

SpringerBriefs in Statistics

JSS Research Series in Statistics

Editors-in-Chief

Naoto Kunitomo, Economics, Meiji University, Chiyoda-ku, Tokyo, Tokyo, Japan

Akimichi Takemura, The Center for Data Science Education and Research, Shiga University, Bunkyo-ku, Tokyo, Japan

Series Editors

Genshiro Kitagawa, Meiji Institute for Advanced Study of Mathematical Sciences, Nakano-ku, Tokyo, Japan

Tomoyuki Higuchi, Faculty of Science and Engineering, Chuo University, Tokyo, Japan

Toshimitsu Hamasaki, Office of Biostatistics and Data Management, National Cerebral and Cardiovascular Center, Suita, Osaka, Japan

Shigeyuki Matsui, Graduate School of Medicine, Nagoya University, Nagoya, Aichi, Japan

Manabu Iwasaki, School of Data Science, Yokohama City University, Yokohama, Tokyo, Japan

Yasuhiro Omori, Graduate School of Economics, The University of Tokyo, Bunkyo-ku, Tokyo, Japan

Masafumi Akahira, Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki, Japan

Takahiro Hoshino, Department of Economics, Keio University, Tokyo, Japan

Masanobu Taniguchi, Department of Mathematical Sciences/School, Waseda University/Science & Engineering, Shinjuku-ku, Japan

The current research of statistics in Japan has expanded in several directions in line with recent trends in academic activities in the area of statistics and statistical sciences over the globe. The core of these research activities in statistics in Japan has been the Japan Statistical Society (JSS). This society, the oldest and largest academic organization for statistics in Japan, was founded in 1931 by a handful of pioneer statisticians and economists and now has a history of about 80 years. Many distinguished scholars have been members, including the influential statistician Hirotugu Akaike, who was a past president of JSS, and the notable mathematician Kiyosi Itô, who was an earlier member of the Institute of Statistical Mathematics (ISM), which has been a closely related organization since the establishment of ISM. The society has two academic journals: the Journal of the Japan Statistical Society (English Series) and the Journal of the Japan Statistical Society (Japanese Series). The membership of JSS consists of researchers, teachers, and professional statisticians in many different fields including mathematics, statistics, engineering, medical sciences, government statistics, economics, business, psychology, education, and many other natural, biological, and social sciences. The JSS Series of Statistics aims to publish recent results of current research activities in the areas of statistics and statistical sciences in Japan that otherwise would not be available in English; they are complementary to the two JSS academic journals, both English and Japanese. Because the scope of a research paper in academic journals inevitably has become narrowly focused and condensed in recent years, this series is intended to fill the gap between academic research activities and the form of a single academic paper. The series will be of great interest to a wide audience of researchers, teachers, professional statisticians, and graduate students in many countries who are interested in statistics and statistical sciences, in statistical theory, and in various areas of statistical applications.

More information about this subseries at <http://www.springer.com/series/13497>

Hisayuki Tsukuma · Tatsuya Kubokawa

Shrinkage Estimation for Mean and Covariance Matrices

 Springer

Hisayuki Tsukuma
Faculty of Medicine
Toho University
Tokyo, Japan

Tatsuya Kubokawa
Faculty of Economics
University of Tokyo
Tokyo, Japan

ISSN 2191-544X

SpringerBriefs in Statistics

ISSN 2364-0057

JSS Research Series in Statistics

ISBN 978-981-15-1595-8

<https://doi.org/10.1007/978-981-15-1596-5>

ISSN 2191-5458 (electronic)

ISSN 2364-0065 (electronic)

ISBN 978-981-15-1596-5 (eBook)

© The Author(s), under exclusive license to Springer Nature Singapore Pte Ltd. 2020

This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd. The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

The rapid development of computer technology has started to yield many types of high-dimensional data and to enable us to deal with them well. Indeed, high-dimensional data appear in numerous fields such as web data science, genomics, telecommunication, atmospheric science, financial engineering, and others. With such a background, theory of statistical inference with high dimension has received much attention in recent years.

