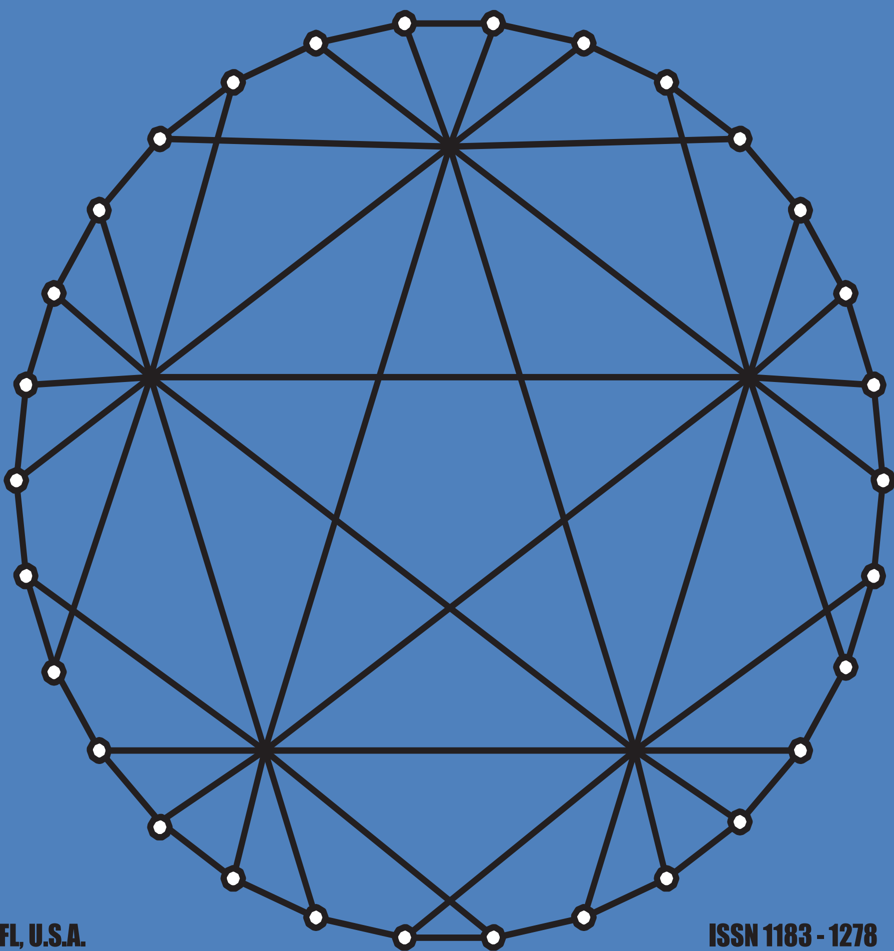
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 79
January 2017



Boca Raton, FL, U.S.A.

ISSN 1183 - 1278

BULLETIN OF THE ICA
Volume **79** (2017), Pages 66–81

Color Stable Graphs

Ronald D. Dutton

Department of Computer Science
University of Central Florida
Orlando FL 32816
U.S.A.

Robert C. Brigham

Department of Mathematics
University of Central Florida
Orlando FL 32816
U.S.A.

Abstract

Many years ago Mycielski [8] showed, for any positive integer k, the existence of a triangle-free graph with chromatic number k. An obvious question concerns the minimum number of vertices such a graph G must have. Every vertex v in a minimum G is k-color critical, that is, $\chi(G-v) = k-1$. A vertex v for which $\chi(G-v) = k$ is called k-color stable. This paper addresses relationships between the minimum number of vertices in triangle-free graphs in which every vertex is k-color critical and those in which every vertex is k-color stable.

Keywords: graph coloring, triangle-free graphs, color critical vertices and graphs, color stable vertices and graphs.

1 Introduction

Let G be a graph with vertex set $V(G)$. The following notation is standard, and it and other terminology can be found in Chartrand and Zhang [1] or Gross and Yellen [5]. The minimum and maximum degrees of G are, respectively, $\delta(G)$ and $\Delta(G)$. The open and closed neighborhoods of a vertex v in graph G are, respectively, $N_G(v)$ and $N_G[v]$, and if it is clear what graph is under consideration, we simply use $N(v)$ and $N[v]$. The graph with vertex v removed from G is denoted $G - v$, and $G - X$, where $X \subseteq V(G)$, is the graph resulting from removing all vertices of X from G. The expression $G + v$ indicates a graph obtained from G by adding a vertex v, in which case $N(v)$ must be specified. The chromatic number of G is $\chi(G)$. When convenient a k-coloring of a graph will be expressed as a function $c : V(G) \to \{1, 2, \ldots, k\}$ so $c(v)$ is the color assigned to vertex v. A *color class* of a k-coloring c is a maximal subset $W \subseteq V(G)$ such that $c(w)$ has the same value for all $w \in W$.

The problem of determining the smallest number of vertices in a triangle-free graph having a given chromatic number k has received some interest. Early mentions appear in Erdös [4] and Chvátal [3]. Here, this number of vertices is denoted $m(k)$. It is easy to determine these numbers when $1 \leq k \leq 3$: $m(1) = 1$ (uniquely achieved by K_1), $m(2) = 2$ (K_2), and $m(3) = 5$ (C_5). Chvátal [3] showed that $m(4) = 11$ and the Grötzsch graph shown in Figure 1 is the unique such graph. As reported by Jenson and Royle [7], Toft in 1988 asked for the value of $m(5)$, and they, using a computer search, discovered that $m(5) = 22$ and there are 80 such graphs.

