Removing Phase Variables from Biped Robot Parametric Gaits

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***Abstract*— Hybrid zero dynamics-based control is a promising framework for controlling underactuated biped robots and powered prosthetic legs. In this control paradigm, stable walking gaits are implicitly encoded in polynomial output functions of the robot configuration variables, which are to be zeroed via feedback. The biped output functions are parameterized by a suitable mechanical phasing variable whose evolution determines the biped gait progression during each step. Determining a proper phase variable, however, might not always be a trivial task. In this paper, we present a method for generating output functions from given parametric walking gaits without any explicit knowledge of the phase variables. Our elimination method is based on computing the resultant of polynomials, an algebraic tool widely used in computer algebra.**

I. Introduction

Hybrid zero dynamics-based (HZD) control is a promising framework for controlling underactuated biped robots [1], [2], [3], [4], [5]. In this paradigm, stable biped walking gaits are encoded as relations between the biped generalized coordinates that can be re-programmed on the fly. Recently, HZD-based controllers have also been used for controlling powered prosthetic legs for amputees [6], [7].

Walking gaits in the HZD-based control framework are trajectories in the configuration space of the robot. These trajectories are parameterized by means of phase variables that are kinematic quantities whose monotonic evolution determines the robotic gait progression during each step. In order to enforce the HZD-based walking gaits via feedback, they are *implicitly* encoded in the zero level set of polynomial output functions that are invariant with respect to discontinuous impact events (i.e., *hybrid invariant*) [1], [2], [4], [5], [6]. Driving these output functions to zero via feedback corresponds to stabilizing the desired walking gaits.

In some applications such as powered prostheses control, there are numerous phase variable candidates such as the foot center of pressure and the hip phase angle [8], [9], [10]. It has been observed that the choice of phase variable affects the walking robustness with respect to disturbances [6], [9]. Indeed, some parameterizations provide more human-like transient responses than the others [9]. However, generating output functions with closed-form expression from stable parametric walking gaits without any explicit knowledge of the phase variable is not a trivial problem.

Contributions of the paper. In this paper, we present an elimination method for removing phase variables from given parametric relations, which represent stable walking gait trajectories in the biped configuration space. Our method can be used for generating output functions with closed-form expressions that are suitable for feedback implementation. We also provide a necessary and sufficient condition for the generated outputs to have well-defined vector relative degree. The key ingredient used in our elimination method is based on computing the *resultant of polynomials*, a well-known algebraic tool widely used in computer algebra that is used for eliminating one variable from a system of two polynomial equations [11]. This tool has been used in a few control applications such as contouring control of multi-axis motion systems [12] and generating symmetric output functions from parametric virtual holonomic constraints [13, Chatper 4]. To the best of our knowledge, this paper employs the resultant of polynomials in the context of legged locomotion control for the first time.

The rest of this paper is organized as follows. Section II reviews preliminaries from biped robot modeling as well as some results from the HZD-based control framework. We also discuss the relationship between the implicit representation of walking gaits via output functions and their parametric representations. The formal problem statement is presented in Section III. In Section IV we find the implicit relationship between two given parametric polynomials. Next, we present our gait implicitization method for biped robots and provide a necessary and sufficient condition for the generated output functions to have well-defined relative degree in Section V. We then present simulation studies in Section VI. Concluding remarks are provided in Section VII.

Notation. Given two vectors (matrices) a, b of suitable dimensions, we denote by $[a\,;\,b]$ the vector (matrix) $[a^\top, b^\top]^\top$.

II. Biped Robot Hybrid Dynamics

In this section we present the dynamic model of underactuated planar biped robots and review some standard material from the HZD-based control framework [1], [2]. Also, we present the relationship between parametric and implicit representations of walking gaits.

A. Hybrid dynamical model of biped robots

Given an underactuated planar biped robot with point feet (see Figure 1), its equations of motion during **swing phase**, using the method of Lagrange, can be written as (see [2,

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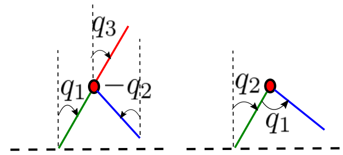

Fig. 1: Three-link and two-link planar bipeds with point feet.

Chapter 3])

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu, \quad (q,\dot{q}) \notin \mathcal{S}, \tag{1}$$

where the vectors $q = [q_1, \cdots, q_N]^\top \in \mathcal{Q}$ and $\dot{q} = [\dot{q}_1, \cdots, \dot{q}_N]^\top \in \mathbb{R}^N$ denote the joint angles and the joint velocities, respectively. The set $\mathcal{Q}$, called the biped **configuration space**, is assumed to be an open and connected subset of the Euclidean space $\mathbb{R}^N$. Therefore, the state $(q,\dot{q})$ of dynamical system (1), belongs to the state space $T\mathcal{Q} := \mathcal{Q} \times \mathbb{R}^N$. Moreover, $D(q)$, $C(q,\dot{q})$, and $G(q)$, denote the inertia matrix, the matrix of Coriolis/centrifugal forces, and the vector of gravitational forces, respectively. The vector of control inputs u belongs to $\mathbb{R}^{N-1}$ and the control input matrix $B \in \mathbb{R}^{N\times(N-1)}$ is assumed to be constant and of full rank $N-1$. We say that system (1) has **one degree of underactuation**. Without loss of generality, we assume that

