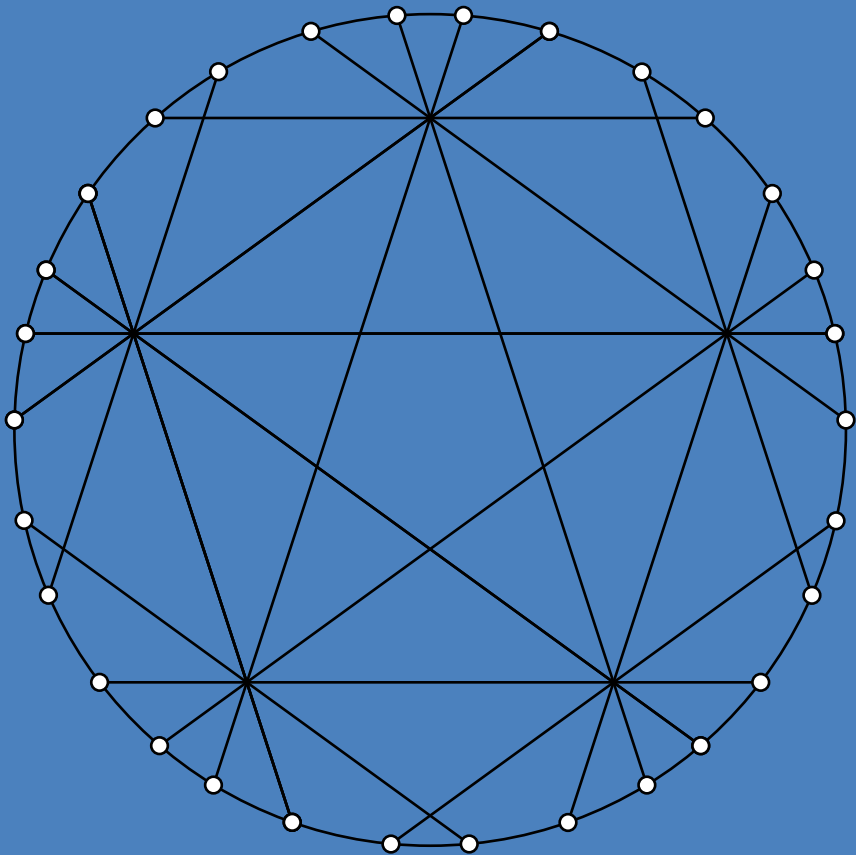# BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

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Diagram versus bundle equivalence for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices

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Abstract: One of the most promising structural approaches to resolving the Hadamard Conjecture uses the family of cocyclic matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$. Two types of equivalence relations for classifying cocyclic matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ have been independently found. Any cocyclic matrix equivalent by either of these relations to a Hadamard matrix will also be Hadamard.

Bundle equivalence is based on algebraic relations between cocycles over any finite group. Diagram equivalence is based on geometric relations between diagrammatic visualisations of cocyclic matrices over the group

$\mathbb{Z}_t \times \mathbb{Z}_2^2$. Here we reconcile the two. We show the group $Bund(t)$ generated by bundle equivalence operations is isomorphic to a subgroup of index 2 in the group $Diag(t)$ generated by diagram equivalence operations, and that $Diag(t) \cong \langle Bund(t), \top \rangle$, where $\top$ is the geometric translation of matrix transposition.

Keywords: Hadamard matrix, cocyclic matrix, shift equivalence, bundle, Williamson-type matrix.

1 Introduction

A Hadamard matrix of order m is a square matrix $[h(i, j)]$ with entries $h(i, j) = \pm 1$, $1 \leq i, j \leq m$, whose row vectors are pairwise orthogonal. A Hadamard matrix must have order 1, 2 or a multiple of 4, but no other restrictions on the order are known, and the century-old Hadamard Conjecture proposes that a Hadamard matrix of order m exists for every $m \equiv 0$ (mod 4).

About 20 years ago, the use of cocycles and cocyclic matrices was introduced by Horadam and de Launey [10] as a structural approach to resolving the Hadamard Conjecture. Its advantages led to the cocyclic Hadamard conjecture: that a cocyclic Hadamard matrix exists for every $m = 4t$. We can restrict the conjecture to odd $t \geq 1$. The study and use of cocyclic matrices has since expanded substantially, to include generalised Hadamard matrices [8, 9] and pairwise combinatorial designs [5].

If G is a group and C is an abelian group, a (2-dimensional, normalized) cocycle ψ from G to C is a mapping $\psi : G \times G \to C$ satisfying $\psi(1, 1) = \psi(g, 1) = \psi(1, g) = 1$, $g \in G$ and the cocycle equation:

$$\psi(g, h)\, \psi(gh, k) \;=\; \psi(g, hk)\, \psi(h, k), \quad g, h, k \in G. \tag{1}$$

The set of cocycles from G to C forms an abelian group $Z^2(G, C)$ under pointwise multiplication. The simplest cocycles are the coboundaries ∂f, defined for any function $f : G \to C$ by $\partial f(g, h) = f(g)^{-1} f(h)^{-1} f(gh)$.

