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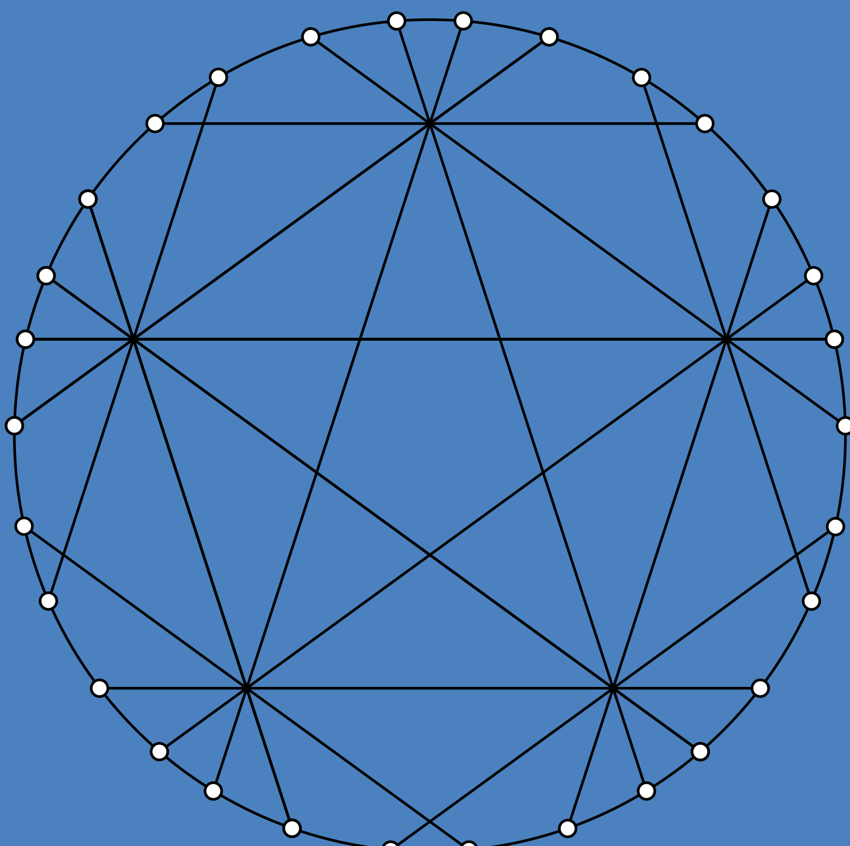

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On Edge extremal properties of Hamilton laceable bigraphs

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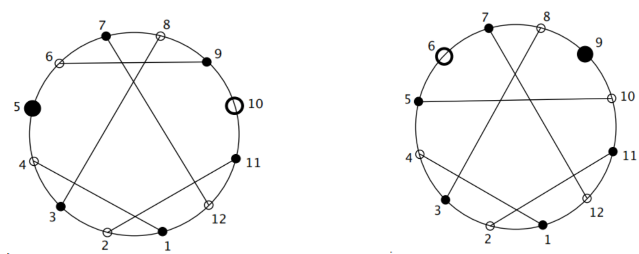
Abstract:

Hamilton laceable is the analogous property for bipartite graphs to Hamilton connected for non-bipartite graphs. Both are concerned with the global connectivity of a graph. Knowing how many or how few edges are required for a graph to exhibit edge extremal behaviors, such as edge minimal, edge critical or edge stable with respect to Hamilton laceablility, is vital to understanding the property. It is therefore important to have the best possible answers to these questions for Hamilton laceable bigraphs. For equitable bigraphs on 2m vertices, edge critical and edge stable share a well populated 3m edge upper bound for the one and lower bound for the other. The question of how far from this common bound extremal cases exist is partially answered by constructions showing suitably redefined edge stable graphs can have at least 4m - 4 edges and that edge critical graphs exist with as few as 3m - O(m/3) edges.

1: Introduction:

For any graph property, Hamilton laceability here, a natural question is: What is the least number of edges a graph on a specified vertex set must have in order to exhibit the property? The minimum is taken over all graphs having the property on the vertex set. While this edge minimum is unique, *edge minimal* graphs realizing it may not be. The Hamilton laceable graphs in Figure 1 illustrate this. They each have the minimum number, 17, of edges a graph on 12 vertices must have to be Hamilton laceable, but in spite of their similar appearance cannot be isomorphic since graph 1a hosts 125 Hamilton paths while graph 1b only hosts 106.



a b

Figure 1

Given a graph which has a defining property there are a pair of mutually exclusive edge extremal attributes it could also have. If no edge can be deleted without causing the punctured graph to cease to have the defining property the

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graph is said to be *edge critical* with respect to the property. Conversely, if the punctured graph still has the property no matter which edge is deleted, the graph is said to be *edge stable* with respect to the property. If a graph is edge minimal, it is by definition edge critical. The converse need not be true as illustrated by the graph in Figure 2 which is edge critical, but not edge minimal since it has 18 edges instead of the minimum of 17.

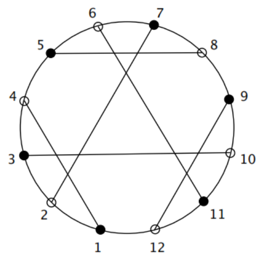
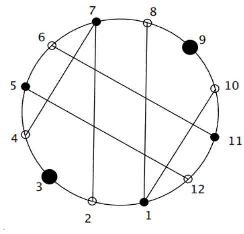


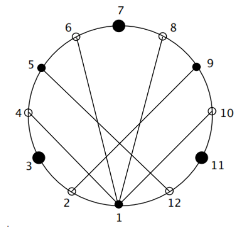
Figure 2

Hamilton paths can only exist in a bipartite graph if the parts have the same cardinality, *equitable*, or differ by one, *nearly equitable*. We will only be concerned with equitable bigraphs. For equitable bigraphs in which each part contains m vertices, 3m appears to be a hard boundary between the number of edges edge critical or edge stable graphs can or must have respectively in order for the graph to be Hamilton laceable. There are many cubic bigraphs realizing these graph specific properties. For example, m-prisms when $V \equiv 0 \bmod 4$ and m-Mobius ladders when $V \equiv 2 \bmod 4$ provide edge stable examples for all $m \geq 3$. Similarly Weisstein's m-crossed prisms [6] when $V \equiv 0 \bmod 4$ and Simmons' sausage graphs [5] when $V \equiv 2 \bmod 4$ provide edge critical examples for all $m \geq 5$.