$$B = [I_{N-1}; 0_{1\times(N-1)}],$$

where I_{N-1} denotes the identity matrix in $\mathbb{R}^{(N-1)\times(N-1)}$. The above choice of B implies that the biped's N^{th} degree-of-freedom, i.e., q_N, is unactuated. The vertical height from the ground and the horizontal position of the swing leg end, with respect to an inertial coordinate frame, are denoted by $p_2^{\text{v}}(q)$ and $p_2^{\text{h}}(q)$, respectively. The set $\mathcal{S}$, which represents the biped configurations at which impacts with the ground happen, is called the **switching surface** and defined as

$$\mathcal{S} := \{(q,\dot{q}) \in T\mathcal{Q} : p_2^{\text{v}}(q) = 0,\ p_2^{\text{h}}(q) > 0\}, \tag{2}$$

with respect to the inertial coordinate frame origin at $(0,0)$.

The **double support phase** is assumed to be *instantaneous* and modeled by rigid impacts with the ground. In particular, the impact model is given by

$$[q^+; \dot{q}^+] = [\Delta_q q^-; \Delta_{\dot{q}}(q^-)\dot{q}^-],\ (q^-, \dot{q}^-) \in \mathcal{S}, \tag{3}$$

where $(q^-, \dot{q}^-)$ and $(q^+, \dot{q}^+)$ denote the states of the robot just before and after impact, respectively. The complete biped dynamics, subject to rigid impacts with the ground, are described by the **hybrid dynamical system** in (1)–(3).

B. Gait Stabilization in the HZD-based control framework

In the HZD-based control framework, a walking gait is a smooth one-dimensional curve without self-intersections, i.e., a one-dimensional trajectory in the N-dimensional robot configuration space $\mathcal{Q}$. We denote this trajectory by γ_{w}. The trajectory γ_{w}, which determines the biped configurations during each step, connects the post-impact, i.e., q_0^+, and the pre-impact, i.e., q_0^-, biped configurations (see Figure 2).

For the purpose of feedback implementation, stable gaits are *implicitly* encoded in output functions, which are driven to zero via feedback. In particular, an output of the form

$$y = h(q), \tag{4}$$

is considered for the biped hybrid dynamics (1)–(3), where

$$h(q) = \big[h_1(q); \cdots; h_{N-1}(q)\big],$$

is a vector of $N-1$ polynomial functions $h_i : \mathcal{Q} \to \mathbb{R}$, $1 \leq i \leq N-1$. Zeroing the output (4) for the hybrid dynamical system (1)–(3) corresponds to making the biped configuration q converge to the gait trajectory γ_{w}. In other words, the output function in (4) satisfies $h(q) = 0,$ for all $q \in \gamma_{\text{w}}$.

Remark 2.1: Robotic gaits similar to (4), which are encoded as relations between the generalized coordinates of a robot, are called **virtual constraints** [14], [15]. In addition to biped and powered prostheses control, they have also been used for controlling biologically-inspired snake robots [16], [17], [18]. △

In order to guarantee stable walking, the output function given by (4) should be designed such that a number of hypotheses are satisfied (see [2, Chapter 5]). For the development in this paper, two hypotheses are relevant[1]: (H1) there exists an open set $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ such that h has vector relative degree $\{2, \cdots, 2\}$ everywhere on $\tilde{Q}$; (H2) there exists a smooth real-valued function $\theta(q)$ such that $\big[h(q); \theta(q)\big] : \tilde{Q} \to \mathbb{R}^N$ is a diffeomorphism onto its image.

The output function $y = h(q)$ in (4) is designed to be invariant with respect to impacts with the ground. Such an output function is said to be **hybrid invariant**. Hybrid invariance implies that the biped's post-impact configurations belong to the walking trajectory γ_{w} if the biped's pre-impact configurations belong to the walking trajectory, and that the vector of post-impact joint velocities is tangent to γ_{w}.

The function $\theta : \mathcal{Q} \to \mathbb{R}$ in Hypothesis H2 is called the **phase function**. For the biped configurations q belonging to the walking gait trajectory γ_{w}, the parameter

$$\xi = \theta(q), \tag{5}$$

is called the **phase variable**. Knowing the phase variable ξ during walking, when the output function is zeroed, *uniquely* determines the biped configuration (Figure 2).

Under Hypothesis H1, a given hybrid invariant output function can be zeroed for the biped hybrid dynamics (1)–(3) using a proper control input u, e.g., a standard input-output feedback linearizing control law (see [2, Chapter 5]). Once the outputs associated with a stable periodic orbit are zeroed, the resulting closed-loop motion is governed by lower-dimensional dynamics, called the **hybrid zero dynamics (HZD)**. It can be shown that if there exists an exponentially stable periodic orbit in the state space of the biped that is induced by zeroing the output function $h(\cdot)$

[1]Hypotheses H1 and H2 are regularity conditions for the existence of the zero dynamics associated with the output $y = h(q)$ for the biped robot. Exponential stability of the periodic orbit in the HZD framework requires additional conditions, which we assume to hold in this paper.