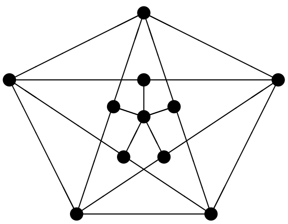


Figure 1: Grötzsch graph

The vertices of a graph achieving the value $m(k)$ are vertex color critical, that is, removal of any vertex reduces the chromatic number by one. We formalize the concept in the following definition. Any coloring considered in this paper is assumed to be proper.

Definition 1. *Let G be a graph with vertex set $V(G)$.*

1. *Vertex $v \in V(G)$ is* vertex color critical *(vcc) if $\chi(G - v) = \chi(G) - 1$.*
2. *G is a k-*vcc graph *if $\chi(G) = k$ and every $v \in V(G)$ is vcc.*
3. *G is a* minimum k-vcc graph *if every k-vcc graph has at least $|V(G)|$ vertices.*

4. *$m(k)$ is the number of vertices of a minimum triangle-free k-vcc graph.*

The standard definition of a graph G being color critical is that $\chi(H) < \chi(G)$ for every proper subgraph H. In particular, the chromatic number decreases if any vertex or any edge is removed. This is confirmed in Chartrand and Zhang [1] on page 175 where they also state that every graph G of chromatic number k has a k-color critical subgraph. This study is concerned only about vertex removal decreasing the chromatic number and hence the term vertex color critical.

If a vertex v in graph G is not vcc, then $\chi(G - v) = \chi(G)$. This paper studies triangle-free graphs for which this property holds for every vertex. The following definition is similar to the one given above.
Definition 2. *Let G be a graph with vertex set $V(G)$.*

1. *Vertex $v \in V(G)$ is* vertex color stable *(vcs) if $\chi(G - v) = \chi(G)$.*
2. *G is a k-*vcs graph *if $\chi(G) = k$ and every $v \in V(G)$ is vcs.*
3. *G is a* minimal k-vcs graph *if, for every vertex $v \in V(G)$, $G - v$ is not a k-vcs graph, that is, $G - v$ contains at least one vcc vertex.*
4. *G is a* minimum k-vcs graph *if every k-vcs graph has at least $|V(G)|$ vertices.*
5. *$M(k)$ is the number of vertices of a minimum triangle-free k-vcs graph.*

The two definitions above aren't completely parallel since minimal vcc graphs are undefined. The reason is every k-vcc graph is minimal since it does not contain a proper k-vcc minimal subgraph and hence is not of interest. Any bipartite graph not equal to $K_{1,n}$, $n \geq 2$, is an example of a 2-vcs graph. It is easy to see by examining all graphs on at most four vertices (see Harary [6]) that $M(1) = 2$ (the unique graph achieving this is $2K_1$) and $M(2) = 4$ (the only graphs achieving this are C_4, P_4, and $2K_2$). The study reported here was motivated by examining the relation between $m(k)$ and $M(k)$.

It is clear that $m(k)$ is the smallest number of vertices in a triangle-free graph of chromatic number k such that every color class has at least one vertex. We will show $M(k)$ is the smallest number of vertices in a triangle-free graph of chromatic number k such that every color class has at least two vertices. This idea can be extended as is described in Problem 7 of Section 6.

Section 2 presents needed preliminary results and Section 3 gives relationships involving $m(k)$ and $M(k)$. Section 4 determines the value of $M(3)$, along with the set of all graphs that achieve it. The determination of $M(k)$,

$k \geq 4$, appears to be a difficult problem and in Section 5 bounds are shown for $M(4)$. The final section presents some open problems.

2 Preliminaries

This section develops the background needed for later results.

Lemma 3. *G is a k-vcs graph if and only if, for every k-coloring of G, every color class has at least two vertices.*

Proof: If a color class has only a single vertex, removal of that vertex reduces the chromatic number by one and G is not k-vcs. If G is not k-vcs, there is a vcc vertex v whose removal reduces the chromatic number. The $k-1$ coloring of $G-v$ can then be extended to a k-coloring of G with v the only vertex of color k. □

The next definition describes three graphs based on graph G that are useful in exploring properties of k-vcc and k-vcs graphs.

Definition 4. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let $W = \{w_1, w_2, \ldots, w_n\}$ and $Z = \{z_1, z_2\}$ be sets of vertices used in the constructions described below.*

1. *$F_0(G)$ has vertex set $V(G) \cup W$ where $V(G)$ induces G and $N(w_i) = N_G(v_i)$ for $1 \leq i \leq n$.*
2. *$F_1(G) = F_0(G) + z_1$ where $N(z_1) = W$.*
3. *$F_2(G) = F_1(G) + z_2$ where $N(z_2) = W$.*

$F_1(G)$ is the well-known Mycielski graph [8] of G.

Observation 5. *Let G be a graph with $\chi(G) = k$,*

1. *$\chi(F_0(G)) = k$ and $\chi(F_1(G)) = \chi(F_2(G)) = k+1$.*
2. *$F_0(G)$ is k-vcs.*
3. *If G is k-vcc, then $F_1(G)$ is $(k+1)$-vcc.*
4. *If G is k-vcs, then $F_2(G)$ is $(k+1)$-vcs.*

Proof:

1. A k-coloring of G can be extended to one of $F_0(G)$ by assigning w_i the same color as v_i, $1 \leq i \leq n$. It is known the Mycielski graph $F_1(G)$ has chromatic number $k+1$ and a $k+1$-coloring of $F_1(G)$ can be extended to one of $F_2(G)$ by assigning z_2 the same color as z_1.

