. Maejima (Mae03), devoted to central and non-central limit theorems, where the asymptotic distribution is not the classical standard normal and the limit process is not the Wiener process. The processes of moving average type are obtained as the limiting ones for increasing sums of some stationary sequences that do not have finite variance. See also the papers (Ho96), (Dec03), (Do03), (Mol03), (PT03), (Taq03), (SW03) from this edition describing stochastic analysis and other aspects of the processes with long memory; papers concerning statistical problems were mentioned above. It is clear from the aforesaid descriptions and citations that there exists the urgent need to systematize the existing results devoted to fractional Brownian motion, to select the best of them (in the author's opinion) and to present them in appropriate form. Also, some well-known results admit generalizations, and it can be done without great technical difficulties. The present book is devoted to the solution of these two problems. Of course, we cannot claim the complete presentation of all the results concerning fractional

Brownian motion; it is impossible as the reader can see from aforesaid list. So, we choose only the following topics: Wiener and stochastic integration, Itô formula, Fubini and Girsanov theorems, stochastic differential equations, filtering in the mixed Brownian–fractional-Brownian models, financial applications, some statistical inferences for fractional Brownian motion and the stochastic calculus of multi-parameter fractional Brownian processes. These fields coincide with the main directions of our own interest in the long-memory effect.

The book consists of six chapters divided into 41 sections. Chapter 1 is devoted to the Wiener integration (when the integrand is nonrandom) with respect to fractional Brownian motion. Section 1.1 is devoted to the principal definitions from fractional calculus. We recall the notions of fractional integrals and derivatives both for finite and infinite intervals, formulate the Hardy–Littlewood theorem, give the Fourier transformation for fractional integrals and derivatives and calculate the values of some important fractional derivatives. Section 1.2 contains some elementary properties of fractional Brownian motion including the simplest spectral representations. Section 1.3 contains the Mandelbrot–van Ness representation of fractional Brownian motion via the Wiener process and some fractional kernels on real axes. These kernels are the prototypes for the future definition of the Wiener integration w.r.t. fBm. Sections 1.4 and 1.5 describe the construction of fractional Brownian motion and fractional noise on white noise space. Such space is convenient for applications since it is possible to consider mixed Brownian–fractional-Brownian processes and linear combinations of fractional Brownian motions with different Hurst indices on such space and to apply Wick calculus to them. It is proved that any fractional noise with $H \in [1/2, 1]$ belongs to the Hida distribution space S^* (we establish the corresponding estimates for the negative norms). The relations between motion and noise are established as in the usual Wick calculus for the Wiener noise. In Section 1.6 we return to fBm on arbitrary space. The section contains the definition of the Wiener integral with respect to fBm and various relations between different "integrable spaces" related to fBm. Section 1.7 is devoted to (non) completeness of the Gaussian spaces generated by fBm, in connection with their norms. Section 1.8 contains the representation of fBm via the Wiener process on any finite interval $[0, T]$ and some representations for auxiliary processes. Sections 1.9 and 1.10 present moment estimates for Wiener integrals w.r.t. fractional Brownian motion. Using the conditions of continuity of the trajectories of Wiener integrals w.r.t. fBm (Section 1.11) we extend in Section 1.12 the upper moment estimates to solutions of very simple stochastic differential equations containing Wiener integrals. Section 1.13 contains the proof of the stochastic Fubini theorem for the Wiener integrals w.r.t. fractional Brownian motion. Section 1.14 deals with such Gaussian processes that can be transformed into martingales with the help of some kernels (fBm can be transformed into the Wiener process with the help of hypergeometric kernels). Section 1.15 is devoted to different convergence schemes, in which fBm is approximated by the sequence of semimartingales, and even by the continuous processes with bounded variation. In the last case Wiener integrals w.r.t. fractional Brownian motion also can be approximated. Section 1.16 demonstrates the Hölder properties of the Wiener integrals w.r.t. fractional Brownian motion. Section 1.17 contains some auxiliary estimates for fractional derivatives of fBm and for the Wiener integrals w.r.t. Wiener process via the Garsia–Rodemich–Rumsey inequality. Section 1.18 contains one- and two-sided bounds for power variations for fBm and Wiener integrals w.r.t. fBm. Section 1.19 contains the result stating that some conditions of quadratic variation of a stochastic process supply that this process is an fBm; it is kind of generalization of the Lévy theorem for the Wiener process. Section 1.20 concludes; it describes Wiener fields on the plane and related fractional integrals and derivatives.