A cocycle ψ may be represented by its matrix of values in C

$$M_\psi = [\psi(g, h)]_{g, h \in G} \tag{2}$$

once an indexing of the elements of G has been chosen.

We set $C = \{\pm 1\} \cong \mathbb{Z}_2$ when searching for cocyclic Hadamard matrices. A cocycle ψ for which the cocyclic matrix M_ψ is Hadamard is termed *orthogonal*. It is computationally easy to check whether M_ψ is a Hadamard matrix, as we only need to check whether the dot product of the first row with each other row is 0. This computational cutdown is one motivation for using cocyclic matrices.

Many known constructions for Hadamard matrices yield cocyclic matrices [8, Ch. 6]. Computationally, the most prolific indexing groups G for producing cocyclic Hadamard matrices appear to be the abelian groups $\mathbb{Z}_t \times \mathbb{Z}_2^2$ and the dihedral groups D_{4t}. The D_{4t} cocyclic Hadamard matrix family, related to the Ito type Hadamard matrices, has been investigated by many researchers including the authors (see [8]). The $\mathbb{Z}_t \times \mathbb{Z}_2^2$ family, related to the Williamson type Hadamard matrices, has also been investigated by the authors [3, 4], and while exhaustive search often finds fewer Hadamard matrices in each order than for D_{4t}, abelian-ness makes the family computationally more tractable.

In parallel with the search for examples of Hadamard matrices in new orders, whether cocyclic or not, has been the attempt to classify them into equivalence classes. Hadamard equivalence of a $\{\pm 1\}$ matrix involves only permutation of rows or columns, and multiplication of a row or column by -1. While the transpose of a Hadamard matrix is a Hadamard matrix, transposition is not a Hadamard equivalence. The total number of Hadamard equivalence classes in small orders grows so rapidly that Orrick [13] uses a coarser Q-equivalence relation on Hadamard matrices which allows extra "switching" operations and leads to a dramatic reduction in the number of classes.

The total number of equivalence classes of cocyclic Hadamard matrices over all indexing groups G is studied by Ó Catháin and Röder [12] and calculated up to $m = 36$. An allied but distinct approach has been to identify equivalences of cocycles that preserve orthogonality. For the $\mathbb{Z}_t \times \mathbb{Z}_2^2$ family, two different types of equivalence of cocycles, both of which preserve orthogonality, have been discovered independently.

The first of these is defined (see [8]) for any G and C by all compositions of a "shift" action and two "automorphism" actions. (For $C = \{\pm 1\}$, one of the automorphism actions is trivial.) The resulting equivalence classes, called *bundles*, are already studied under other names in specific contexts; for example, they correspond to the Extended Affine equivalence classes for cryptographic functions [9]. Shift action is also studied separately, for applications to the search for self-dual codes [14] and, via shift representations,

to classification of pairwise combinatorial designs [6].

The second of these equivalences, independently introduced in [3], is specific to cocycles ψ in $Z^2 := Z^2(\mathbb{Z}_t \times \mathbb{Z}_2^2, \{\pm 1\})$ and arises from detailed investigation of a generating set of cocycles for Z^2. Corresponding to the decomposition of ψ as a product of generators there is a Hadamard product decomposition of M_ψ into generator matrices. Geometric actions on these generator matrices lead to a concise diagrammatic representation of cocycles and geometric equivalences which is very useful for effective computation.

This paper relates and reconciles the two types of equivalence.

The paper is organized as follows. Section 2 describes the two types of equivalence. The group acting on cocycles is determined for each type; the two groups are not isomorphic. Section 3 gives our main results, Theorems 3 and 4, translating shift action and the nontrivial automorphism action into diagram actions, relating the two groups of actions, and showing that the diagram action termed "complement" has no algebraic analogue. In Section 4 this diagram action is shown to be the transposing operation on M_ψ. We summarise and suggest further work.

2   Background

From now on we assume $C = \{\pm 1\}$, $G \cong \mathbb{Z}_t \times \mathbb{Z}_2^2$ with $t > 1$ odd, and $\psi \in Z^2$. Denote the group of units of the ring $\mathbb{Z}_t$ by $\mathbb{Z}_t^*$. Let G have presentation

$$G = \langle x, u, v : \ x^t = u^2 = v^2 = 1, xu = ux, xv = vx, uv = vu \rangle,$$

and ordering

$$(x^i, 1) < (x^i, u) < (x^i, v) < (x^i, uv), 0 \le i < t, \ (x^i, uv) < (x^{i+1}, 1), 0 \le i < t-1.$$

We describe an orthogonality-preserving algebraic action on ψ in the first subsection and an orthogonality-preserving geometric action on ψ in the second.