2. For any vertex v of $F_0(G)$, $F_0(G) - v$ has a subgraph isomorphic to G so has chromatic number k. Therefore $F_0(G)$ is k-vcs.

3. $F_1(G) - z_1$ is isomorphic to $F_0(G)$ so z_1 is vcc in $F_1(G)$. Consider vertex v_i, $1 \le i \le n$. Since G is vcc, there is a k-coloring of the vertices of $V(G)$ such that v_i is the only vertex with color k. In $F_1(G) - v_i$ color every vertex of W with k and color z_1 with 1. Thus $\chi(F_1(G) - v_i) = k$ and v_i is vcc in $F_1(G)$. Finally consider vertex w_i, $1 \le i \le n$. Color the vertices of $F_1(G) - w_i$ so that v_i is the only vertex of $V(G)$ with color k, w_j, $j \ne i$, is colored the same as v_j, and z_1 is colored k. Thus w_i is vcc in $F_1(G)$.

4. For $x \in \{z_1, z_2\}$, $F_2(G) - x$ is isomorphic to $F_1(G)$ so x is vcs in $F_2(G)$. Let $x \in \{v_i, w_i\}$, $1 \le i \le n$. Note that $F_2(G) - \{v_i, w_i\}$ is isomorphic to $F_2(G - v_i)$. Thus $k+1 = \chi(F_2(G)) \ge \chi(F_2(G) - x) \ge \chi(F_2(G) - \{v_i, w_i\}) = \chi(F_2(G - v_i)) = \chi(G - v_i) + 1 = k + 1$ implying x is vcs in $F_2(G)$. □

While the values $m(k)$ and $M(k)$ are defined only for triangle-free graphs, the concept can be considered for all graphs. However, as is shown next, the extension to general graphs may be of little value.

Observation 6. *The minimum number of vertices in a k-vcc graph is k and in a k-vcs graph is $2k$.*

Proof: Any k-vcc graph has at least k vertices and the complete graph K_k is vcc so the first result holds. From Observation 5 Part 2, the $2k$ vertex graph $F_0(K_k)$ is k-vcs. The fact there must be at least $2k$ vertices in any k-vcs graph follows from Lemma 3. □

Theorem 7. *If G is a minimal k-vcs graph, then $\delta(G) \ge k - 1$.*

Proof: The result is obvious if $k \le 2$. Otherwise assume $\delta(G) \le k - 2$ and let v be a minimum degree vertex of G. Since G is minimal k-vcs, $\chi(G - v) = k$ and $G - v$ has a vcc vertex v'. Thus $G - v$ has a k coloring in which v' is the only vertex colored k. Since in G vertex v has no more than $k - 2$ neighbors, the coloring of $G - v$ can be extended to G by giving v some color less than k; hence v' is the only vertex colored k in G and therefore is vcc, a contradiction. □

Theorem 8. *Let graph G have an independent set of vertices B. Then $\chi(G - B) \ge \chi(G) - 1$.*

Proof: The result is immediate when $k = 1$. Otherwise suppose $k \ge 2$ and there is an independent set of vertices B such that $\chi(G - B) \le \chi(G) - 2$. Then any $\chi(G) - 2$ coloring of $G - B$ can be extended to a $\chi(G) - 1$ coloring of G by coloring all the vertices of B with color $\chi(G) - 1$, a contradiction. □

A vertex v of a graph is called *triangle-free* if $N(v)$ is an independent set of vertices.

Lemma 9. *Let $G \neq K_{1,n}$ be a graph with triangle-free vertex v. Then $\chi(G - N[v]) \geq \chi(G) - 1$.*

Proof: The result is immediate if $\chi(G) \leq 2$ since $G \neq K_{1,n}$. Otherwise suppose $\chi(G-N[v]) \leq \chi(G)-2$. Any $\chi(G)-2$ coloring of $G-N[v]$ can be extended to a $\chi(G) - 1$ coloring of G by assigning the vertices of $N(v)$ to color $\chi(G) - 1$ and v to any color less than $\chi(G) - 1$, a contradiction. $\square$

Corollary 10. *Let $G \neq K_{1,n}$ be a triangle-free graph. Then for every vertex $v \in V(G)$, $\chi(G - N[v]) \geq \chi(G) - 1$.*

Lemma 11. *Let $G \neq K_2$ be a graph of chromatic number k with triangle-free vcc vertex v. Then $G - v$ is $(k-1)$-vcs.*

Proof: The result is true for $K_{1,n}$, $n \geq 2$, so assume $G \neq K_{1,n}$. Since v is vcc, $\chi(G-v) = k-1$. If $G-v$ is not $(k-1)$-vcs, there is a vertex v' which is vcc in $G-v$ so $\chi(G-\{v,v'\}) = k-2$. Thus there is a k-coloring of G in which v' is the only vertex colored $k-1$ and v the only vertex colored k. If v and v' are not adjacent, v can be recolored $k-1$, contradicting $\chi(G) = k$. If v and v' are adjacent, $v' \in N(v)$ so $\{v,v'\} \subseteq N[v]$ and $\chi(G - N[v]) \leq \chi(G - \{v,v'\}) = k-2$. Since the vertices of $N(v)$ are independent and $G \neq K_{1,n}$, it follows from Lemma 9 that $\chi(G - N[v]) \geq k-1$, a final contradiction that shows $G - v$ is $(k-1)$-vcs. $\square$