It is meaningful to ask how many edges an edge critical graph with respect to Hamilton laceability can have since $K_{m,m}$ is Hamilton laceable for $m \geq 3$, but is edge stable, not edge critical. As was pointed out earlier there are numerous cubic edge critical examples, i.e. cases with 3m edges, but there is no known example in which the total number of edges is greater than 3m. There are edge critical bigraphs with 3m edges which have vertices of degree greater than 3, i.e. which are non-cubic, but in every known such case there are exactly the required number of vertices of degree 2 to make the total number of edges be 3m. The Hamilton laceable graphs in Figure 3 are both edge critical; 3a has two vertices of degree 4and hosts 156 Hamilton paths while 3b has a single vertex of degree 6 and hosts only 132. In both cases though the vertices of degree greater than 3are offset by the requisite number of degree 2 to make the total number of edges be 18 = 3m.



3a



3b

Figure 3

It is conjectured 3m is a hard upper bound for edge critical bigraphs on 2m vertices [2]. In a later section it will be proven that 3m is a hard lower bound for edge stable bigraphs.

Since any super-graph formed by adding edges to an edge stable graph is also edge stable, it is meaningless to ask how many edges an edge stable graph can have. There is a meaningful edge maximum question though. What is the maximum number of edges an edge stable bigraph can have subject to the condition it does not contain an edge stable sub-graph; in other words that it is not a super-graph of a stable graph? The graph on 10 vertices in Figure 4 is edge stable, has one more edge than a cubic graph would (16 instead of 15) and does not properly contain any edge stable sub-graph. It was purposely constructed to minimize the effort required to show this. The graph has a total of 208 Hamilton paths. If any of the eight edges incident on a vertex of degree 4 is deleted the punctured graph has 112 and if any of the eight edges not incident on a vertex of degree 4 is deleted only 70. The important thing is that the punctured graphs are all Hamilton laceable but none of them are edge stable.

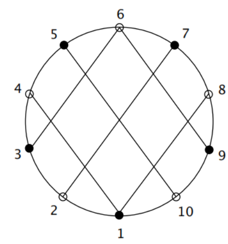


Figure 4

The question of just how large this edge bound can be is a main topic in this paper. The reason for mentioning it in the Introduction though is to justify an extension to common graph nomenclature. The complete description of an edge stable graph which has no proper edge stable sub-graph is that the graph is edge stable with respect to Hamilton laceablility and edge critical with respect to edge stability, in other words it is edge extremal with respect to two different graph properties; Hamilton laceable and edge stable. There is no existing term for this double condition, which we will call critically edge stable. The graph in Figure 4 is critically edge stable.

It should be noted the terminology of critically edge stable is equally applicable to any other graph property. Being edge stable with respect to a graph property is still a meaningful attribute of a specific graph, it is just not meaningful when considering the maximum number of edges such graphs can have. There, edge stable needs to be replaced by critically edge stable.

2: Quasi-cubic bigraphs:

A connected bigraph on 2m vertices is *quasi-cubic* if every edge has at least one endpoint of degree 3 and none of degree less than 3. Of course if both endpoints of every edge are of degree 3 the graph is simply cubic. The graph in Figure 4 is the smallest non-degenerate quasi-cubic bigraph. Given the importance of these graphs to the analysis in the next section, some of their properties will be derived here.

The vertices in parts A and B can be further partitioned into sets of vertices of degree 3, A^3 and B^3, and those of degree greater than 3, A^+ and B^+. Since there can be no edges between parts A^+ and B^+, quasi-cubic bigraphs can be schematically represented:

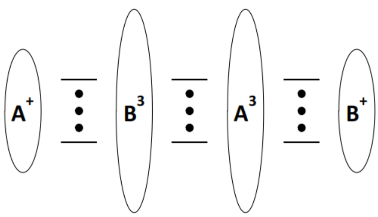


Figure 5

where edges can only connect vertices in adjacent sets. Let a denote the cardinality of A^+ and b the cardinality of B^+. In a degenerate case where one of the + sets is empty it is easy to show that either the other + set is also empty, so the graph is simply a cubic, or else the adjacent set of vertices of degree 3 is empty. Assume $b = 0$ and A^3 has $m\text{-}a > 0$ vertices in it. $3m - 3(m\text{-}a) = 3a$ edges are available to link to the a vertices in A^+. But by the definition of quasi-cubic graphs, every vertex in A^+ has at least 4 edges on it, i.e $\geq 4a$ edges which is a contradiction. Therefore if $b = 0$ and $m\text{-}a > 0$, $a = 0$. The conclusion is that if one of the + sets is empty either both are and the graph degenerates to a simple cubic or else the graph consists of a set of m-a vertices of degree 3 with the 3(m-a) edges on them apportioned to the a vertices in the complementary set in such a way that they all have degree 4 or greater. The smallest such example is $K_{3,4}$. The interesting cases are the non-degenerate ones in which none of the vertex sets are empty. The smallest example, with m = 5 and a = b = 1, is shown in Figure 6;