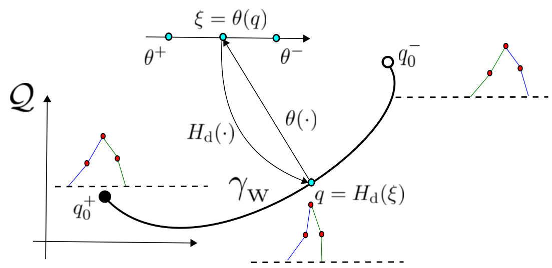

Fig. 2: The walking gait of every biped, in the HZD-based control framework, is a one-dimensional trajectory in the configuration space $\mathcal{Q}$. As the parameter ξ evolves from θ^+ to θ^-, the biped configuration evolves from q_0^+ to q_0^-.

in (4), then the phase variable time trajectories $\xi(t) = \theta(q(t))$, are *strictly monotonic* during each step of the robot ([2, Proposition 5.1]). Thus, $\xi(t)$ achieves its minimum and maximum at the beginning and the end of each single support phase. In particular, it can be shown that $\xi(t)$ varies between the two values

$$\theta^- := \theta(q_0^-),\ \theta^+ := \theta(q_0^+), \tag{6}$$

where q_0^- and $q_0^+ = \Delta_q q_0^-$ are the pre- and post-impact configurations of the biped, respectively. Without loss of generality, we assume that $\theta^+ < \theta^-$.

C. Parametric representations of biped walking gaits

Since every walking gait trajectory $\gamma_{\rm w}$ is a one dimensional smooth curve, it can be represented via parametric relationships. We let

$$q = H_{\rm d}(\xi), \tag{7}$$

be an arbitrary parametric relationship, representing $\gamma_{\rm w}$, where $H_{\rm d} : [\theta^+, \theta^-) \to \gamma_{\rm w}$ is a smooth bijective function (i.e., one-to-one and onto), and θ^- , θ^+, given by (6), are the post-and pre-impact phase variable values, respectively. Additionally, we assume that the parameterization in (7) satisfies $H'_{\rm d}(\xi) \neq 0$ for all $\xi \in [\theta^+, \theta^-)$. This condition guarantees that the curve $\gamma_{\rm w}$ is traversed once and only once as the phase variable ξ evolves from θ^+ to θ^- and that the curve $\gamma_{\rm w}$ does not have any self-intersections (see Figure 2). We call the parametric relation in (7) a **parametric representation** of the walking gait $\gamma_{\rm w}$.

Example 2.2 (Active compass gait biped): Consider the two-link biped in Figure 1. We let the leg length, the leg center of mass (COM) location, the leg mass, and the leg inertia about leg COM be $l = 1$ m, $l_c = 0.8$ m, $m = 0.3$ kg, and $I = 0.03$ kg.m^2, respectively. The two-link biped has one actuated variable q_1 and one unactuated variable q_2. Its hybrid dynamics, which are of the form (1)–(3), can be derived using standard methods (see, e.g., [2, Chapter 2]). The following stable walking gait for the two-link biped with these physical parameters is taken from [2, Chapter 6]:

$$q_1 = H_{\rm d}^1(\xi),\ q_2 = \xi, \tag{8}$$

where $H_{\rm d}^1(\xi)$ is the polynomial

$$\begin{aligned} H_{\rm d}^1(\xi) &= \alpha_0(1-s(\xi))^4 + 4\alpha_1 s(\xi)(1-s(\xi))^3 + \\ &\quad 6\alpha_2 s(\xi)^2(1-s(\xi))^2 + 4\alpha_3 s(\xi)^3(1-s(\xi)) + \\ &\quad \alpha_4 s(\xi)^4, \\ s(\xi) &= \frac{\xi - \theta^+}{\theta^- - \theta^+}, \end{aligned} \tag{9}$$

with coefficients $\alpha_1 = -0.42$, $\alpha_2 = 1.4$, $\alpha_3 = 0.8$, and $\alpha_4 = -\alpha_0 = \pi/7$. As the parameter ξ evolves from θ^+ to θ^-, its normalized form $s(\xi)$ changes from 0 to 1. In order to enforce the gait in (8), the output

$$y = q_1 - H_{\rm d}^1(q_2), \tag{10}$$

should be zeroed via an input-output linearizing feedback control law. △

In summary, walking gaits in the HZD framework can either be represented by zero level set of an output function $y = h(q)$ or by a parametric relationship of the form $q = H_{\rm d}(\xi)$, as in (7).

III. Problem Statement

In most of the HZD-based controllers (see, e.g., [2], [6]), outputs of the form

$$y = H_0 q - h_{\rm d}(c_0 q)$$

are considered, where $c_0 \in \mathbb{R}^{1\times N}$ is a row vector. However, it is possible that a stable walking gait trajectory $\gamma_{\rm w}$ in the biped configuration space, is given by a parametric relationship. Such parametric relationships are encountered in applications such as powered prostheses control [6], [9].