Corollary 12. *Let $G \neq K_2$ be a triangle-free graph of chromatic number k. Then for every vcc vertex $v \in V(G)$, $G - v$ is $(k-1)$-vcs.*

Lemma 13. *Let G be a k-vcs triangle-free graph, $k \geq 3$, with a vertex v such that $\chi(G - N[v]) = k-1$. Then $G - N[v]$ is a $(k-1)$-vcs graph.*

Proof: If $G - N[v]$ is not a $(k-1)$-vcs graph, it contains a vcc vertex w and has a $(k-1)$-coloring in which w is the only vertex colored $k-1$. This coloring can be extended to a k-coloring of G by coloring the vertices of $N(v)$ with k and v with any color less than $k-1$, which is possible since $k \geq 3$. Now w is the only vertex colored $k-1$ and hence is vcc in G, contradicting the fact that G is a k-vcs graph. $\square$

3 Relationships involving $m(k)$ and $M(k)$

For the remainder of this paper, all graphs are assumed to be triangle-free. This section determines bounds on $m(k)$ and $M(k)$. The first shows $M(k)$ lies strictly between $m(k)$ and $m(k+1)$ and hence implies $m(k)$ lies strictly between $M(k-1)$ and $M(k)$.

Theorem 14. *For $k \geq 2$, $m(k) < M(k) < m(k+1)$.*

Proof: Suppose G is a minimum k-vcs graph. For every vertex $v \in V(G)$, $\chi(G-v) = k$. Then $m(k) \leq |V(G-v)| < |V(G)| = M(k)$. Next assume G is a minimum $(k+1)$-vcc graph. From Corollary 12, $G-v$ is a k-vcs graph for any vertex $v \in V(G)$. Thus $M(k) \leq |V(G-v)| < |V(G)| = m(k+1)$. $\square$

The next result presents similar relationships between vcc values and vcs values, further indicating they have similar growth rates.

Theorem 15.

1. *For $k \geq 4$, $m(k-1) + k + 1 \leq m(k) \leq 2m(k-1) + 1$.*
2. *For $k \geq 3$, $M(k-1) + k + 1 \leq M(k) \leq 2M(k-1) + 2$.*

Proof:

1. Let G be a minimum k-vcc graph and $v \in V(G)$ a vertex of maximum degree $\Delta(G)$. It follows from Corollary 10 that $\chi(G - N[v]) = k - 1$. Since G is triangle-free, it is not complete. Since $k \geq 4$, it is not an odd cycle. Thus, using Brooks' Theorem, $k = \chi(G) \leq \Delta(G) = |N[v]| - 1$. Therefore $m(k-1) + k + 1 \leq |V(G)| - |N[v]| + k + 1 \leq m(k) - |N[v]| + |N[v]| = m(k)$. Next let G be a minimum $(k-1)$-vcc graph. By Observation 5 Parts 1 and 3, $F_1(G)$ is k-vcc so $m(k) \leq |V(F_1(G))| = 2|V(G)| + 1 = 2m(k-1) + 1$.
2. Let G be a minimum k-vcs graph and $v \in V(G)$ a vertex of maximum degree. G is not complete and it is not an odd cycle since such cycles are not vcs. From Corollary 10, $\chi(G-N[v]) \geq k-1$. If $\chi(G-N[v]) = k-1$, Lemma 13 and Brooks' Theorem imply $M(k-1) + k + 1 \leq |V(G-N[v])| + k + 1 = |V(G)| - (\Delta(G)+1) + k + 1 \leq M(k) - (k+1) + k + 1 = M(k)$. If $\chi(G-N[v]) = k$, Theorem 14 and Brook's Theorem imply $M(k-1) + k + 1 < m(k) + k + 1 \leq |V(G - N[v])| + k + 1 = |V(G)| - (\Delta(G) + 1) + k + 1 \leq M(k) - (k+1) + k + 1 = M(k)$. Now assume G is a minimum $(k-1)$-vcs graph. From Observation 5 Part 4, $F_2(G)$ is a k-vcs graph. Therefore $M(k) \leq |F_2(G)| = 2|V(G)| + 2 = 2M(k-1) + 2$. $\square$

The left inequality of Theorem 15 Part 1 was previously found by Chvátal [3]. The following theorem presents an upper bound on $M(k)$ that is helpful in determining $M(3)$ but is less useful for higher chromatic numbers.

Theorem 16. *For $k \geq 3$, $M(k) \leq 3m(k-1) + 2$.*

Proof: Let G be a minimum $(k-1)$-vcc graph. We construct a vcs graph H with $3|V(G)| + 2$ vertices. From Observation 5, $F_1(G)$ has $2|V(G)| + 1$ vertices and chromatic number k. Now H is formed by adding $n =$

$|V(G)|$ vertices $X = \{x_1, x_2, \ldots, x_n\}$ and a single vertex z_3 so H will have $3|V(G)| + 2$ vertices. The additional edges are given, for $1 \leq i \leq n$, by $N(x_i) = N_{F_1(G)}(v_i) \cup \{z_3\}$. It is easy to see $\chi(H) = k$. We must show H is vcs. Let $u \in V(H)$. If $u \in N[z_1] \cup N[z_3]$, $H - u$ contains a subgraph isomorphic to $F_1(G)$ and thus has chromatic number k. If $u = v_i$, $1 \leq i \leq n$, $(V(G) - \{v_i\}) \cup \{w_i\} \cup X \cup \{z_3\}$ induces a subgraph isomorphic to $F_1(G)$ so it has chromatic number k. Thus H is vcs and the theorem follows. □

4 $M(3)$

The value of $M(3)$ is determined and then all graphs that achieve it are found.