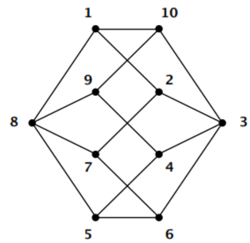
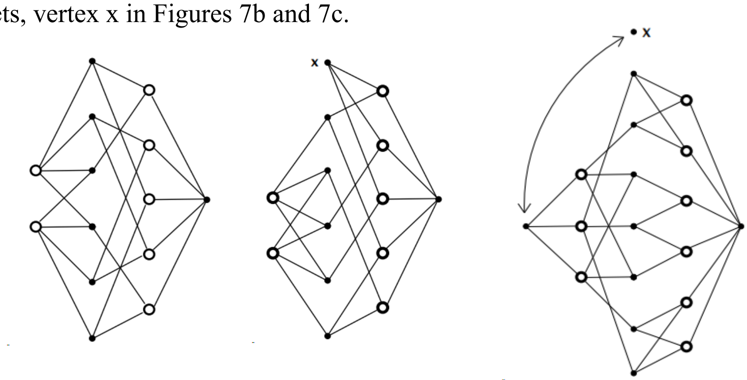


Figure 6 (G)

which is just an alternative representation of the graph in Figure 4.

An important parameter for quasi-cubic graphs is the total number of edges, E, in the graph. The key to computing E is the sum of the excess degrees of the vertices, i.e. exceeding 3, in A^+ and B^+. Since the degree of vertices in A^+ and B^+ is at least 4, $\Sigma_A \geq a$ and $\Sigma_B \geq b$. In G, Figure 6, $\Sigma_A = a = \Sigma_B = b = 1$.

There are $3a + \Sigma_A$ edges between A^+ and B^3. The number of edges on vertices in A^3 is 3(m-a) so the total number of edges in the graph is
$E = 3a + \Sigma_A + 3(m\text{-}a) = 3m + \Sigma_A$. By symmetry, $E = 3m + \Sigma_B$ and hence
$\Sigma_A = \Sigma_B = \Sigma$. Therefore, in order for a non-degenerate quasi-cubic bigraph to exist on a vertex partition as shown in Figure 5 the sums of the excess degrees of vertices must be the same for A^+ and B^+. The number of edges in the graph will be $3m + \Sigma$. Quasi-cubic bigraphs can be quite complex, even though the conditions defining them are simple; a and b can differ, Figures 7a, 7b and 7c, or the three edges on a vertex in either A^3 or B^3 can have all their endpoints in a single one of the adjacent sets, vertex x in Figures 7b and 7c.



a b

m = 7, a = 2, b = 1, $\Sigma = 2$

c. m = 9, a = 3, b = 1, $\Sigma = 3$

Figure 7

Using only these simple observations it is easy to show that as m gets large the maximum number of edges a quasi-cubic bigraph on 2m vertices can have is 6m - 19. By symmetry, in a bigraph with a maximum number of edges, a = b. To maximize Σ connect every vertex but one in A^+ to all m - b vertices in B^3 and that one to m-b-1. This leaves a single edge between A^3 and B^3 to connect the graph. Since the vertices in A^3 are of degree 3, a = 3. $\Sigma = a(m-b-3) - 1$, so $E = -a^2 + (m-3)a + 3m - 1$ which is maximized at $E = 6m - 19$ for all $m \geq 8$. Only this specific quasi-cubic bigraph has 6m - 19 edges which is clearly not Hamilton laceable since the endpoints of the single connecting edge between A^3 and B^3 are a cut-set in the graph.

At the opposite extreme, since $\Sigma \geq 1$, a quasi-cubic bigraph on 2m vertices must have at least 3m + 1 edges. Bigraphs realizing this bound for all $m \geq 5$ are easily constructed using the graph G in Figure 6. The edge pair 1-10 and 5-6 in G hosts ladders [3], meaning that if the end rungs of the ladder L_k are spliced into edges 1-10 and 5-6 the resulting graph, $G \cup L_k$, on 2(m + k) vertices will also be Hamilton laceable. If k is odd the ladder must be spliced with a half twist and if k is even with no twist to preserve parity of the vertices in parts A and B [3]. In addition, $G \cup L_k$ is edge stable since both G and L_k are.

Determining whether a particular set of parameters correspond to a realizable quasi-cubic graph is essentially a problem of constrained partitions. The degree of any vertex in A^+ is at least 4 and no greater than m-b. Similarly the degree of any vertex in B^+ is at least 4 and no greater than m-a. $3a + \Sigma$ edges connect vertices in A^+ to vertices in B^3 which leaves $3(m-b) - (3a + \Sigma) = 3(m-a-b) - \Sigma$ edges to connect between vertices in A^3 and B^3. These edges must be apportioned so that all of the vertices in A^3 and B^3 have degree 3. Any constrained partition satisfying these conditions defines a realizable quasi-cubic bigraph. Generally there will be many non-isomorphic bigraphs realizing a given set of parameters. For example the parameter set m = 5, a = b =1 has two realizations; the one shown in Figure 6 and the one in Figure 8. The first is Hamilton laceable and the latter is not since the vertices of degree 4 are a cut-set.

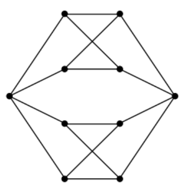


Figure 8

Of interest here are quasi-cubic bigraphs that are also Hamilton laceable, like the one shown in Figure 6. Some are easily shown to not be Hamilton laceable, i.e. the cut-set argument for the maximal edge cases or the graph in Figure 8, but in general the only way to show one is Hamilton laceable is to exhibit constructions for Hamilton paths between all pairs of vertices in different parts. An exception are the graphs obtained by splicing ladders into host pairs of edges in Hamilton laceable bigraphs when that is possible.

3: Critically edge stable bigraphs:

The object in this section is to show that for m $\geq$ 5 critically edge stable bigraphs can have as many as 4m-4 edges. This doesn't rule out the possibility they could have more, only that they can have this many.