2.1   Bundle action on cocycles

For any $a \in G$, the *shift* $\psi \cdot a$ of ψ is the cocycle $(\psi \cdot a)(g, h) = \psi(ag, h)\psi(a, h)^{-1}$. It is orthogonal if ψ is orthogonal. For any automorphism $\theta \in \mathrm{Aut}(G)$, the cocycle $\psi \circ (\theta \times \theta)$, defined by $\psi \circ (\theta \times \theta)(g, h) = \psi(\theta(g), \theta(h))$, is orthogonal if ψ is. When the two actions are combined, the result is an action called *bundle action* under which the orbit of ψ is its *bundle*

$$\mathcal{B}(\psi) = \{(\psi \cdot a) \circ (\theta \times \theta) : a \in G, \ \theta \in \mathrm{Aut}(G)\}. \tag{3}$$

The group acting on Z^2 is $G \rtimes \mathrm{Aut}(G)$, where the semidirect product is defined for $a, b \in G$, $\theta_1, \theta_2 \in \mathrm{Aut}(G)$ by $a\theta_1 \circ b\theta_2 = a\theta_1^{-1}(b)\theta_1\theta_2$ [8, Ch. 8].

The Hadamard equivalence operations on M_ψ corresponding to shift and automorphism action can be easily described. $M_{\psi \cdot a}$ is Hadamard equivalent to M_ψ by first permuting the rows of M_ψ with respect to the row index permutation $g \mapsto g' = ag$, $g \in G$, obtaining $M' = [\psi(ag, h)]_{g, h \in G}$. The first row of M' is the ath row of M_ψ. Then obtain $M_{\psi \cdot a}$ from M' by multiplying every column of M' by its first entry. $M_{\psi \circ (\theta \times \theta)}$ is Hadamard equivalent to M_ψ by permuting rows and columns under θ.

We complete this subsection by computing $G \rtimes \mathrm{Aut}(G)$ for $G = \mathbb{Z}_t \times \mathbb{Z}_2^2$.

Theorem 1 *The group $Bund(t)$ defined by bundle action on Z^2 is $Bund(t) \cong [\mathbb{Z}_t \rtimes \mathbb{Z}_t^*] \times [\mathbb{Z}_2^2 \rtimes S_3]$. Its order is $24\, t\, \phi(t)$, where ϕ is the Euler function.*

A generating set for $Bund(t)$ is $\{x, u, v, h_r, r \in \mathbb{Z}_t^*, h_{23}, h_{243}\}$, where x, u and v are shift actions and $h_{23} : x \mapsto x, u \mapsto v, v \mapsto u$; $h_{243} : x \mapsto x, u \mapsto uv, v \mapsto u$ and $h_r : x \mapsto x^r, u \mapsto u, v \mapsto v$ are automorphism actions.

Proof. Since t is odd, $\mathrm{Aut}(\mathbb{Z}_t \times \mathbb{Z}_2^2) \cong \mathrm{Aut}(\mathbb{Z}_t) \times \mathrm{Aut}(\mathbb{Z}_2^2) \cong \mathbb{Z}_t^* \times S_3$. Under the identification $1 \leftrightarrow 1, u \leftrightarrow 2, v \leftrightarrow 3, uv \leftrightarrow 4$, $\mathrm{Aut}(\mathbb{Z}_2^2)$ is the subgroup of S_4 which fixes 1. Then $\{\mathrm{Id}\} \times \mathrm{Aut}(\mathbb{Z}_2^2)$ is generated by h_{23} and h_{243}. Thus $Bund(t) = [\mathbb{Z}_t \times \mathbb{Z}_2^2] \rtimes [\mathbb{Z}_t^* \times S_3]$, with the listed generating set. Since $h_{23}(x) = h_{243}(x) = x$, $\mathbb{Z}_t$ commutes with S_3 and since $h_r(u) = u$, $h_r(v) = v$, $\mathbb{Z}_2^2$ commutes with $\mathbb{Z}_t^*$. Hence $Bund(t) \cong [\mathbb{Z}_t \rtimes \mathbb{Z}_t^*] \times [\mathbb{Z}_2^2 \rtimes S_3]$. $\qquad \square$

2.2 Geometric action on cocycle diagrams

The group of cocycles Z^2 has a (non minimal, see Section 2 in [3]) generating set $\mathcal{Z} = \{\partial_1, \ldots, \partial_{4t}, \beta_1, \beta_2, \kappa\}$ consisting of $4t$ coboundaries $\partial_i := \partial \delta_i$, where δ_i is the Kronecker delta function of the ith-element in G in the given ordering, and three representative cocycles β_1, β_2, κ, all of which are explicitly described in [2, 3]. Every 2-cocycle over G admits a (non unique) representation as a product of the generators in $\mathcal{Z}$. The identity of Z^2 is the trivial cocycle $\mathbf{1}$ for which $M_{\mathbf{1}} = J_{4t}$ is the all-ones matrix. All orthogonal cocycles known so far (cf. [4, 3]) contain the factor $\rho = \beta_1 \beta_2 \kappa$, where

$$
M_\rho = J_t \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \tag{4}
$$

and J_t denotes the $t \times t$ matrix all of 1s. It is conjectured this must always be true [8, Research Problem 37]. For the remainder of the paper, we work with cocycles of this type. That is, $\psi = \partial_1^{\epsilon_1} \ldots \partial_{4t}^{\epsilon_{4t}} \rho$, $\epsilon_i \in \{0, 1\}$, so $\psi = \partial_{d_1} \ldots \partial_{d_k} \rho$ for some $d_1 < \cdots < d_k$, $d_i \in \{1, \ldots, 4t\}$.