High-dimensional data in general are hard to handle, and ordinary or traditional methods in statistics are frequently inapplicable for them. This has inspired statisticians to develop new methodology in high dimension from both theoretical and practical aspects. Most statisticians' interests seem to be in development of efficient algorithms for statistical inference and in investigation of their asymptotic properties with the dimension going to infinity. On the other hand, there does not exist much literature in high-dimensional problems from a decision-theoretic point of view.

Statistical decision theory is the study of how to make decisions in the presence of statistical knowledge under uncertainty. It has been studied from around the 1940s and the researchers have already been produced many important and interesting results. Probably the most surprising result in decision-theoretic estimation is the inadmissibility of the sample mean vector to estimate a multivariate normal population mean. In the multivariate normal mean estimation, the sample mean vector is the maximum likelihood estimator and the uniformly minimum variance unbiased estimator, and thus it has been recognized to be optimal for a long time. However, in 1956, Charles Stein showed that the sample mean vector is admissible for the one- and two-dimensional cases but inadmissible for three or more dimensional cases. A little after that, a specific estimator, called a shrinkage estimator, was provided for exactly dominating the sample mean vector. To this day, various extensions of shrinkage estimation have been achieving in other statistical models.

The purpose of this book is to give a brief overview of shrinkage estimation in matrix-variate normal distribution model. More specifically, it includes recent techniques and results in estimation of mean and covariance matrices with a

high-dimensional setting that implies singularity of the sample covariance matrix. Such a high-dimensional model can really be analyzed by using the same arguments as for a low-dimensional model. Thus this book takes a unified approach to both high- and low-dimensional shrinkage estimation.

Theory of shrinkage estimation for matrix parameters needs many mathematical tools. In Chap. 1, we begin by briefly introducing basic terminology of decision-theoretic estimation and a mathematical technique in shrinkage estimation. Chapter 2 defines the notation with respect to matrix algebra and collects useful results in terms of the Moore-Penrose inverse, the Kronecker product and matrix decompositions. Chapter 3 provides the definition and some properties of matrix-variate normal distribution and related distributions, including the Wishart distribution and joint distributions corresponding to the Cholesky and the eigenvalue decompositions of the Wishart matrix. With a unified treatment for high- and low-dimensional cases, some related distributions are discussed. Chapter 4 introduces a multivariate linear model and derives its canonical form. To find decision-theoretically optimal estimators, we usually direct our attention to several classes of invariant estimators. Therefore Chap. 4 briefly explains group invariance in the canonical form as well. A key tool in shrinkage estimation is an integration by parts formula, called the Stein identity. Chapter 5 gives a generalized Stein identity on matrix-variate normal distribution. Moreover we list some results on matrix differential operators and in particular show useful differentiation formulae concerning the Moore-Penrose inverse. Chapter 6 addresses the problem of estimating the mean matrix in matrix-variate normal distribution model. A unified result on matricial shrinkage estimation is presented, and extensions and applications are given for more general models. Chapter 7 deals with the problem of estimating the covariance matrix relative to an extended Stein loss and provides various unified estimation procedures for high- and low-dimensional cases. Some related topics to covariance estimation are also touched.

The authors would like to thank Prof. M. Akahira for giving us the opportunity of publishing this book. The work of the first author was supported in part by Grant-in-Aid for Scientific Research (18K11201) from the Japan Society for the Promotion of Science (JSPF). The work of the second author was supported in part by Grant-in-Aid for Scientific Research (18K11188) from the JSPF.