Example 3.1 (Active compass gait biped): Consider the two-link biped in Example 2.2 and its stable walking gait given by (8)–(9). In [2], the phase variable $\theta(q) = q_2$ is considered, which corresponds to the linear progression of the unactuated degree-of-freedom q_2 with the phase variable $\xi = \theta(q)$. However, it is also possible to consider the more general relation

$$q_2 = H_{\rm d}^2(\xi),$$

where $H_{\rm d}^2(\xi)$ is some polynomial in the phase variable ξ of order greater than one, such that $\theta^- = H_{\rm d}^2(\theta^-)$ and $\theta^+ = H_{\rm d}^2(\theta^+)$. For instance, if

$$H_{\rm d}^2(\xi) = \frac{1}{2}(\theta^- - \theta^+)\big(s(\xi) + s^2(\xi)\big) + \theta^+, \tag{11}$$

where $s(\xi) := (\xi - \theta^+)/(\theta^- - \theta^+)$, then the unactuated variable q_2 is a nonlinear quadratic function of ξ. △

The underlying challenge for stabilizing a walking gait in parametric form, is that finding the output function $y = h(q)$ for the biped hybrid dynamics (1)–(3) requires solving a collection of N nonlinear polynomial equations to obtain $N-1$ output functions, which are independent of the parameter ξ. Although, solving such a nonlinear equation using numerical methods is often possible, the resulting solution is not suitable for feedback implementation. Rather it is desired to find an output function $y = h(q)$ with a closed-form expression. Finding such a closed-form expression, in general, is impossible. In order to address this challenge, we solve the following problem in this paper.

Parametric Gait Implicitization Problem. Consider

$$q = H_{\rm d}(\xi), \tag{12}$$

where $H_{\rm d} : [\theta^+, \theta^-) \to \mathcal{Q}$ is a smooth function, whose image represents the gait trajectory $\gamma_{\rm w}$ given by

$$\gamma_{\rm w} = H_d\big([\theta^+, \theta^-)\big). \tag{13}$$

Suppose that the components of the function $H_{\rm d}(\cdot)$ are

$$H_{\rm d}^i(\xi) = \sum_{k=0}^{n_i} b_k^i \xi^k,\ 1 \le i \le N, \tag{14}$$

which are N polynomials of the real variable ξ such that the degree of the $i^{\rm th}$ polynomial is equal to n_i. Find an output function $y = h(q)$, $h(q) = [h_1(q); \cdots ; h_{N-1}(q)]$, such that it becomes zero on the gait trajectory $\gamma_{\rm w}$. In other words,

$$h\big(H_{\rm d}(\xi)\big) = 0, \tag{15}$$

for all $\xi \in [\theta^+, \theta^-)$. Furthermore, determine necessary and sufficient conditions for the output $y = h(q)$ to have vector relative degree $\{2, \cdots , 2\}$ for the biped dynamics. △

Finding the output $y = h(q)$, which *implicitly* represents the walking gait trajectory $\gamma_{\rm w}$ through (15), enables us to enforce the given parametric representation in (12) by zeroing the output $y = h(q)$. We call the process of bringing a parametric gait to its implicit form **gait implicitization**.

Solution Strategy. Our strategy for solving the parametric gait implicitization problem unfolds in two steps. In the first step, presented in Section IV, we use a symbolic algebraic 2-by-2 elimination method, which is based on computing the resultant of polynomials, to eliminate the phase variable ξ and find the implicit relationship between $H_{\rm d}^i(\xi)$ and $H_{\rm d}^j(\xi)$, $i \ne j$, in terms of the configuration variables q_i and q_j. Next, in Section V, we construct an output vector function $y = h(q)$ with $N-1$ components such that it becomes zero whenever q belongs to the walking gait $\gamma_{\rm w}$. The generated output *implicitly* represents the walking gait. We also find a necessary and sufficient condition for the constructed output function $y = h(q)$ to have well-defined vector relative degree. △

IV. Implicitization of Two Parametric Polynomials

This section will establish the implicit relationship between any two given biped configuration variables that are given by parametric polynomials. We achieve this goal by removing the phase variable using *resultant of polynomials*, a tool which is frequently used in computer algebra. Necessary preliminaries are provided in the Appendix.

Consider an arbitrary biped configuration represented by the symbolic variable q and the walking gait curve $\gamma_{\rm w}$ in (13) with parametric polynomial representation (14). We define the polynomials

$$P_i^{q_i}(\xi) := H_{\rm d}^i(\xi) - q_i,\ 1 \le i \le N. \tag{16}$$

in the real variable ξ, where q_i is the $i^{\rm th}$ element of the vector q. Indeed, $b_0^i - q_i$ is the constant term of the polynomial $P_i^{q_i}(\xi)$, i.e., the coefficient of ξ^0 in (14). The parametric relationship

$$P_i^{q_i}(\xi) = 0 \Longrightarrow q_i = H_{\rm d}^i(\xi),$$

gives the trajectory of the $i^{\rm th}$ joint variable during each walking step, as the parameter ξ varies in the interval $[\theta^+, \theta^-)$. Now, we consider two arbitrary polynomials $P_i^{q_i}(\xi)$ and $P_j^{q_j}(\xi)$, $1 \le i, j \le N$, from the collection of polynomials in (16). In order to remove the phase variable ξ and to find the implicit relationship between the joint variables q_i and q_j during each step of the walking gait, we compute

$$\tilde{h}_{ij}(q) = {\rm Res}(H_{\rm d}^i(\xi) - q_i, H_{\rm d}^j(\xi) - q_j), \tag{17}$$

where ${\rm Res}(\cdot, \cdot)$, defined by (A-2) in the Appendix, is the resultant of the polynomials $P_i^{q_i}(\xi)$ and $P_j^{q_j}(\xi)$ in (16).