Theorem 17. $M(3) = 8$.

Proof: Since $m(2) = 2$ and $M(2) = 4$, Theorem 15 Part 2 shows $M(3) \geq 8$ and Theorem 16 yields $M(3) \leq 8$. □

Lemma 18. *Let G be a minimum 3-vcs graph. Then* $2 \leq \delta(G) \leq \Delta(G) = 3$.

Proof: From Theorem 7, $\delta(G) \geq 2$. Let v be a vertex of maximum degree. By Corollary 10, $2 \leq \chi(G - N[v])$ and, since $\chi(G) = 3$, $\chi(G - N[v]) \leq 3$. If $\chi(G - N[v]) = 3$, $|V(G)| - (\Delta(G) + 1) \geq 5$ implying $\Delta(G) \leq 2$, meaning G is bipartite, a contradiction to G being 3-vcs. Thus $\chi(G - N[v]) = 2$ and, by Lemma 13, $G - N[v]$ is 2-vcs. Thus $G - N[v]$ has at least $4 = M(2)$ vertices so $|V(G) - N[v]| = 8 - \Delta(G) - 1 \geq 4$ so $\Delta(G) \leq 3$. Since G is not an odd cycle, $\Delta(G) = 3$. □

Lemma 19. *Let G be a minimum 3-vcs graph. Then G has a subgraph that is an 8-cycle.*

Proof: Lemma 18 and its proof show G has a degree 3 vertex so $G - N[v]$ has four vertices, has chromatic number 2, and is 2-vcs. Therefore, since $M(2) = 4$, $G - N[v]$ is a minimum 2-vcs graph. From the discussion following Definition 2, $G - N[v]$ is C_4, P_4, or $2K_2$. We examine each separately. Let $N(v) = W = \{w_0, w_1, w_2\}$ and $V(G - N[v]) = X = \{x_0, x_1, x_2, x_3\}$.

Suppose $G - N[v] = C_4$. The vertices of X can be 2-colored. Since there are no triangles, the vertices in W must be adjacent in X only to vertices of the same color so the same two colors suffice for W. Thus G can be 3-colored with only vertex v colored 3, contradicting that G is 3-vcs.

Next assume $G - N[v] = P_4$ where the path is given by $< x_0, x_1, x_2, x_3 >$. If x_0 and x_3 do not have a common neighbor, then X can be 2-colored and a contradiction arises as in the preceding case. Otherwise assume w_1 has neighbors x_0 and x_3. Since w_0 and w_2 are not adjacent, $G - N[w_1]$ must induce the path $< w_0, x_1, x_2, w_2 >$ or $< w_2, x_1, x_2, w_0 >$. Without loss of generality, assume the former so G contains the 8-cycle $< v, w_0, x_1, x_0, w_1, x_3, x_2, w_2 >$.

Finally let $G - N[v] = 2K_2$ where x_0 is adjacent to x_1 and x_2 is adjacent to x_3. At least one vertex in each K_2 must be adjacent to a common vertex in W. Without loss of generality assume w_1 is adjacent to x_1 and x_2. $G - N[w_1]$ is $2K_2$, since other possibilities have already been considered in the previous two cases, with vertex set $\{w_0, w_2, x_0, x_3\}$. Since w_0 and w_2 are not adjacent and x_0 and x_3 are not adjacent, we may assume without loss of generality that w_0 and x_0 are adjacent and so are w_2 and x_3. Then G contains the 8-cycle $< v, w_0, x_0, x_1, w_1, x_2, x_3, w_2 >$. $\Box$

The next theorem gives a characterization of minimum 3-vcs graphs. Given an 8-cycle with vertices in order $< v_0, v_1, v_2, v_3, v_4, v_5, v_6.v_7 >$, a *bisecting chord* is an edge $v_i v_{i+4}$ where arithmetic is modulo 8.

Theorem 20. *Graph G is a minimum 3-vcs graph if and only if the eight vertices of $V(G)$ contain an 8-cycle subgraph, $2 \leq \delta(G) \leq \Delta(G) = 3$, and the cycle has at least two bisecting chords.*

Proof: Suppose G is a minimum 3-vcs graph. From Lemmas 18 and 19, G has an 8-cycle subgraph and $2 \leq \delta(G) \leq \Delta(G) = 3$. If G has no bisecting chords it can be 2-colored since any chords must be of the form $v_i v_{i+3}$, a contradiction to $\chi(G) = 3$. If G has only one bisecting chord $v_i v_{i+4}$, removal of v_i leaves a bipartite graph, contradicting G being 3-vcs. Thus G must have at least two bisecting chords.

Now assume G has an 8-cycle subgraph, $2 \leq \delta(G) \leq \Delta(G) = 3$, and at least two bisecting chords. Since a bisecting chord creates 5-cycles and $8 < 11 = m(4)$, $\chi(G) = 3$. For every vertex $v \in V(G)$, $G - v$ contains a 5-cycle so $\chi(G - v) = 3$ and G is 3-vcs. $\Box$

We now show explicitly all minimum 3-vcs graphs.