We describe ψ in the concise notation of [3], which allows one to determine orthogonality much more easily, but here we adopt a more natural ordering for clarity. First, partition the set $\{d_1, \ldots, d_k\}$ according to the equivalence classes modulo 4, in class order $1, 2, 3, 4$ and in descending order within each class. Denote this ordered set of coboundary indices by

$$
\{\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4}\} = \{\{d_{1+4j_1}\}, \{d_{2+4j_2}\}, \{d_{3+4j_3}\}, \{d_{4j_4}\}\}. \tag{5}
$$

For example, for $t = 7$, the cocycle $\psi = \partial_4 \partial_6 \partial_9 \partial_{10} \partial_{11} \partial_{12} \partial_{14} \partial_{20} \partial_{21} \partial_{25} \rho$ is orthogonal, and is represented as

$$
\{\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4}\} = \{\{25, 21, 9\}, \{14, 10, 6\}, \{11\}, \{20, 12, 4\}\}. \tag{6}
$$

Second, write the integers $1, \ldots, 4t$, in descending order, by equivalence classes modulo 4, as the rows of a $4 \times t$ matrix (treated as a cylinder, i.e. left and right edges are identified) and mark out only the entries occurring in $\{d_1, \ldots, d_k\}$.

Definition 1 (cf. [3]) *The diagram D_ψ of $\psi = \partial_{d_1} \ldots \partial_{d_k} \rho$ is a $4 \times t$ matrix A, such that $a_{ij} = \times$ if $4(t - j) + i \in \{d_1, \ldots, d_k\}$ and $a_{ij} = -$ elsewhere.*

The diagram for the example in (6) above is

$$
A = \begin{vmatrix}
\times & \times & - & - & \times & - & - \\
- & - & - & \times & \times & \times & - \\
- & - & - & - & \times & - & - \\
- & - & \times & - & \times & - & \times
\end{vmatrix}
\tag{7}
$$

We now list four types of orthogonality-preserving operations on ψ. We adopt the notation $[m]_n$ for $m \mod n$ for brevity.

Definition 2 *Let* $\{c_1, c_2, c_3, c_4\}$ *represent a set of coboundaries. Denote the columns of its diagram* A *by* $(\mathcal{C}_{t-1}, \cdots, \mathcal{C}_0)$. *Let* $c_j + k$ *denote the set of coboundaries obtained by adding k to each element of* c_j *modulo* $4t$.

1. *The* complement $C_2(\{c_1, c_2, c_3, c_4\})$ *of this set is the set* $\{c_1, \overline{c_2}, c_3, c_4\}$ *where* $\overline{c_2}$ *is complement of* c_2 *in the equivalence class 2 modulo 4.*

2. *Six elementary* swapping *operations are possible on this set, of which we list three here:*

   - $s_{12}(\{c_1, c_2, c_3, c_4\}) = \{c_2 - 1, c_1 + 1, c_3, c_4\}$.
   - $s_{23}(\{c_1, c_2, c_3, c_4\}) = \{c_1, c_3 - 1, c_2 + 1, c_4\}$.
   - $s_{34}(\{c_1, c_2, c_3, c_4\}) = \{c_1, c_2, c_4 - 1, c_3 + 1\}$.

3. *The i-rotation* $T_i(\{c_1, c_2, c_3, c_4\})$, $0 \le i \le t - 1$, *of this set is the set*

$$
\{c_1 - 4i, c_2 - 4i, c_3 - 4i, c_4 - 4i\}.
$$

4. *The r-th dilatation* $V_r(\{c_1, c_2, c_3, c_4\})$, *for $r \in \mathbb{Z}_t^*$, is the set with diagram* $V_r(A)$, *where* $V_r(\mathcal{C}_j) = \mathcal{C}_{[jr]_t}$, $0 \le j \le t - 1$.

Complementation at row 2 is chosen since by Lemma 4 in [3] any other row complementation reduces to it.

Clearly the order of C_2 is 2 and $\langle C_2 \rangle \cong \mathbb{Z}_2$. The swappings each have order 2 and generate a group $\cong S_4$. The rotations are generated by T_1 so $\langle T_1 \rangle \cong \mathbb{Z}_t$; and $\langle V_r, r \in \mathbb{Z}_t^* \rangle \cong \mathbb{Z}_t^*$.

In terms of diagrams, C_2 complements the second row of A; s_{ij} swaps rows corresponding to c_i and c_j; T_i cyclically shifts columns i places to the right;

and V_r permutes columns according to multiplication of column index by the invertible element r (so $\mathcal{C}_0$ is always fixed).