According to the definition of resultant in (A-2), the functions $\tilde{h}_{ij}(\cdot)$ in (17) are independent of the phase variable ξ and only depend on the coefficients of the polynomials $P_i^{q_i}(\xi)$ and $P_j^{q_j}(\xi)$ in (16), i.e., b_k^i, b_k^j, q_i, and q_j. The coefficients b_k^i, b_k^j are numerical, while q_i, q_j are symbolic variables. Computer algebra systems such as the MATLAB Symbolic Math Toolbox are capable of computing (17) symbolically (see Example 4.1). The resultant of the polynomials $P_i^{q_i}(\xi)$ and $P_j^{q_j}(\xi)$ has the form

$$\tilde{h}_{ij}(q) = \sum_{k,l} \beta_{kl} q_i^k q_j^l,$$

which is a symbolic bivariate polynomial (i.e., of two variables), independent of the phase variable ξ.

Example 4.1 (Active compass gait biped): Consider the biped gait walking trajectory in Example 3.1. Consider a parametric representation of the walking gait curve with $q_1 = H_{\rm d}^1(\xi)$ and $q_2 = H_{\rm d}^2(\xi)$, where $H_{\rm d}^1(\cdot)$ and $H_{\rm d}^2(\cdot)$ are given by (9) and (11), respectively. Consider the two polynomials $P_1(\xi)$ and $P_2(\xi)$, defined by (16), in the real variable ξ. The variables q_1 and q_2, which are considered to be the coefficients of ξ^0, are symbolic. Computing the resultant of the two polynomials $P_1(\cdot)$ and $P_2(\cdot)$ would remove the parameter ξ and give us a bivariate polynomial in the joint variables q_1 and q_2. The implicit function $\tilde{h}_{12}(\cdot)$ in (17) is

$$\begin{aligned}\tilde{h}_{12}(q_1, q_2) \;=\;& 0.003 + 0.27q_1^2 + 1.49q_1q_2^2 \\ & -4.99q_1q_2 - 13.85q_1 + 47.13q_2^4 \\ & -0.36q_2^3 - 1.41q_2^2 - 0.68q_2,\end{aligned} \tag{18}$$

which is a function of the symbolic variables q_1 and q_2, and independent of the parameter ξ. △

The functions $\tilde{h}_{ij}(\cdot)$ defined by (17) satisfy a *fundamental property* at the biped configurations that belong to the walking gait curve (13), as stated in the following proposition.

Proposition 4.2: Consider the biped walking gait curve $\gamma_{\rm w}$ in (13) with parametric polynomial representation (14). Consider the collection of polynomials in (16), arbitrary integers $1 \leq i,j \leq N$, and the output function $\tilde{h}_{ij}(q)$ given by (17). Then, $\tilde{h}_{ij}\big(H_\mathrm{d}(\xi)\big) = 0,$ for all $\xi \in [\theta^+, \theta^-)$.

Proof: Suppose that $(q_i,\, q_j) = (H_\mathrm{d}^i(\xi_0),\, H_\mathrm{d}^j(\xi_0))$, for an arbitrary $\xi_0 \in [\theta^+, \theta^-)$. Therefore, ξ_0 is a common root of the two polynomials $P_i^{q_i}(\xi)$ and $P_j^{q_j}(\xi)$ defined by (16). By Part 2 of Lemma A1 in the Appendix, $\tilde{h}_{ij}(H_\mathrm{d}(\xi_0)) = 0$, because the two polynomials $P_i^{q_i}$ and $P_j^{q_j}$ have a common root at $\xi = \xi_0$. ■

Proposition 4.2 states that the functions $\tilde{h}_{ij}(q)$ in (17), which are generated by taking the resultant of the parametric polynomials $P_i(\xi)$ and $P_j(\xi)$, become zero whenever the configuration q belongs to the walking gait trajectory γ_w. Thus, the function $\tilde{h}_{ij}(q)$ can be considered as an output for the biped and driven to zero via feedback. Driving $\tilde{h}_{ij}(q)$ to zero corresponds to making the desired relationship between q_i and q_j, which is prescribed by the given parametric representation $H_\mathrm{d}(\cdot)$, hold during each walking step.

V. Solution to the Gait Implicitization Problem

In this section we solve the parametric gait implicitization problem formulated in Section III. In particular, using the functions obtained in Section IV, we construct an output vector function with $N-1$ components such that it becomes zero on the walking gait curve γ_w given by (13). Next, we provide a necessary and sufficient condition for the generated output function to have well-defined vector relative degree.

Given the walking gait curve γ_w in (13) with parametric polynomial representation (14), we construct $N-1$ functions

$$h_k(q) = \tilde{h}_{k,k+1}(q),\ 1 \leq k \leq N-1, \tag{19}$$

where the functions $\tilde{h}_{k,k+1}(\cdot)$ are defined by (17). Using the functions $h_k(q)$ in (19), we construct the output function $y = h(q)$, where

$$h(q) := \big[h_1(q);\, \cdots ;\, h_{N-1}(q)\big], \tag{20}$$

for the biped hybrid dynamics (1)–(3).