Theorem 21. *Graph G is a minimum 3-vcs graph if and only if it is isomorphic to one of the graphs in Figure 2.*

Proof: From Theorem 20, the graphs of Figure 2 are minimum 3-vcs graphs and any other minimum 3-vcs graph G must contain one of these as a proper subgraph. The only way this might be possible is if chords are added to one of these graphs. This is impossible in Figure 2(a) since $\Delta(G) = 3$ and adding an edge to the graph of (b) results in that of (a). In the graph of (c) any permissible additional chord is bisecting and adding one or two produces either graph (b) or graph (a). Possible additional chords to the graph of (d) are the bisecting v_2v_6 and v_3v_7 and nonbisecting chords v_2v_7 and v_3v_6. Without loss of generality, add chord v_2v_7 and observe that the result contains the 8-cycle $\{v_0, v_4, v_3, v_2, v_1, v_5, v_6, v_7\}$ with bisecting chords v_0v_1, v_2v_7, and v_4v_5 and the graph is isomorphic to that in (b). $\square$

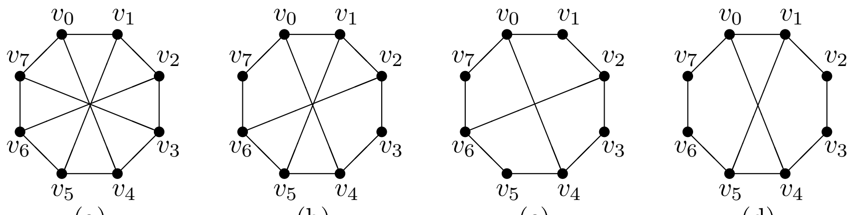


Figure 2: Minimum 3-vcs graphs

5 $M(4)$

In this section lower and upper bounds for $M(4)$ are obtained. Theorem 15 Part 2 implies, using $M(3) = 8$, that $13 \leq M(4)$. Employing Theorem 16 and $m(3) = 5$ gives $M(4) \leq 17$. The two following subsections improve each of these bounds by one.

5.1 $M(4) \geq 14$

We first demonstrate the structure of a 13-vertex 4-vcs graph, if such a graph exists, is highly restricted. Then Theorem 23 shows there are no such graphs.

Lemma 22. *If G is a 4-vcs graph with 13 vertices, G is 4-regular.*

Proof: Since $M(4) \geq 13$, G must be minimum 4-vcs so, by Theorem 7, $\delta(G) \geq 3$. Therefore, for every $v \in V(G)$, $|V(G - N[v])| \leq 9 < 11 = m(4)$, implying $\chi(G - N[v]) = 3$. By Lemma 13, $G - N[v]$ is a 3-vcs graph, so $|V(G)| - (\Delta(G) + 1) \geq M(3) = 8$ and thus $\Delta(G) \leq 4$. Suppose $v \in V(G)$ has degree 3 and w is any neighbor of v. Since G is 4-vcs, $\chi(G - w) = 4$ and v has degree 2 in $G - w$. Thus removing v from $G - w$ does not lower the chromatic number so the 11-vertex graph $G - \{v, w\}$ has chromatic number 4. But the Grötzsch graph is the only triangle-free 4-chromatic graph with 11 vertices and it has a degree 5 vertex, a contradiction. We conclude G has no degree 3 vertices and hence must be 4-regular. $\square$

Let G be a 13-vertex 4-vcs graph. Since it is 4-regular, its structure can be described as follows for any vertex $v \in V(G)$. Let $W = \{w_1, w_2, w_3, w_4\}$ be the set of four vertices adjacent to v. Each w_i, $1 \leq i \leq 4$, has three edges to the 8-vertex graph $G - N[v]$ which is 3-vcs as shown in the previous proof. These facts are assumed in the proof of the following theorem.

Theorem 23. $M(4) \geq 14$.

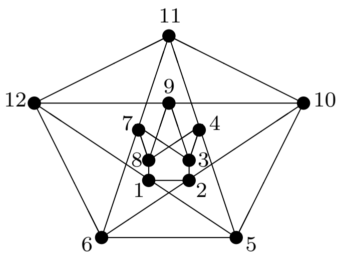
Proof: If $M(4) = 13$, a smallest vcs-graph G is 4-regular from Lemma 22 and has 26 edges. Four of these edges join v to W and another 12 connect W with $G - N[v]$ for a total of 16, leaving 10 for $G - N[v]$. Since $G - N[v]$ is 3-vcs and $M(3) = 8$, $G - N[v]$ must be one of the graphs (c) and (d) in Figure 2. Note that each degree 3 vertex in these graphs has exactly one edge to W and at most two w_i's can have edges to two degree three vertices.

Consider first that $G - N[v]$ is isomorphic to Figure 2(c). If, for some i, $1 \leq i \leq 4$, w_i is adjacent to two degree three vertices, we may assume without loss of generality they are v_0 and v_6. Now let c be a coloring of $G - N[v]$ defined as follows: $c(v_0) = c(v_6) = 1$, $c(v_2) = c(v_4) = 2$, and $c(v_1) = c(v_3) = c(v_5) = c(v_7) = 3$. Then w_i is adjacent only to vertices of $G - N[v]$ colored 1 or 3 so it can be colored 2. If a second vertex w_j of W is adjacent to two degree 3 vertices, they must be v_2 and v_4 so w_j is adjacent only to vertices of $G - N[v]$ colored 2 or 3 and hence can be colored 1. Any vertex of W adjacent to at least two degree 2 vertices of $G - N[v]$ can be colored 1 or 2. Therefore v can be colored 3 and G is 3-colorable, a contradiction.