For instance, if A is the diagram in (7),

$$C_2(A) = \begin{vmatrix} \times & \times & - & - & \times & - & - \\ \times & \times & \times & - & - & - & \times \\ - & - & - & - & \times & - & - \\ - & - & \times & - & \times & - & \times \end{vmatrix}, \quad T_2(A) = \begin{vmatrix} - & - & \times & \times & - & - & \times \\ \times & - & - & - & - & \times & \times \\ - & - & - & - & - & - & \times \\ - & \times & - & - & \times & - & \times \end{vmatrix},$$

$$s_{23}(A) = \begin{vmatrix} \times & \times & - & - & \times & - & - \\ - & - & - & - & \times & - & - \\ - & - & - & \times & \times & \times & - \\ - & - & \times & - & \times & - & \times \end{vmatrix}, \quad V_2(A) = \begin{vmatrix} - & \times & \times & \times & - & - & - \\ \times & - & \times & - & \times & - & - \\ - & - & \times & - & - & - & - \\ - & - & \times & - & - & \times & \times \end{vmatrix}.$$

Since Z^2 is abelian, we can identify the action of C_2 on coboundaries directly.

Lemma 1 $C_2(\partial_{d_1} \dots \partial_{d_k}) = \partial_{d_1} \dots \partial_{d_k} \prod_{i=0}^{t-1} \partial_{2+4i}$. $\qquad\square$

We complete this subsection by identifying the group generated by the operations above.

Theorem 2 *The group $Diag(t)$ defined by diagrammatic action on Z^2 is $Diag(t) \cong [\mathbb{Z}_t \rtimes \mathbb{Z}_t^*] \times S_4 \times \mathbb{Z}_2$, of order $48\,t\,\phi(t)$. A generating set for $Diag(t)$ is $\{T_1, V_r, r \in \mathbb{Z}_t^*,\ s_{12}, s_{23}, s_{34}, C_2\}$.*

Proof. The composition $V_r^{-1} T_1 V_r$ acts on column $[j]_t$ of A to give column $[(jr-1)r^{-1}]_t = [j - r^{-1}]_t$, so $V_r^{-1} T_1 V_r = T_{r^{-1}}$. Define a homomorphism $\mu : \mathbb{Z}_t^* \to \mathrm{Aut}(\mathbb{Z}_t)$ by $\mu(V_r)(T_1) = T_{r^{-1}}$. Consequently, $\langle T_1, V_r, r \in \mathbb{Z}_t^* \rangle \cong \mathbb{Z}_t \rtimes_\mu \mathbb{Z}_t^*$. Each swapping s_{ij} is shown in [3] to commute with rotations T_i and dilatations V_r, but it permutes rows while rotations and dilatations permute columns, so $\langle s_{12}, s_{23}, s_{34} \rangle \cong S_4$ does not intersect $\langle T_1, V_r, r \in \mathbb{Z}_t^* \rangle$. All combinations of swapping, rotation and dilatation preserve the total number of coboundaries but complement C_2 does not, so C_2 is not in the subgroup of $Diag(t)$ generated by rotations, swappings and dilatations. It is shown in [3] that C_2 commutes with all other operations. $\qquad\square$

3   Bundle actions as Diagram actions

In this section we express the bundle actions on Z^2 in terms of the diagrammatic operations and identify the role of the diagrammatic action C_2. Subsection 3.1 is given to proving the following theorem.

Theorem 3 *In the notation of Theorems 1 and 2*

1. *The shift actions by x, u and v, respectively, on ψ, are the diagrammatic actions T_1, $s_{12}s_{34}$ and $s_{13}s_{24}$, respectively.*

2. *The automorphism actions by h_r, h_{23} and h_{243}, respectively, on ψ, are the diagrammatic actions $V_{r^{-1}}$, C_2s_{23} and $s_{234} := s_{23}s_{24}$, respectively.*

From Theorem 3 we obtain our main result.

Theorem 4 *Bundle action by $Bund(t)$ on $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices corresponds to diagrammatic action by the subgroup*

$$Diag(t)^* = \langle T_1, \, V_r, r \in \mathbb{Z}_t^*, \, s_{12}s_{34}, \, s_{13}s_{24}, \, C_2s_{23}, \, s_{23}s_{24} \rangle \cong (\mathbb{Z}_t \rtimes \mathbb{Z}_t^*) \times S_4$$

of index 2 in $Diag(t)$. The operation C_2 is not in $Diag(t)^*$.

Proof. Define a homomorphism $\alpha : Bund(t) \rightarrowtail Diag(t)$ by $x \mapsto T_1, h_r \mapsto V_{r^{-1}}, u \mapsto s_{12}s_{34}, v \mapsto s_{13}s_{24}, h_{23} \mapsto C_2s_{23}$ and $h_{243} \mapsto s_{23}s_{24}$. By Theorem 2 and Theorem 3, $\alpha(\langle x, h_r, r \in \mathbb{Z}_t^* \rangle) = \langle T_1, V_{r^{-1}}, r \in \mathbb{Z}_t^* \rangle \cong \mathbb{Z}_t \rtimes \mathbb{Z}_t^*$ is an isomorphism.