Theorem 1: If G is edge stable with respect to Hamilton laceability the cardinality of each of the parts and the degree of each of the vertices is at least 3.
Proof:
If a part contained only a single vertex, deleting any edge would disconnect the graph so the punctured graph could not be Hamilton laceable. Therefore each part contains at least two vertices. Assume one contains exactly two, a and b. Every vertex in the other part must be connected by an edge to both a and to b, otherwise there would be an edge whose deletion would disconnect the graph etc. Delete an arbitrary edge, say a-x. The degree of x in the punctured graph will be 1, i.e. it can not be an interior point in any path. But there is at least one other vertex in the part containing x which by the assumption that G is edge stable must have at least one Hamilton path with it as an endpoint. Therefore the cardinality of each part must be at least 3 in order for G to be edge stable. Exactly the same argument suffices to show that no vertex can be of degree 2. Since the degree of each vertex is at least 3 and edge stable cubic bigraphs exist for all m $\geq$ 3, 3m is a hard lower bound for the number of edges.
Corollary: An edge stable quasi-cubic bigraph is critically edge stable since the deletion of any edge will result in a graph with at least one vertex of degree 2.

The utility graph is easily shown to be edge stable and is therefore the smallest example of an edge stable graph. By the Corollary it is also critically edge stable. The m-prisms, m-Mobius ladders, m-crossed prisms and sausage graphs invoked earlier to show that the 3m bound was achievable for edge stable and edge critical graphs are also critically edge stable for the same reason since they are all cubic.

The example of a critically edge stable graph in Figure 4 is clearly invariant under reflections and a 180° rotation but has other less obvious symmetries. As was remarked earlier, there are 70 Hamilton paths in each of the sub-graphs obtained by deleting any one of the eight edges not on a vertex of degree 4. This combined with the fact that the eight vertices of degree 3 all have the same number of Hamilton paths on them, 44, suggests they are likely transitive under an appropriate transformation. Figure 9 makes clear what that transformation is.

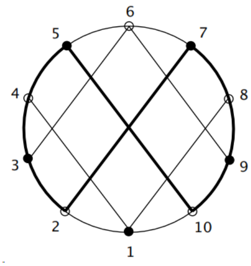


Figure 9

The bold path is a cycle on the eight vertices of degree 3. Vertices 1 and 6, which are in opposite parts, are each connected to the four vertices in the other part on C_8; this is even more readily apparent in the alternative representation of the graph as a quasi-cubic in Figure 6. The generalization would be a cycle C_{2n}, n = m-1, and a pair of vertices, one in each part, connected to all the vertices in the other part on $C_{2n.}$ As we will show these bigraphs, which we will call bi-wheel graphs, not because they are bigraphs but because of their structural analogy to a bicycle wheel[1], all have m-4 more edges than the 3m bound.

Theorem 2: Bi-wheel graphs are critically edge stable with respect to Hamilton laceability.

Note concerning notation: The path endpoints will always be labeled x and y. For consistency, in the figures vertex x will always be in the part denoted by small circles. Hub vertices, unless they are endpoints, will be labeled a and b. If only a single path endpoint is a hub vertex the convention will be that it is x and is hub vertex a. Vertices identified in the cycle C_{2n} to define paths in a construction will be labeled 1, 2, 3 etc.

Proof:
The first step is to prove the graphs are Hamilton laceable. There are three cases to be considered determined by whether the endpoints of the path are both hub vertices, one is or neither are.

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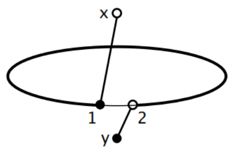
1. The structure of a bicycle wheel is different than that of a wagon wheel – after which wheel graphs are named. A wagon wheel connects points on the rim (a cycle) with spokes to a hub (a central point). A bicycle wheel has an even number of points on the rim which are alternately connected with spokes to two ends of a cylinder, i.e. to two hubs. By the very nature of the construction bi-wheels are bipartite, just as wheel graphs cannot be since they contain triangles. Bi-wheel graphs are not to be confused with the double-wheel graphs of Heule and Walsh [7] which connect the vertices on two cycles to a common hub vertex.

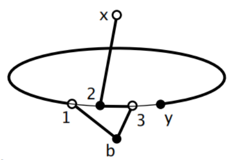
Case 1: *Both endpoints are hub vertices*.
Let 1-2 be an arbitrary edge in C_{2n}. The path x-1—2-y, where 1—2 indicates a traverse of $C_{2n,}$ is a Hamilton path.

Figure 9



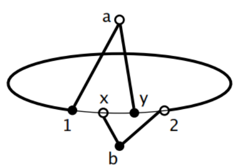
Case 2: *Only one endpoint is a hub vertex.*
Choose edges 1-2-3-y to be a contiguous run of three edges in C_{2n} ending on endpoint y. The path x-2-3-b-1—y is a Hamilton path, where again 1—y indicates a traverse of the unused vertices in C_{2n}.

Figure 10



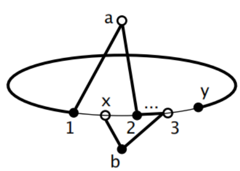
Case 3a: *Neither endpoint is a hub vertex*; *x and y are adjacent*
Vertices 1 and 2 are adjacent to x and to y respectively. Path x-b-2—1-a-y in Figure11a is a Hamilton path

Figure 11a



Case 3b: *Neither endpoint is a hub vertex*, *x and y are not adjacent*
There will be an even number of vertices between x and y; the path linking 2 to 3. Vertex 1 is adjacent to x. Path x-b-3—2-a-1—y in Figure 11b is a Hamilton path

Figure11b



If the orientation of x and y is reversed, use the same constructions on the arc linking them in the reverse direction.