The output function (20) has the property that it becomes zero whenever $q = H_\mathrm{d}(\xi)$, due to Proposition 4.2. If this output also satisfies a certain rank condition, then it can be zeroed using an input-output feedback linearizing control law, as stated in the following proposition.

Proposition 5.1: Consider the gait curve γ_w in (13) with parametric polynomial representation (14). Consider the $N-1$ functions $h_k(q)$, $1 \leq k \leq N-1$, in (19) and the output function $y = h(q)$, where $h(q)$ is given by (20). Suppose that $B^\perp D(H_{_\mathrm{d}}(\xi))H'_{_\mathrm{d}}(\xi) \neq 0$ for all $\xi \in [\theta^+, \theta^-]$, where $B^\perp = [0_{1\times(N-1)}\ 1]$. If

$$\mathrm{rank}\big(\frac{\partial h}{\partial q}\big) = N-1, \tag{21}$$

for all $q \in \gamma_\mathrm{w}$, then the output $y = h(q)$ can be zeroed using

$$u = \big(\frac{\partial h}{\partial q}D^{-1}(q)B\big)^{-1}\Big\{v(y,\dot{y}) - \frac{\partial}{\partial q}\big(\frac{\partial h}{\partial q}\dot{q}\big)\dot{q} + \frac{\partial h}{\partial q}D^{-1}(q)[C(q,\dot{q})\dot{q} + G(q)]\Big\}, \tag{22}$$

for the biped dynamics and $v(y,\dot{y})$ is a high-gain PD feedback or a continuous finite time stabilizer of the double integrator $\ddot{y} = v(y,\dot{y})$[2].

The proof is omitted for the sake of brevity.

Remark 5.2: If the rank condition (21) is not satisfied for a generated output $y = h(q)$ associated with a given parametric representation $q = H_\mathrm{d}(\xi)$ for some configurations $q \in \gamma_\mathrm{w}$, it is still possible to zero the output using the *constraint augmentation approach* introduced in [2, Chapter 5]. △

VI. Simulation Studies

Two-link walker. Consider the active two-link biped robot in Example 2.2 and the stable walking gait trajectory γ_w in Figure 1. The polynomial $H_\mathrm{d}^1(\xi)$, given by (9), determines the desired evolution of the joint variable q_1 during each step. The unactuated variable is q_2, which varies between $\theta^+ = -0.22$ radians and $\theta^- = 0.22$ radians during each walking step. We consider the following three parameterizations for the unactuated variable

$$\begin{aligned} H_\mathrm{d}^{2\mathrm{a}}(\xi) &= \xi, \\ H_\mathrm{d}^{2\mathrm{b}}(\xi) &= \frac{1}{2}(\theta^- - \theta^+)\big(s(\xi) + s^2(\xi)\big) + \theta^+, \\ H_\mathrm{d}^{2\mathrm{c}}(\xi) &= \frac{1}{3}(\theta^- - \theta^+)\big(s(\xi) + 2s^3(\xi)\big) + \theta^+, \end{aligned} \tag{23}$$

where $s(\xi)$ is defined in (11). Also, $H_\mathrm{d}^{2\mathrm{a}}(\cdot)$, $H_\mathrm{d}^{2\mathrm{b}}(\cdot)$, and $H_\mathrm{d}^{2\mathrm{c}}(\cdot)$ correspond to a linear, a quadratic from (11), and a cubic parameterization of the unactuated variable q_2 by the parameter ξ, respectively.

In order to be able to enforce each of these parametric representations, we need to find an output with closed-form expression for each of the above parametric representations. For the linear parametric representation $H_\mathrm{d}^{2\mathrm{a}}(\cdot)$, one can easily set $q_2 = \xi$ and find its associated output $y_\mathrm{a} = h_\mathrm{a}(q)$ given by (10). For the other two parametric representations, we use the methodology presented in the paper. First, we form the two polynomials $P_{2\mathrm{b}}(\cdot)$ and $P_{2\mathrm{c}}(\cdot)$ given by (16). Next, using (17) and (19), we obtain two different outputs $y = h_\mathrm{b}(q)$ and $y = h_\mathrm{c}(q)$, associated with the parametric representations $H_\mathrm{d}^{2\mathrm{b}}(\cdot)$ and $H_\mathrm{d}^{2\mathrm{c}}(\cdot)$. The function $h_\mathrm{b}(q)$ is given by (18). The function $h_\mathrm{c}(q)$, whose expression has been omitted for the sake of brevity, can also be computed using any symbolic computer algebra system, similar to Example 4.1. All of these outputs satisfy the rank condition in Theorem 5.1. Therefore, they can be zeroed via an input-output feedback linearizing control input. The time profiles of the biped joints and their associated phase portraits are demonstrated in Figures 3 and 4, respectively. In this example, different parameterizations of the unactuated degree of freedom results in different biped walking speeds. △

Three-link walker. Consider the three-link biped robot shown in Figure 1. We let the torso length, the leg length, the torso mass, the hip mass, and the leg mass to be $l = 0.5$ m, $r = 1.0$ m, $M_T = 10$ kg, $M_H = 15$ kg, and $m = 5$ kg, respectively. The three-link biped has two actuated variables

[2]A possible choice for $v(y,\dot{y})$ is the Bhat-Bernstein's continuous time double integrator in [19], which is used on biped robots in [2].