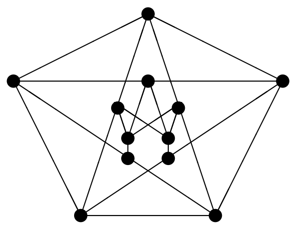
Suppose next that $G - N[v]$ is isomorphic to Figure 2(d). No w_i can be adjacent to three degree 2 vertices of $G - N[v]$ so each must be adjacent to exactly one degree 3 vertex. Without loss of generality, assume w_1 is adjacent to v_0, w_2 to v_5, w_3 to v_1, and w_4 to v_4. Each w_i must be adjacent to one vertex from $\{v_2, v_3\}$ and one from $\{v_6, v_7\}$. Note that, since w_1 is adjacent to v_0, it must be adjacent to v_6. Similarly w_2 must be adjacent to v_7, w_3 to v_3, and w_4 to v_2. We now define a coloring c of $G - N[v]$. Set $c(v_0) = c(v_5) = 3$, $c(v_6) = 2$, and $c(v_7) = 1$. If w_1 is adjacent to v_2, set $c(v_2) = c(v_4) = 2$ and $c(v_1) = c(v_3) = 1$. If w_1 is adjacent to v_3, set $c(v_2) = c(v_4) = 1$ and $c(v_1) = c(v_3) = 2$. In either case we may extend coloring c to G by $c(w_1) = 1$, $c(w_2) = 2$, $c(w_3) = c(w_4) = 3$, and $c(v) = 4$. Thus either $\chi(G) = 3$ or $\chi(G) = 4$ and there is a coloring in which v is the only vertex colored 4 meaning v is vcc in G. Each provides the final contradiction to G being 4-vcs, so $M(4) \geq 14$. $\square$

5.2 $M(4) \leq 16$

To show $M(4) \leq 16$, it is sufficient to demonstrate a 16-vertex graph of chromatic number four where the removal of any vertex leaves a graph also of chromatic number four. The graph used for this purpose is based on the Chvátal graph shown in Figure 3(a) which first appeared in Chvátal [2]. This graph is the smallest 4-regular triangle-free graph having chromatic number four. The same reference indicates that the graph of Figure 3(b), obtained by removing from the Chvátal graph the edge between vertices 1 and 2 of Figure 3(a), still has chromatic number four.



(a) Chvátal graph



(b) Chvátal graph with edge 12 removed

Figure 3: Two 12-vertex triangle-free graphs of chromatic number 4

There are several alternative depictions of the Chvátal graph, one of which is that shown in Figure 4(a). The equivalence of the two representations is shown by the vertex labelings in Figures 3(a) and 4(a). The graph in Figure 4(b) corresponds to that in Figure 3(b).

The graph of Figure 5(a) is a supergraph of the Chvátal graph and represents the first step in developing the vcs graph. Let $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ form an 8-cycle in that order with four chords $v_i v_{i+4}$ for $0 \leq i \leq 3$, $W = \{w_0, w_1, w_2, w_3\}$ induce a 4-cycle, and $X = \{x_0, x_1, x_2, x_3\}$ induce a second 4-cycle. Arithmetic on the indices of V is modulo 8 while on the indices of W and X it is modulo 4. The remaining edges are given by the open neighborhoods $N(w_i) = \{v_{2i}, v_{2i+3}, w_{i-1}, w_{i+1}\}$ and $N(x_i) = \{v_{2i+1}, v_{2i+4}, x_{i-1}, x_{i+1}\}$. The graph of Figure 5(b), which is the one we need, has the three additional edges w_0x_3, w_1x_1, and w_2x_2, shown dashed in the figure. Call this graph G. The chromatic number of G is four because it contains the Chvátal graph and it can't be larger since $m(5) = 22$.

We now show, for any vertex u of G, that $G - u$ has chromatic number four.

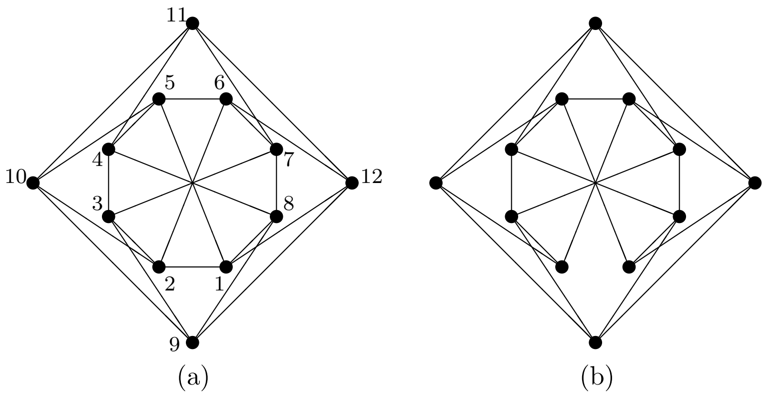


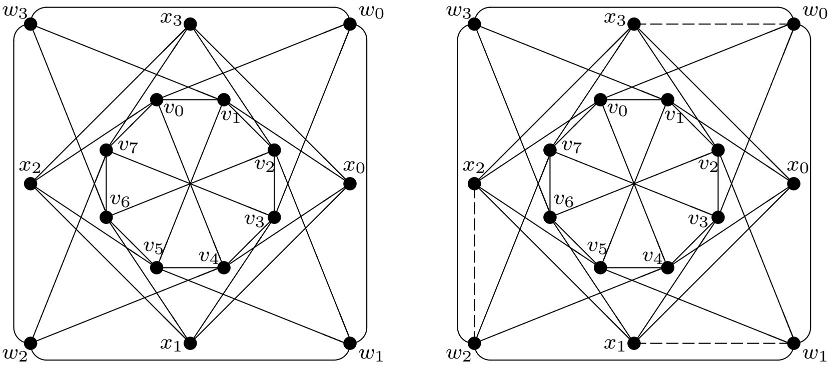
Figure 4: Alternative renderings of Chvátal graph with and without one edge

If $u \in W$, $V \cup X$ induces the Chvátal graph. Similarly, if $u \in X$, $V \cup W$ induces the Chvátal graph.