Therefore bi-wheels are Hamilton laceable since we have exhibited a Hamilton path between every vertex in the one part and every vertex in the other.

The next step is to show the graph remains Hamilton laceable after an arbitrary edge is deleted. There are only two possibilities; either the deleted edge has a hub vertex as an endpoint or it doesn't, i.e. it is an edge in C_{2n}.

Case 4:
Both endpoints are hub vertices.
4a: *The deleted edge lies on a hub vertex.*
Edge 1-2 used in the path construction in Case 1was arbitrary. It is always possible to choose an edge not incident on the deleted edge, so the construction for a Hamilton path still holds.
4b. *The deleted edge is in C_{2n}.*
Let vertices 1 and 2 in Figure 6 be the endpoints of the deleted edge. The construction for a Hamilton path still holds.

Case 5:
Only one endpoint is a hub vertex.
5a. *The deleted edge lies on a hub vertex.*
There are two possibilities; either the deleted edge lies on x or else it lies on b. The same argument applies in either case. If the deleted edge is not one of the three connecting a hub vertex to vertices 1,2 and 3 in Figure 10, the Hamilton path construction still holds. If it is one of those three edges, simply make the same construction on the traverse of C_{2n} going in the opposite direction.
5b. *The deleted edge is in C_{2n}.*
If the deleted edge is on vertex y, then the Hamilton path construction in Figure 11, or the same construction on the opposite traverse of C_{2n}, will avoid the deleted edge and the Hamilton path construction still holds.
If the deleted edge 1-2 is not on vertex y assign the vertex labels to the deleted edge so that 1 is in the same part as y and vertex 3 is adjacent to y in the arc connecting 1 and y. The path x-1---3-b-2—y will be a Hamilton path. Note that the vertex labels on the deleted edge will be reversed depending on its orientation but the labeled path will be the same.

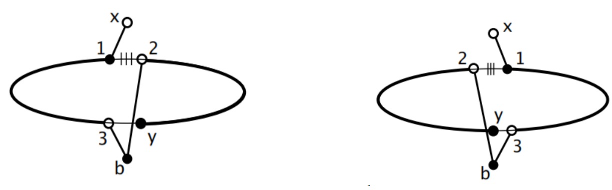


Figure 12

Case 6: *Neither endpoint is a hub vertex.*
There are several sub-cases to this case.
6a. *The deleted edge lies on a hub vertex and x and y are not adjacent.*
If the deleted edge is either of the edges b-2 or a-3 in Figure 11a , simply use the same construction on the other arc connecting x and y. If the deleted edge is either of the edges a-x or b-1 connect x by two edges to a point in the same part at distance 2 from it and use the same construction.
6b. *The deleted edge lies on a hub vertex and x and y are adjacent.*
If the deleted edge is either a-x or b-y in Figure 11b use the same construction on the other arc connecting x and y. If the deleted edge is either a-2 or b-1 connect x or y as needed to a vertex in the same part at distance 2 and use the extended path. For example if the deleted edge is a-2, use vertices 2, 3 and 4 to the right of y. The Hamilton path will be x-a-4—b-3-2-y.
6c. *The deleted edge is in C_{2n} and lies on at least one of x and y.*
The construction in Figure 11b still forms a Hamilton path. If the deleted edge lies on just one of x or y it is simple to cause one of the unused edges in Figures 11a or 11b to be the deleted edge. This may require using the same construction on the other arc connecting x and y.

The delicate case is when the deleted edge does not lie on either of x or y. There are two cases to be considered depending on whether x and y are adjacent or not.
6d. *The deleted edge is in C_{2n} and does not lie on either x or y; x and y are not adjacent.*
There are two paths, depending on the orientation of the deleted edge with respect to the orientation of the arc between x and y: x—1-a-3—4-b-2—y is a Hamilton path in the one case and x—1-b-4—3-a-2—y is in the other

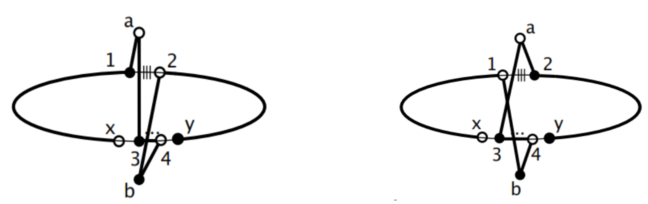


Figure 13

6e. *The deleted edge is in C_{2n} and does not lie on either x or y; x and y are adjacent.*
The key to the path construction in this case is to choose an edge 3-4 with the same orientation as the deleted edge. The paths x—4-a-2—3-b-1—y and x—4-b-2—3-a-1—y are Hamilton paths in the two cases. Note they are the same labeled sequence except the hub labels are interchanged.

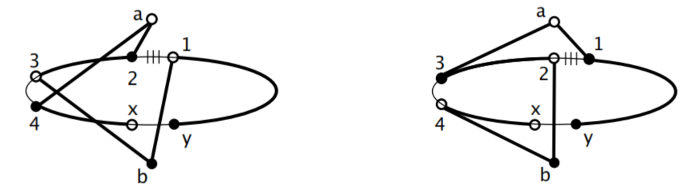


Figure 14

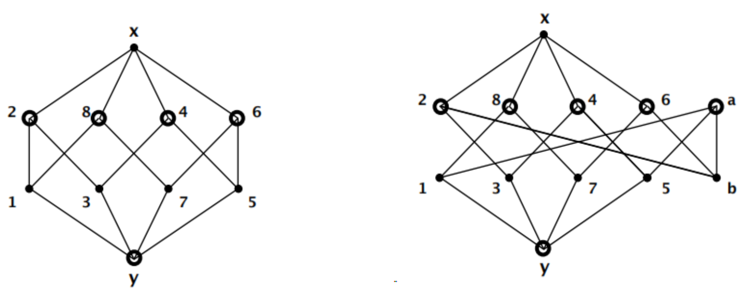
Therefore bi-wheels are edge stable with respect to Hamilton laceability. That they are critically edge stable follows from the Corollary to Theorem 1 since every edge in a bi-wheel lies on at least one vertex of degree 3.