Let CS_4 be the subgroup of $Diag(t)$ which is generated by the 6 order-2 elements C_2s_{ij}; it is isomorphic to S_4. Since C_2 and s_{ij} commute, products corresponding to even permutations in S_4 will appear unchanged, while those corresponding to odd permutations in S_4 will be multiplied by C_2. Then, from Theorem 1 and Theorem 3, $\alpha(\mathbb{Z}_2^2 \rtimes S_3)$ is generated by $C_2s_{12}C_2s_{34} = s_{12}s_{34}$ and $C_2s_{13}C_2s_{24} = s_{13}s_{24}$ (shift action, isomorphic to $\mathbb{Z}_2^2$), and C_2s_{23} and $C_2s_{23}C_2s_{24} = s_{23}s_{24}$ (automorphism action, isomorphic to S_3). Direct calculation shows that α maps $\mathbb{Z}_2^2 \rtimes S_3$ onto CS_4, so α is an isomorphism. Thus $Diag(t)^* \cong (\mathbb{Z}_t \rtimes \mathbb{Z}_t^*) \times S_4$, and $\alpha(Bund(t))$ does not contain C_2. $\qquad\square$

3.1 Proof of Theorem 3

Every cocyclic matrix M_ψ admits a decomposition as the Hadamard (pointwise) product of the cocyclic matrices corresponding to the generators. That is, $M_\psi = M_{\partial_1}^{\epsilon_1} \dots M_{\partial_{4t}}^{\epsilon_{4t}} M_\rho$, $\epsilon_i \in \{0, 1\}$. Each coboundary matrix M_{∂_i} is symmetric.

Let M_i the matrix obtained from M_{∂_i} by negating the ith row and the ith column. Each M_i is a 4×4-block back diagonal square matrix of order $4t$ (see [2] for details). The first block row has a 4×4 matrix $A_{[i]_4}$ as the $\lceil \frac{i}{4} \rceil^{th}$ block and J_4 blocks in the other $t - 1$ positions. The remaining block rows are obtained by successively back-cycling the first.

Let $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ so $DR = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. The 4×4-blocks $A_{[i]_4}$ depend on the equivalence class of i modulo 4, as follows:

$A_0 = \begin{pmatrix} J_2 & DR \\ DR & J_2 \end{pmatrix}, A_1 = \begin{pmatrix} D & J_2 \\ J_2 & D \end{pmatrix}, A_2 = \begin{pmatrix} DR & J_2 \\ J_2 & DR \end{pmatrix}, A_3 = \begin{pmatrix} J_2 & D \\ D & J_2 \end{pmatrix}$.

It may be checked that bundle action by each of x, u, v, h_r and h_{243} leaves M_ρ invariant. Only action by h_{23} alters M_ρ. In terms of identifying diagram actions, it does not matter whether we work with M_{∂_i} or M_i so we use the latter. We determine each bundle action on M_i in the subsections below, concluding with the action of h_{23} on M_ρ.

3.1.1 Shift action of x

First, we change the order of the elements in the group to $g' = xg$, obtaining

$$(x, 1) < (x, u) < \dots < (x^{t-1}, uv) < (1, 1) < \dots < (1, uv)$$

that is, we put the first block of 4 elements at the end of the list.

For an individual coboundary ∂_i, the reordering takes the first four rows to the last four, moving the other rows upwards. Now the blocks $A_{[i]_4}$ start from the $\lceil \frac{i}{4} \rceil - 4^{th}$-column, the negated row is the $i - 4^{th}$ row, and the negated column is still the ith column. Next we perform the pointwise product of the first row and the others. This first row (the former 5th) has

two negative entries, at positions i and $i-4$, so we have to negate these columns, getting the coboundary ∂_{i-4}.

So, the action of x on the cocyclic matrix is the 1-rotation T_1 on D_ψ.

3.1.2 Shift action of u (resp. v)

First, we change the order of the elements in the group to $g' = ug$ (resp. $g' = vg$), obtaining

$$(x^i, u) < (x^i, 1) < (x^i, uv) < (x^i, v),\ 0 \le i < t,\ (x^i, v) < (x^{i+1}, u),\ 0 \le i < t-1,$$

or respectively,

$$(x^i, v) < (x^i, uv) < (x^i, 1) < (x^i, u),\ 0 \le i < t,\ (x^i, u) < (x^{i+1}, v)\ 0 \le i < t-1);$$

that is, we reorder every block of 4 elements by means of the permutation $\sigma = (12)(34)$ (resp. $\sigma = (13)(24)$).