Tokyo, Japan
March 2020

Hisayuki Tsukuma
Tatsuya Kubokawa

Contents

| | | |
|----------|---|----|
| 1 | Decision-Theoretic Approach to Estimation | 1 |
| 1.1 | Decision-Theoretic Framework for Estimation | 1 |
| 1.2 | James-Stein's Shrinkage Estimator | 2 |
| 1.3 | Unbiased Risk Estimate and Stein's Identity | 3 |
| | References | 4 |
| 2 | Matrix Algebra | 7 |
| 2.1 | Notation | 7 |
| 2.2 | Nonsingular Matrix and the Moore-Penrose Inverse | 9 |
| 2.3 | Kronecker Product and Vec Operator | 10 |
| 2.4 | Matrix Decompositions | 11 |
| | References | 12 |
| 3 | Matrix-Variate Distributions | 13 |
| 3.1 | Preliminaries | 13 |
| 3.1.1 | The Multivariate Normal Distribution | 13 |
| 3.1.2 | Jacobians of Matrix Transformations | 14 |
| 3.1.3 | The Multivariate Gamma Function | 16 |
| 3.2 | The Matrix-Variate Normal Distribution | 17 |
| 3.3 | The Wishart Distribution | 21 |
| 3.4 | The Cholesky Decomposition of the Wishart Matrix | 23 |
| | References | 26 |
| 4 | Multivariate Linear Model and Group Invariance | 27 |
| 4.1 | Multivariate Linear Model | 27 |
| 4.2 | A Canonical Form | 30 |
| 4.3 | Group Invariance | 31 |
| | References | 33 |

| | | |
|----------|--|-----------|
| 5 | A Generalized Stein Identity and Matrix Differential Operators | 35 |
| 5.1 | Stein's Identity in Matrix-Variate Normal Distribution | 35 |
| 5.2 | Some Useful Results on Matrix Differential Operators | 37 |
| | Appendix | 40 |
| | References | 42 |
| 6 | Estimation of the Mean Matrix | 45 |
| 6.1 | Introduction | 45 |
| 6.2 | The Unified Efron-Morris Type Estimators Including Singular Cases | 48 |
| 6.2.1 | Empirical Bayes Methods | 48 |
| 6.2.2 | The Unified Efron-Morris Type Estimator | 49 |
| 6.3 | A Unified Class of Matricial Shrinkage Estimators | 50 |
| 6.4 | Unbiased Risk Estimate | 53 |
| 6.5 | Examples for Specific Estimators | 55 |
| 6.5.1 | The Unified Efron-Morris Type Estimator | 55 |
| 6.5.2 | A Modified Stein-Type Estimator | 56 |
| 6.5.3 | Modified Efron-Morris Type Estimator | 58 |
| 6.6 | Related Topics | 59 |
| 6.6.1 | Positive-Part Rule Estimators | 59 |
| 6.6.2 | Shrinkage Estimation with a Loss Matrix | 62 |
| 6.6.3 | Application to a GMANOVA Model | 63 |
| 6.6.4 | Generalization in an Elliptically Contoured Model | 67 |
| | Appendix | 68 |
| | References | 74 |
| 7 | Estimation of the Covariance Matrix | 75 |
| 7.1 | Introduction | 75 |
| 7.2 | Scale Invariant Estimators | 77 |
| 7.3 | Triangular Invariant Estimators and the James-Stein Estimator . . . | 79 |
| 7.3.1 | The James-Stein Estimator | 79 |
| 7.3.2 | Improvement Using a Subgroup Invariance | 81 |
| 7.4 | Orthogonally Invariant Estimators | 84 |
| 7.4.1 | Class of Orthogonally Invariant Estimators | 84 |
| 7.4.2 | Unbiased Risk Estimate | 84 |
| 7.4.3 | Examples | 86 |
| 7.5 | Improvement Using Information on Mean Statistic | 96 |
| 7.5.1 | A Class of Estimators and Its Risk Function | 97 |
| 7.5.2 | Examples of Improved Estimators | 98 |
| 7.5.3 | Further Improvements with a Truncation Rule | 100 |
| 7.6 | Related Topics | 102 |
| 7.6.1 | Decomposition of the Estimation Problem | 102 |
| 7.6.2 | Decision-Theoretic Studies Under Quadratic Losses | 104 |

| | |
|--|------------|
| 7.6.3 Estimation of the Generalized Variance | 105 |
| 7.6.4 Estimation of the Precision Matrix | 106 |
| References | 108 |
| Index | 111 |