Corollary:

There exist critically edge-stable quasi-cubic bigraphs on 2m vertices with $E = 4m - (k + 4)$ edges for all $m \geq 5 + k$ and all $k \geq 0$.

Proof:

The result is based on splicing ladders into bi-wheels [3]. For $k = 0$ the construction is simply the bi-wheel on $C_{2(m-1)}$ which was just shown to be critically edge stable with $4m - 4$ edges. Any pair of diametrically opposite edges in $C_{2(m-1)}$ will host ladders, so for $k > 1$ one end rung of the ladder is spliced into one of the host edges and the other end rung into the other. The case $k = 1$ would be a ladder with only one rung and no rails. To be logically consistent, this single rung would have to be spliced into both host edges. For example, in the bi-wheel on C_8, Figure 15a, let the pair of diametrically opposite edges in the cycle be edges 1-2 and 5-6 and let the single rung be an edge a-b. To splice a-b into both host edges edge 1-2 is broken into parts, 1-a and 2-b, which are joined to the rung as shown in Figure 15b. Similarly edge 5-6 is broken into parts 5-a and 6-b and joined to the rung. The resulting bigraph hosts 390 Hamilton paths and is critically edge stable. The same extension can be made to any other bi-wheel, adding two vertices and three edges while preserving the edge extremal properties.



a b

Figure 15

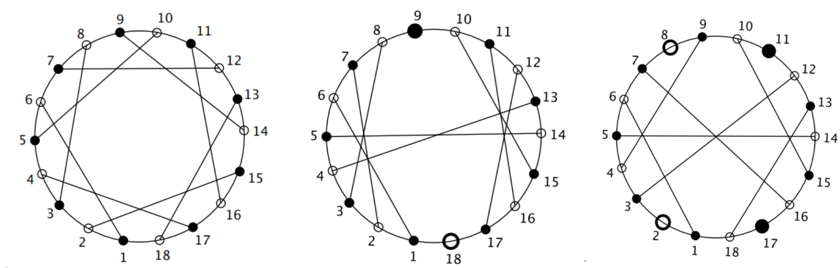
The significance of bi-wheel graphs being critically edge stable is that establishes an achievable bound for how many edges a critically edge stable graph can have. A bi-wheel on $C_{2(m-1)}$ vertices will have 2(m-1) vertices of degree 3 and two of degree m-1 or 4(m-1) edges in total, i.e. an excess of m-4 edges over the 3m lower bound for all m ≥ 5.

There are many edge-stable quasi-cubic bigraphs, and hence many that are critically edge stable, but none are known which have more than 4m - 4 edges, so none of them improve the bi-wheel bound.

3: Sparse, edge critical bigraphs:

For all m ≥3 there exist edge critical bigraphs on 2m vertices with only 3m - O(m/3) edges [4]. Since an important related result will be proven here a brief description of how the bound is derived is warranted.

The three graphs in Figure 16 are all edge critical; 16a since it is the polygonal bigraph $P_{9,2}$, 16b by direct verification using backtracking and 16c ($\mathbb{Q}$) since it will be shown to be edge minimal. Together they illustrate that bigraphs on the same vertex set can be edge critical with the number of missing edges ranging from none to the maximum possible. Graph 16c, $\mathbb{Q}$, which has four vertices of degree 2, i.e. has only 3m - 2 edges, is the key to constructing edge sparse bigraphs.



a b c ($\mathbb{Q}$)

Figure 16

$\mathbb{Q}$ can be redrawn in the form shown in Figure 17 to emphasize the sub-graph Q lying between the dashed vertical lines.

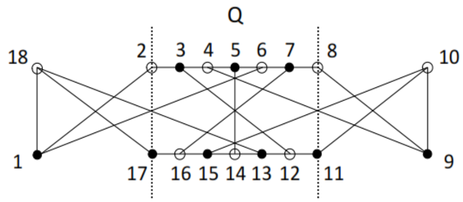


Figure 17

In [4] it was shown that n copies of Q can be concatenated by sharing the vertices of degree 2 on the dashed lines to form a Hamilton laceable bigraph on $2m = 12n + 6$ vertices with $2(n+1)$ of them being of degree 2, i.e. to form a bigraph having $17n + 8$ edges instead of $3m = 18n + 9$. The construction is only defined for an integer number of copies of Q so m is incremented by 6 for each additional copy of Q. However k-rung ladders, L_k, can be spliced into any pair of edges incident on a pair of matching vertices of degree 2 in Q^n to form Hamilton laceable bigraphs on $12n + 6 + 2k$ vertices, $2(n+1)$ of which will be of degree 2. This is where the $O(m/3)$ term in the bound comes from.

In a rare bit of serendipity, given the vital role of Q in constructing bigraphs with many vertices of degree 2, it is possible to prove that $\mathbb{Q}$ is not only edge critical, but also edge minimal. Ordinarily, to prove a bigraph is edge minimal it is necessary to carry out an exponentially difficult backtracking computation on each of sub-factorially many candidate bigraphs: $m = 9$ may be the only non-trivial exception.

Theorem 3:

$\mathbb{Q}$ is edge-minimal with respect to Hamilton laceability.