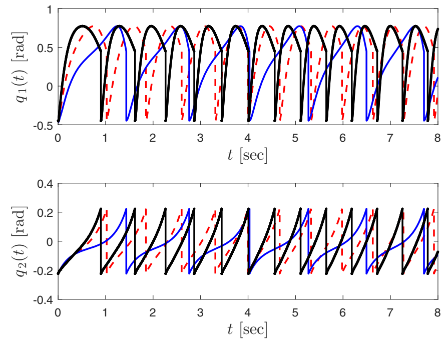

Fig. 3: Temporal progression of the two-link biped configuration variables. The blue, red, and black curves correspond to zeroing the outputs $y_a = h_a(q)$, $y_b = h_b(q)$, and $y_c = h_c(q)$, respectively.

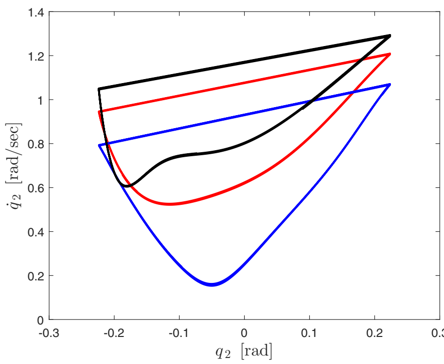

Fig. 4: Phase portraits of the two-link biped resulting from zeroing three different outputs. The blue, red, and black orbits correspond to the outputs $y_a = h_a(q)$, $y_b = h_b(q)$, and $y_c = h_c(q)$, respectively.

and one unactuated variable. Its hybrid dynamics have the form (1)– (3) and can be derived using standard methods (see, e.g., [2, Chapter 2]). The following stable walking gait for the three link biped with these physical parameters is taken from [2, Chapter 6]

$$\begin{aligned} H_d^{1a}(\xi) = \xi,\ H_d^2(\xi) = a_1^0 + \cdots + a_1^3\xi^3, \\ H_d^3(\xi) = -\xi_1 + (a_2^0 + \cdots + a_2^3\xi^3)(\xi + q_1^d)(\xi - q_1^d), \end{aligned} \tag{24}$$

where the vector of coefficients $a_0 := [a_0^1; \cdots ; a_3^1]$ and $a_2 := [a_0^2; \cdots ; a_3^2]$ are given by $a_0 = [0.512; 0.073; 0.035; -0.819]$ and $a_1 = [-2.27; 3.26; 3.11; 1.89]$, respectively. The parametric representation in (24) corresponds to the linear parameterization of q_1 with the phase variable ξ. The outputs associated with this parametric representation can be readily found to be $y_a = [h_1^a(q); h_2^a(q)] = [q_2 - H_d^2(q_1); q_3 - H_d^3(q_1)]$. Now, we consider the following nonlinear parameterizations of the joint variable q_1 with the phase variable ξ

$$\begin{aligned} H_d^{1b}(\xi) = \frac{1}{4}(\theta^- - \theta^+)\big(s(\xi) + 3s^2(\xi)\big) + \theta^+, \\ H_d^{1c}(\xi) = \frac{1}{5}(\theta^- - \theta^+)\big(s(\xi) + s(\xi)^2 + 3s^3(\xi)\big) + \theta^+, \end{aligned} \tag{25}$$

where $s(\xi) := (\xi - \theta^+)/(\theta^- - \theta^+)$ is defined in (11), and $H_d^{1b}(\cdot)$ and $H_d^{1c}(\cdot)$ respectively correspond to a quadratic and a cubic progression of the joint variable q_1, along the walking trajectory γ_w. Using the methodology presented in the paper, we find outputs associated with the parametric representations $H_d^{2b}(\cdot)$ and $H_d^{2c}(\cdot)$ of the walking gait trajectory γ_w. The time profiles of the biped joints and their associated phase portraits are shown in Figures 5 and 6, respectively. $\triangle$

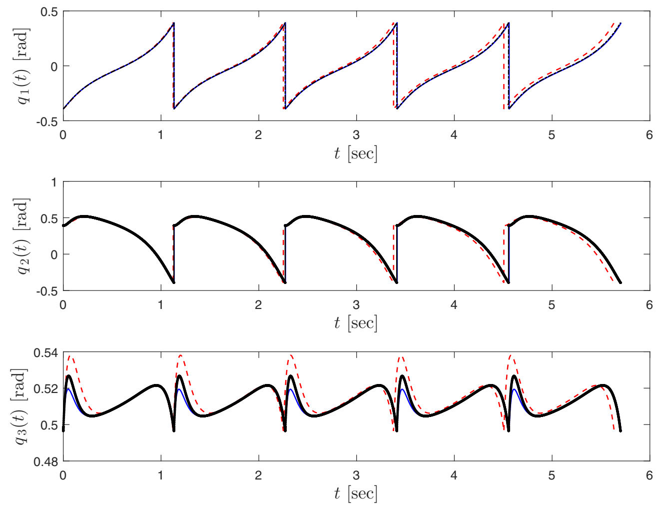

Fig. 5: Temporal progression of the three-link biped configuration variables. The blue, red, and black orbits correspond to the outputs $y_a = h_a(q)$, $y_b = h_b(q)$, and $y_c = h_c(q)$, respectively.