Chapter 2 is devoted to stochastic integration w.r.t. fractional Brownian motion and other aspects of stochastic calculus of fBm. There exist several approaches to stochastic integration w.r.t. fractional Brownian motion: pathwise integration, Wick integration, Skorohod integration, isometric integration and some others that are not mentioned here. Pathwise stochastic integration in fractional Sobolev-type spaces and in fractional Besov-type spaces is described in Section 2.1 and is generalized to fBm fields in Section 2.2. Wick integration is considered in Section 2.3 and is reduced to the integration w.r.t. white noise. Two approaches to the Skorohod integration and their connections with forward, backward and symmetric integration are discussed in Section 2.4. Isometric integration is the subject of section 2.5. The stochastic Fubini theorem and various versions of the Itô formula and the Girsanov theorem are contained in Sections 2.6–2.8 which conclude Chapter 2.

Chapter 3 is devoted to different properties of stochastic differential equations involving fBm. Section 3.1 contains the conditions of existence and uniqueness of solution of a "pure" stochastic differential equation containing a pathwise integral w.r.t. fBm and the estimates of its solution. Most of the theorems are stated in the spirit of the paper (NR00) but the results of Zähle (Zah99) on existence of local solutions are also presented since they are used later for construction of global solutions in the cases when other results cannot help. Some properties of SDEs with stationary coefficients including differentiability and local differentiability of the solutions are presented in Subsection 3.1.4. Existence and uniqueness of solutions of SDEs with two-parameter fractional Brownian fields is contained in Subsection 3.1.6. Semilinear "pure" and "mixed" SDEs are considered in detail in Subsections 3.1.5 and 3.2.1. The rate of convergence of Euler approximations of solutions of SDEs involving fBm is the subject to Section 3.4. SDEs with fractional white noise are considered in Section 3.3, and a detailed discussion of SDEs with additive Wiener integrals w.r.t. fBm is presented in Section 3.5.

Chapter 4 is devoted to filtering problems in the mixed fractional models. Section 4.1 considers the case when the signal process is modeled by mixed stochastic differential equations involving both fractional Brownian motion and the Wiener process and the observation process is the sum of the fractional

Brownian integral and the term of bounded variation. Optimal filtering in conditionally Gaussian linear systems with mixed signals and fractional Brownian observation is studied in Section 4.2. In these sections we consider only nonrandom integrands in all the stochastic integrals. In Section 4.3 we make an attempt to generalize the model and consider polynomial integrands depending on fBm.

Chapter 5 is devoted to financial models involving fBm. In general, financial markets fairly often have a long memory and it is a natural idea to model them with the help of fBm or with the help of some of its modifications. Nevertheless, it is not so easy to do this because the market model is "good" when it does not admit arbitrage and the models involving fractional Brownian motion are not arbitrage-free. So, this chapter is devoted to some methods of construction of the long-memory arbitrage-free models and to the discussion of different approaches to this problem. In Section 5.1 we introduce the mixed Brownian–fractional-Brownian model and establish conditions that ensure the absence of arbitrage in such a model. In Section 5.2 we consider a fractional version of the Black–Scholes equation for the mixed Brownian-fractional Brownian model which contains pathwise integrals w.r.t. fBm, discuss possible applications of Wick products in fractional financial models and produce Black–Scholes equation for the fractional model involving Wick product w.r.t. fBm.

Chapter 6 is devoted to the solution of some statistical problems involving fBm. The choice of the first problem which is solved in Sections 6.1 and 6.2 was evoked by some financial reasonings considered in Chapter 5. More exactly, we try to determine which of the two geometric Brownian motions from (5.2.6) serves as the better model for the real financial market, i.e. we test the complex hypothesis concerning the shifts in the geometric fBm; one of the shifts corresponds to the pathwise integral, and another to the Wick integral. In Section 6.3 we consider the existence and the properties of estimates of the shift parameter in different "pure" and "mixed" models involving fBm and, possibly, the Wiener process, which can be independent of or, conversely, "linearly dependent" on fractional Brownian motion.

I am grateful to Esko Valkeila who invited me several times to Helsinki University during the period of 1997-2005 and presented a possibility for fruitful work and discussion of the problems connected to fractional Brownian motion and related topics. Also, I am grateful to David Nualart for inviting me to Barcelona University during 2001–2003 when we discussed the problems connected to stochastic differential equations involving fBm. My thanks to all my other coauthors, with whom we have written the series of papers devoted to the stochastic calculus for fractional Brownian motion, especially to Jean Memin, Alexander Kukush, Georgij Shevchenko and Taras Androshchuk. My special thanks to Murad Taqqu and Christian Bender for their useful suggestions concerning contents of the minicourse of the lectures devoted to the stochastic calculus for fBm that I delivered in Helsinki Technology University

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Kiev, *Yuliya Mishura*
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