For an individual coboundary ∂_i, the reordering permutes rows in the same way. This permutation transforms the blocks $A_{[i]_4}$ in the same way, under $(A_1 A_2)(A_3 A_0)$ (resp. $(A_1 A_3)(A_2 A_0)$), the negated row is the $\sigma(i)^{th}$ and the negated column is the ith. The first row (the former 2nd, resp. 3rd) has two negative entries, at positions i and $\sigma(i)$. After negating these columns, we get the coboundary $\partial_{\sigma(i)}$.

So, the action of u (resp. v) on the cocyclic matrix is the composition of swappings $s_{12}s_{34}$ (resp. $s_{13}s_{24}$).

3.1.3 Automorphism action of h_r

A straightforward algebraic calculation shows that $h_r(\partial_k) = V_{r^{-1}}(\partial_k)$, for each $k = x^{kx} u^{ku} v^{kv}$. Set $\delta(i,j) = -1$ if $i = j$, and $\delta(i,j) = 1$ otherwise.

On one hand, $h_r(\partial_k)(x^{ix} u^{iu} v^{iv},\ x^{jx} u^{ju} v^{jv})$

$$
\begin{aligned}
&= \partial_k(x^{r \cdot ix \bmod t}\, u^{iu}\, v^{iv},\ x^{r \cdot jx \bmod t}\, u^{ju}\, v^{jv}) \hspace{2cm} (8)\\
&= \delta(x^{kx} u^{ku} v^{kv},\ x^{r \cdot ix \bmod t} u^{iu} v^{iv})\ \delta(x^{kx} u^{ku} v^{kv}, x^{r \cdot jx \bmod t} u^{ju} v^{jv})\\
&\quad \delta(x^{kx} u^{ku} v^{kv},\ x^{r \cdot (ix+jx) \bmod t} u^{(iu+ju) \bmod 2} v^{(iv+jv) \bmod 2}).
\end{aligned}
$$

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On the other hand, $V_{r^{-1}}(\partial_k)(x^{ix}u^{iu}v^{iv}, \; x^{jx}u^{ju}v^{jv})$

$$
\begin{aligned}
&= \partial_{x^{kx \cdot r^{-1} \bmod t} u^{ku} v^{kv}}(x^{ix} u^{iu} v^{iv}, \; x^{jx} u^{ju} v^{jv}) && (9)\\
&= \delta(x^{kx \cdot r^{-1} \bmod t} u^{ku} v^{kv}, \; x^{ix} u^{iu} v^{iv}) \, \delta(x^{kx \cdot r^{-1} \bmod t} u^{ku} v^{kv}, \; x^{jx} u^{ju} v^{jv})\\
&\quad \delta(x^{kx \cdot r^{-1} \bmod t} u^{ku} v^{kv}, \; x^{(ix+jx) \bmod t} u^{(iu+ju) \bmod 2} v^{(iv+jv) \bmod 2}).
\end{aligned}
$$

Since r is invertible in $\mathbb{Z}_t$, these equations are equal term by term. Consequently, $h_r = V_{r^{-1}}$, for all $r \in \mathbb{Z}_t^*$.

3.1.4 Automorphism action of h_{243}

The automorphism h_{243} shifts the second, third and fourth positions of the elements in G cyclically to the right, in each block of 4, leaving the first element unchanged. So the action on the cocycles will be the same permutation of every second, third and fourth rows and columns in every block of four.

For an individual coboundary ∂_i, this reordering transforms the blocks $A_{[i]_4}$ in the same way, giving the permutation $(A_2 A_3 A_0)$, and the negated row/column remains unchanged if $[i]_4$ is 1 and is interchanged cyclically between cosets 2, 3 and 0, so we get the coboundary $s_{234}(\partial_i)$.

Hence, the action of h_{243} on any cocyclic matrix gives us the operation s_{234}.

3.1.5 Automorphism action of h_{23}

The action of the automorphism h_{23} on the cocyclic matrix will be the permutation of second and third rows and columns in every block of four.

For an individual coboundary ∂_i, this reordering transforms the blocks $A_{[i]_4}$ in the same way, giving the permutation $(A_2 A_3)$, and the negated row/column remains unchanged if $[i]_4$ is 0 or 1 and interchanged between cosets 2 and 3, so we get the coboundary $s_{23}(\partial_i)$.

The action of this reordering on matrix M_ρ applies the same permutation

to its rows and columns, so the 4×4 blocks in (4) become

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

This expression coincides with the pointwise product of the 4×4 block in (4) and the block A_2 with the second row and column negated, so the action of the automorphism h_{23} on M_ρ gives us $M_\rho \cdot M_{\partial_2} \cdot M_{\partial_6} \ldots M_{\partial_{4t-2}}$, the product with all coboundaries whose index is congruent to 2 modulo 4. Hence, by Lemma 1, $h_{23}(\partial_{d_1} \ldots \partial_{d_k} \rho) = s_{23}(\partial_{d_1} \ldots \partial_{d_k}) \left(\prod_{i=0}^{t-1} \partial_{2+4i} \right) \rho = C_2\big(s_{23}(\partial_{d_1} \ldots \partial_{d_k})\big) \rho$.