VII. Concluding Remarks and Future Research

Using the resultant of polynomials, we presented a method for removing phase variables from given stable parametric walking gaits and generating output functions suitable for feedback implementation. We provided a necessary and sufficient condition for the generated output to have well-defined vector relative degree. In the next step, we plan to examine the applicability of our proposed methodology for powered prostheses control.

Appendix. Mathematical Preliminaries

Given two polynomials

$$P_1 = \sum_{i=0}^{n_1} a_i^1 \xi^i,\ P_2 = \sum_{i=0}^{n_2} a_i^2 \xi^i, \tag{A-1}$$

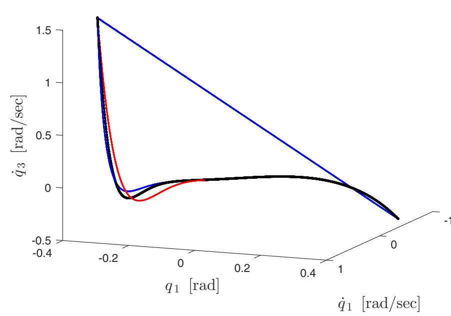

Fig. 6: Phase portraits of the three-link biped resulting from zeroing three different outputs. The blue, red, and black orbits correspond to the outputs $y_a = h_a(q)$, $y_b = h_b(q)$, and $y_c = h_c(q)$, respectively.

$$\mathrm{Syl}(P_1,P_2)=\left(\begin{array}{ccccccccc} a^1_{n_1} & a^1_{n_1-1} & \cdots & \cdots & \cdots & a^1_0 & & & \\ & a^1_{n_1} & a^1_{n_1-1} & \cdots & \cdots & \cdots & a^1_0 & & \\ & & \cdots & & & & & & \\ & & & a^1_{n_1} & a^1_{n_1-1} & \cdots & \cdots & \cdots & a^1_0 \\ -- & -- & -- & -- & -- & -- & -- & -- & -- \\ a^2_{n_2} & a^2_{n_2-1} & \cdots & \cdots & \cdots & a^2_0 & & & \\ & a^2_{n_2} & a^2_{n_2-1} & \cdots & \cdots & \cdots & a^2_0 & & \\ & & \cdots & & & & & & \\ & & & a^2_{n_2} & a^2_{n_2-1} & \cdots & \cdots & \cdots & a^2_0 \end{array}\right)\begin{array}{l} n_2 \text{ rows} \\ \\ n_1 \text{ rows}\end{array} \qquad (\star)$$

Sylvester matrix associated with the two polynomials $P_1=\sum_{i=0}^{n_1} a^1_i \xi^i$ and $P_2=\sum_{i=0}^{n_2} a^2_i \xi^i$.

in the real variable ξ, with $a^1_{n_1} \neq 0$ and $a^2_{n_2} \neq 0$, their associated **Sylvester matrix**, denoted by $\mathrm{Syl}(P_1, P_2)$, is given by ($\star$). The **resultant** of the two polynomials P_1 and P_2 in (A-1) is a function of the coefficients of the two polynomials, and defined as

$$\mathrm{Res}(P_1,P_2) := \det\big(\mathrm{Syl}(P_1,P_2)\big). \qquad \text{(A-2)}$$

Note that the resultant of polynomials given by (A-2) is *independent* of the parameterizing variable ξ, since the Sylvester matrix in ($\star$) is independent of the variable ξ.

Lemma A1 ([20]): Consider the two polynomials $P_1(\xi)$ and $P_2(\xi)$ in (A-1) of degrees n_1 and n_2, respectively. Let $R(P_1)$ and $R(P_2)$ be the sets of real roots of $P_1(\xi)$ and $P_2(\xi)$, respectively. Let

$$P_i(\xi) = a_{n_i} \prod_{\xi_i^k \in R(P_i)} (\xi - \xi_i)^{r_i^k},$$

be the factorization of $P_i(\xi)$, $i = 1, 2$, over the field of real numbers, where r_i^k is the multiplicity of the root $\xi_i^k \in R(P_i)$. Then,

1) the resultant of $P_1(\xi)$ and $P_2(\xi)$, defined in (A-2), satisfies

$$\begin{aligned}\mathrm{Res}(P_1,P_2) &= \big(a_{n_1}\big)^{n_2} \prod_{\xi_1^k \in R(P_1)} \big(P_2(\xi_1^k)\big)^{r_1^k} \\ &= (-1)^{n_1 n_2}\big(a_{n_2}\big)^{n_1} \prod_{\xi_2^k \in R(P_2)} \big(P_1(\xi_2^k)\big)^{r_2^k}, \qquad \text{(A-3)}\end{aligned}$$

2) $\mathrm{Res}(P_1, P_2) = 0$ if and only if $P_1(\xi)$ and $P_2(\xi)$ have at least a common root.

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