When $u \in V$, the argument is more complicated. For six of the v_i there is a similar discussion. For illustration, consider $G - v_0$. Notice that v_0 is adjacent to x_2 and w_0, to v_4, and to v_1 and v_7. We search for a vertex in $W \cup X$ adjacent to one of the first two, v_4, and one of the last two. Vertex w_2 is adjacent to x_2, v_4, and v_7, so it satisfies the requirement. We now replace v_0 by w_2. The subgraph induced by $(V - \{v_0\}) \cup \{w_2\} \cup X$ is isomorphic to the graph of Figure 4(b) and hence has chromatic number four. We will indicate this by the vertex set that induces the desired subgraph, that is, $(V - \{v_0\}) \cup \{w_2\} \cup X$ in this case. The remaining five results are $(V - \{v_2\}) \cup \{x_1\} \cup W$,$(V - \{v_3\}) \cup \{x_3\} \cup W$, $(V - \{v_4\}) \cup \{x_2\} \cup W$, $(V - \{v_6\}) \cup \{w_1\} \cup X$, and $(V - \{v_7\}) \cup \{w_0\} \cup X$.

The graphs $G - v_1$ and $G - v_5$ appear to be more difficult to check and our approach has been to exhaustively show four colors are required. In each case we select a 5-cycle and color it with each of five 3-colorings that together represent all possibilities. For each of these five colorings we give a sequence, not necessarily the shortest, of additional vertices that, in the given order, have the coloring forced upon them.

For $G - v_1$ there is a small deviation from the above approach for one of the colorings of the 5-cycle. A decision must be made when reaching a vertex whose color is not forced. Then that vertex must be tested with two different colors so there are two sequences necessary to prove some vertex needs four colors. The selected 5-cycle is $< v_4, v_5, v_6, v_7, w_2 >$. Each of the



(a) 16-vertex supergraph of Chvátal graph (b) 16-vertex *vcs* graph

Figure 5: Supergraphs of Chvátal graph

following lines lists the colors assigned to those vertices in order, then the sequence of additional vertices whose color is forced until the final vertex which must be colored 4.

12123 v_3, x_1, x_0, x_2, x_3

31212 x_2, x_1, x_3, x_0

23121 $x_2, x_1, w_1, v_2, x_0, x_3$

12312 $w_3, x_1 = 1, x_2, x_3, w_0, w_1$ and $w_3, x_1 = 2, v_3, w_0, x_0, x_3$

21231 $v_3, x_1, x_0, x_3, w_0, w_3$

For $G - v_5$ the selected 5-cycle is $< v_0, v_1, v_2, v_3, w_0 >$.

12123 v_4, x_0, x_1, x_2, x_3

31212 v_4, x_0, x_3, x_1, x_2

23121 $w_3, v_6, v_7, x_1, x_2, w_2$

12312 $w_1, x_3, x_0, x_1, v_4, x_2, w_2$

21231 w_1, v_7, v_6, w_3, w_2

Since the five possible colorings represent all cases, it follows that the chromatic numbers of $G - v_1$ and $G - v_5$ must both be four. In summary we have the following:

Theorem 24. *$M(4) \leq 16$.*

6 Open Problems

1. Determine $M(4)$.
2. Improve the bounds of Theorems 14 and 15.
3. Find an analytical characterization of minimum 4-vcs graphs.
4. Find an analytical characterization of minimum 5-vcc graphs and minimum 5-vcs graphs.
5. More generally, find $m(k)$ and $M(k)$ for $k \geq 6$.
6. Find bounds on the number of edges in minimum k-vcs graphs and minimum k-vcc graphs.
7. The value $m(k)$ can be interpreted as the minimum number of vertices in a k-chromatic triangle-free graph such that there is at least one vertex of every color. In view of Lemma 3, it is seen that $M(k)$ is the minimum number of vertices in a k-chromatic triangle-free graph such that there are at least two vertices of every color. This suggests defining $M_i(k)$ as the minimum number of vertices in a k-chromatic triangle-free graph such that there are at least i vertices of every color. Thus $M_1(k) = m(k)$ and $M_2(k) = M(k)$. Find bounds on $M_i(k)$ and any structural interpretation.

References

[1] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, CRC Press, Boca Raton, 2009.

[2] V. Chvátal, The smallest triangle-free 4-chromatic 4-regular graph, *Journal of Combinatorial Theory* 9 (1970) 93-94.

[3] V. Chvátal, The minimality of the Mycielski graph, in *Graphs and Combinatorics, Proceedings of the Capital Conference on Graph Theory and Combinatorics*, R. A. Bari and F. Harary (eds.), George Washington University, June 18-22, 1973, *Lecture Notes in Mathematics* 406 (1974) Springer, Berlin, pp 243-246.

[4] P. Erdös, Graph theory and probability, *Canadian Journal of Mathematics* 11 (1959), 34-38.

[5] J. L. Gross and J. Yellen (Eds.), *Handbook of Graph Theorey* Second Edition, CRC Press, Boca Raton, 2014.

[6] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1969.

[7] T. Jensen and G. F. Royle, Small graphs with chromatic number 5: a computer search, *Journal of Graph Theory* 19 (1995) 107-116.

[8] J. Mycielski, Sur le coloriage des graphs, *Colloquium Mathematicum* 3 (1955) 161-162.