Hence, the action of h_{23} on any cocyclic matrix gives us the operation $C_2 s_{23}$.

4   Complement

Next we demonstrate that complementation corresponds to matrix transposition and gives the matrix of the transpose cocycle.

Theorem 5 *The operation* C_2 *on* M_ψ *coincides with transposition:* $C_2(M_\psi) = (M_\psi)^\top = M_{\psi^\top}$.

Proof. Consider $M_\psi = M_{\partial_{d_1}} \ldots M_{\partial_{d_k}} M_\rho$. Since transposition commutes with pointwise products, $M_\psi^\top = M_{\partial_{d_1}} \ldots M_{\partial_{d_k}} M_\rho^\top$. By (4)

$$M_\rho^\top = J_t \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = \left(J_t \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \right) \cdot M_\rho.$$

By Lemma 1, $M_\psi^\top = M_{\partial_{d_1}} \ldots M_{\partial_{d_k}} \left(\prod_{i=0}^{t-1} M_{\partial_{2+4i}} \right) M_\rho = C_2(M_\psi)$, as claimed.

Since $G = \mathbb{Z}_t \times \mathbb{Z}_2^2$ is abelian, the transpose $\psi^\top$ of ψ, with $\psi^\top(g, h) = \psi(h, g)$, is a cocycle [8, (6.10)], and $(M_\psi)^\top = M_{\psi^\top}$. $\square$

In summary, we have shown that the diagrammatic operations which can be implemented for effective calculation of cocyclic Hadamard matrices over $G = \mathbb{Z}_t \times \mathbb{Z}_2^2$, can all be interpreted as compositions of known algebraic equivalences, with the exception of complementation, which corresponds to matrix transposition. Ó Catháin [11] has used the algebraic equivalences together with transposition to determine classes of cocyclic matrices of order $4t$ over various G. He then checks any transposes lying in such a class to partition them into Hadamard inequivalence classes. He coins the term *strong inequivalence* for Hadamard matrices H and H' for which H' is not Hadamard equivalent to H or to $H^\top$. So, this approach using diagrammatic operations may be computationally effective.

One might wonder if useful diagrams and diagram operations can be found for cocycles over other groups. This has been the case for cocycles over D_{4t} [1]. It would also be interesting to investigate whether there are diagrammatic operations which correspond to Orrick's switching operations.

References

[1] V. Álvarez, J. A. Armario, M. D. Frau, F. Gudiel, M. B. Güemes, and A. Osuna, "On D_{4t}-cocyclic Hadamard matrices", *J. Comb. Des.*, vol. 24, pp. 352–368, 2016.

[2] V. Álvarez, J. A. Armario, M. D. Frau and P. Real, "A system of equations for describing cocyclic Hadamard matrices", *J. Comb. Des.*, vol. 16, pp. 276–290, 2008.

[3] V. Álvarez, F. Gudiel and M. B. Güemes, "On $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices", *J. Comb. Des.*, vol. 23, pp. 352–368, 2015.

[4] A. Baliga (= A. Rao) and K.J. Horadam, "Cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$", *Australas. J. Combin.*, vol. 11, pp. 123–134, 1995.

[5] W. de Launey and D. L. Flannery, *Algebraic design theory*, Mathematical Surveys and Monographs 175. American Mathematical Society, Providence, RI, 2011.

[6] D.L. Flannery and R. Egan, "On linear shift representations", *J. Pure Appl. Algebra*, vol. 219, pp. 3482–3494, 2015.

[7] K. J. Horadam, "The shift action on 2-cocycles", *J. Pure Appl. Algebra*, vol. 188, pp. 127–143, 2004.

[8] K. J. Horadam, *Hadamard matrices and their applications*. Princeton: Princeton University Press, 2007.

[9] K. J. Horadam, "Hadamard matrices and their applications: Progress 2007–2010", *J Cryptogr. Commun.*, vol. 2 (2), pp. 129–154, 2010.

[10] K. J. Horadam and W. de Launey, "Generation of cocyclic Hadamard matrices", in: *Proc. Computational Algebra and Number Theory*, Sydney, 1992, in *Math. Appl.*, vol. 325, Kluwer Acad. Publ., Dordrecht, pp. 279–290, 1995.

[11] P. Ó Catháin, *Automorphisms of pairwise combinatorial designs*, PhD Thesis, National University of Ireland-Galway, 2011.

[12] P. Ó Catháin and M. Röder, "The cocyclic Hadamard matrices of order less than 40", *Des. Codes Cryptogr.*, vol. 58 (1), pp. 73–88, 2011.

[13] W. P. Orrick, "Switching operations for Hadamard matrices", *SIAM J. Discrete Math.*, vol. 22 (1), pp. 31–50, 2008.

[14] A. Rao, "Shift equivalence and cocyclic self dual codes", *J. Combin. Math. Combin. Comput*, vol. 54, pp. 175–185, 2005.