Proof:

We assume there exists a Hamilton laceable bigraph, H, on 18 vertices with only 24 edges and show this leads to a contradiction. H is Hamiltonian by assumption: adjoin any edge to a Hamilton path between the endpoints of the edge to form a cycle. Choose an arbitrary cycle in H. All 18 of the vertices are on the cycle and 18 of the edges, leaving 6 edges to be placed as chords to form H. In [1] it was shown that in every cycle in such a graph every pair of vertices of degree 2 must be separated by at least two of degree ≥ 3. Irrespective of which cycle is chosen the structure shown in Figure 18 is forced in which six triples of vertices are arranged around the cycle , the middle one of which is of degree 2 and the end ones are of degree 3 and in the opposite part. The proof reduces to showing it is impossible to place the six chords to form a Hamilton laceable bigraph.

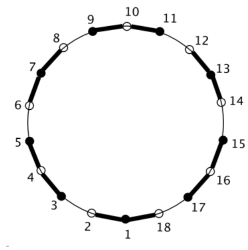


Figure 18

Consider the triple 2-1-18. Endpoints 2 and 18 will each have a chord incident on them whose other endpoint could be any one of five non-adjacent vertices in the other part. Restrict attention for the moment to chords whose other endpoint is in the lower half of the cycle, 5 --- 15. There are five ways this can be done, excluding two mirror images about an axis of symmetry. Figure 19 shows two of these arrangements.

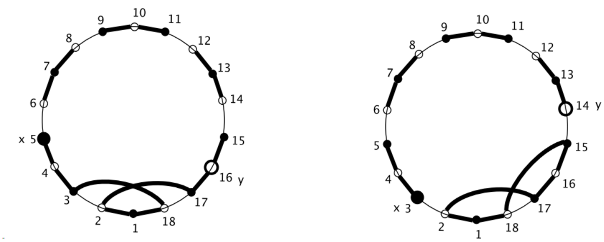


Figure 19

In both cases the indicated endpoints x and y cannot have a Hamilton path between them since they are a cut-set isolating the bold sub-path from the balance of the graph.

The other three ways the two chords on vertices 2 and 18 can end in the lower half cycle are shown in Figure 20,

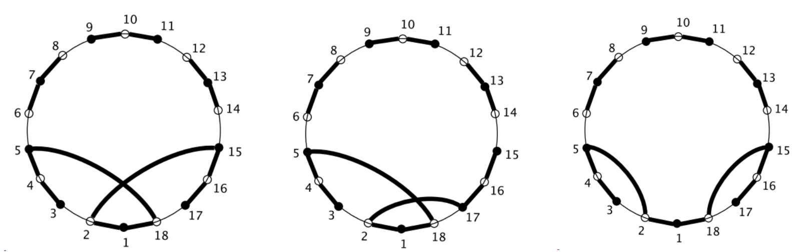


Figure 20

In all three cases there are two vertices which must either be an endpoint or else have a chord on them connecting to a vertex in the upper half cycle. Let x = 3 be an endpoint, which forces the path to begin 3-4-5 in order to have vertex 4 be in the path. The path is then forced to continue to the triple 2-1-18 if those vertices are to be in the path and to triple 15-16-17 for the same reason. So if a Hamilton path is to exist with 3 as an endpoint it is forced to follow the bold paths shown in the drawings ending on either vertex 15 or 17. Irrespective of the endpoint of the forced sub-path, 16 is always the penultimate vertex. There cannot be a Hamilton path between endpoints 3 and 16 in any of the cases in Figure 20 since the forced path does not include vertices in the upper half cycle.

What has been proven is that if the endpoints of the chords on the ends of any bold triple in Figure 18 are both in the half cycle centered on the triple the graph cannot be Hamilton laceable. We next prove the endpoints can't both be in the opposite half cycle either.

Call pairs of diametrically opposite triples matches or matching. For any triple the only vertices in the opposite half cycle which could be endpoints for chords on its endpoints are the endpoints of the matching triple. There are two ways chords can connect the matching triples 2-1-18 and 9-10-11; either the chords are parallel or they cross, edges 2-9 and 11-18 or else 2-11 and 9-18.

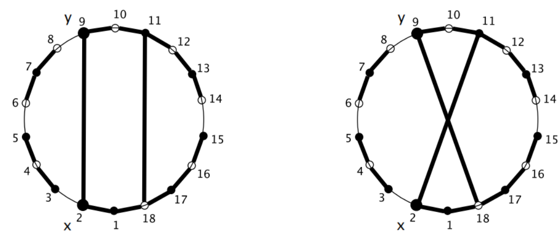


Figure 21

Let the endpoints of an assumed Hamilton path be x = 2 and y = 9. Irrespective of whether the chords are parallel or crossed the paths are forced to start 2-1-18-17-16-15 and 9-10-11-12-13-14 in order to include vertices 1, 16, 10 and 13. Neither path can continue with edge 14-15 since that would close the paths prematurely. The only way the paths can continue is for at least one edge from each of the pairs 3-14 or 5-14 and 6-15 or 8-15 be in the graph. All four of the possible pairings of chords have spanning completions as shown in Figure 22.

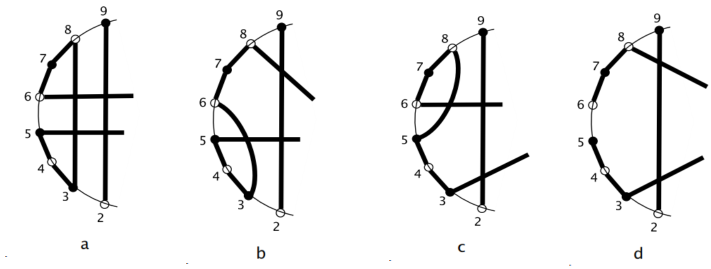


Figure 22

If there is to be a Hamilton path connecting vertices 2 and 9, then one of these four sub-graphs is forced. However, in all four cases the left hand endpoints of the chords are a cut-set isolating the subgraph in the arc 3-4-5-6-7-8 from the balance of the graph. Therefore it is not possible for there to be Hamilton paths between both the vertex pair 2 and 9 and the endpoints of the two chords.

What has been proven thus far is that if the endpoints of the chords on the ends of any bold triple in Figure 18 are either both in the half cycle centered on the triple or both on the opposite half cycle the graph cannot be Hamilton laceable. This says that if there is a Hamilton laceable bigraph with 24 edges on 18 vertices then for every triple one chord must have an endpoint in one half cycle and one in the other. We next show this is impossible.

There are four ways a chord can connect a triple and its match, C and P as shown in Figure 23 and the reflections about the vertical axis denoted C’ and P’.

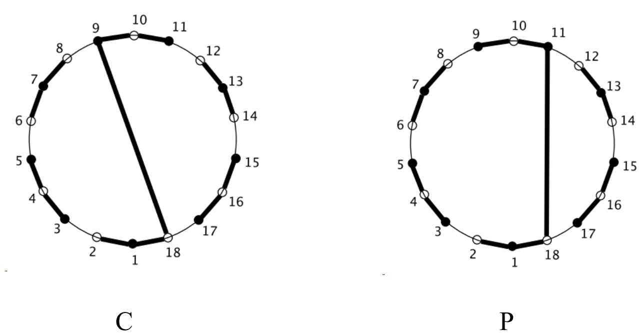


Figure 23

Since there are three matching triple pairs there are $4^3 = 64$ ways to assign the chords, but only eight equivalence classes under rotations and reflections represented by CCC, CCC’, CCP, CC’P, CPP, CPP’, PPP and PP’P. The six endpoints remaining must be connected by chords in the containing half cycle. This is not possible for CCC’ or CC’P, but has a unique solution in the other six cases. All but one of these configurations force at least one chord to connect to the previous triple as shown in Figure 24.

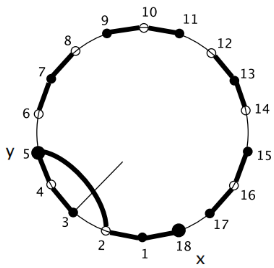
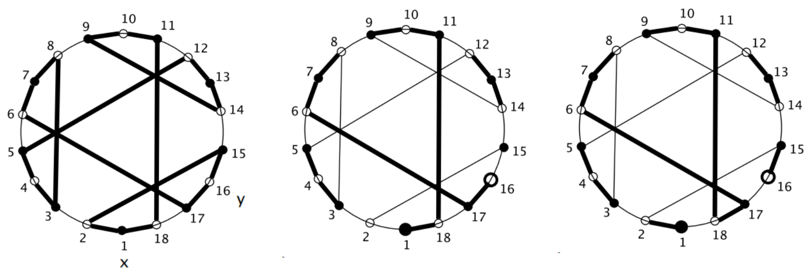


Figure 24

Irrespective of what the rest of the graph may be, it is easy to see that no Hamilton path is possible between x = 18 and y = 5 since the paths must start 5-4-3 and 18-1-2 in order for 1 and 4 to be in the path and both continuations of 18-1-2 close the path prematurely. Only the case PP’P remains as shown in Figure 25a.



a (PP’P)   b   c

Figure 25

Choose endpoints x = 1 and y = 16. The paths can only start 1-18 and 16-17, Figure 25b, or else 1-2 and 16-15, Figure 25c. In the first case, the forced path continuations 1-18-11-10-9 and 16-17-6-7-8 cannot continue with edge 8-9 since that would close the path prematurely. But if they continue using edges 8-3 and 9-14 they are also forced to close prematurely with vertices 2 and 15 not in the path. Similarly if the paths start 1-2 and 16-15 the only way for vertices 17 and 18 to be in the assumed Hamilton path is if sub-path -8-7-6-17-18-11-10-10-9- is. But this path is also forced to close prematurely without including all of the vertices.

Since $\mathbb{Q}$ is Hamilton laceable and has 25 edges and no Hamilton laceable bigraph exists with 24 edges on 18 vertices, $\mathbb{Q}$ is edge minimal.

Edge criticality is inherited by the Q^n concatenations. Unfortunately there is no basis to predict, and virtually no hope of proving, whether any of these bigraphs, other than $\mathbb{Q}$, are edge minimal

4: Conclusion:

Critically edge stable and edge critical bigraphs on 2m vertices are separated from below in the first case and from above in the latter by a bound of 3m edges: hard in the first case and conjectured to be hard in the second. By construction there are critically edge stable bigraphs with as many as 4m-4 edges and edge critical bigraphs with as few as 3m- O(m/3). Since an edge minimal graph is by definition edge critical, 3m- O(m/3) is an upper bound for the number of edges an edge minimal bigraph Hamilton laceable bigraph on 2m vertices can have.

Acknowledgement:
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References:

1. G. J. Simmons, Minimal Hamilton-laceable graphs, Congressus Numerantium, Vol 29 (1980), pp 893-900
2. G. J. Simmons, A possible upper bound on the number of edges an edge-critical Hamilton- laceable bigraph can have, Bulletin of the ICA, Vol 75 (2015), pp 128
3. G. J. Simmons, A modest improvement to the bound on the minimal number of edges in equitable Hamilton laceable bigraphs, Bulletin of the ICA, Vol 70 (2014) pp 97-103
4. G. J. Simmons, Bounding the edge deficit in equitable Hamilton-laceable bigraphs, Bulletin of the ICA, Vol 77 (2016), pp 43-56
5. G. J. Simmons, Cubic, edge critical Hamilton laceable bigraphs, Ars Combinatoria, Vol 123 (1980), pp 291-302
6. E. W. Weisstein, Crossed prism graph, Math World – A Wolfram Web Resource http://mathworld.wolfram.com/CrossedPrismGraph.html
7. M. Heule and T. Walsh, Symmetry within solutions, Proceedings of AAAI, Vol 10 (2010), pp 77-82