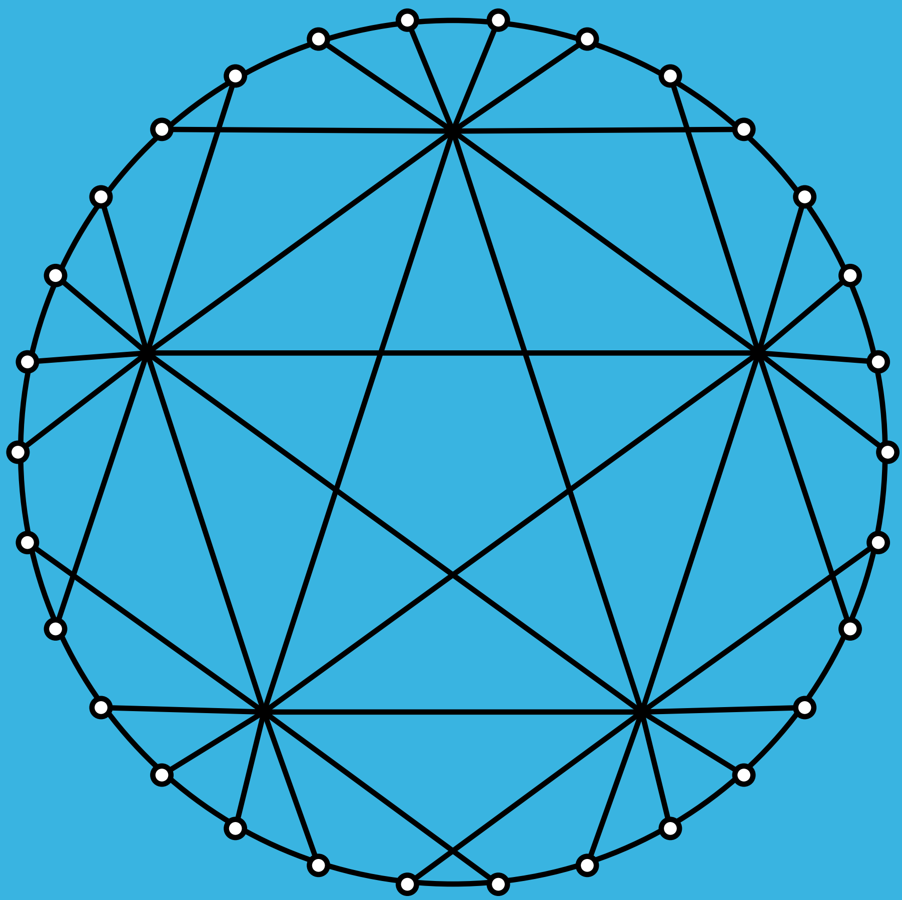
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 84
October 2018



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **84** (2018), Pages 98–108

You can't always get what you want...

Dalibor Froncek

University of Minnesota Duluth, Duluth, MN, USA.
dalibor@d.umn.edu

Abstract: No, you can't always get what you want even if you are the Director of Sports Operations of the Czech National Football (Soccer) League.

1 History

Since its inception after the split of Czechoslovakia in 1993, the Czech National Football (Soccer) League has been played as a two-leg round-robin tournament of 16 teams. For the first few seasons, the Football Association of the Czech Republic (then under the name Czech-Moravian Football Association) was using fixtures based on $GK(16)$, probably the best-known one-factorization of K_{16}, introduced by Kirkman in 1847 [6]. $GK(2n)$ is widely used in sports scheduling, including soccer; see, e.g., [5]. For more on usage of one-factorizations in sports scheduling, we refer the reader to [1] and [2].

Since the 2001–2002 season, the league has been scheduled by the author, with the use of sets of fixtures prepared by Mariusz Meszka. Besides the soccer league, the author has over the years scheduled the Czech National Hockey League, Czech National Basketball League (see [4]), several NCAA

Key words and phrases: Tournament scheduling, one-factorization, directed one-factors

AMS (MOS) Subject Classifications: 05C70

Received: 17 July 2018
Accepted: 4 September 2018

football and basketball conferences, and with Jeff Dinitz the original XFL in 2001 (see [3]).

In 2016, the newly established Football League Association took over the CNFL operation and decided to add post-season tournaments, starting in 2018.

2 Post-season tournament

After playing the two-leg round-robin, the teams are split into three divisions: top six, middle four, and bottom six. The middle four play a two-leg (home-away) single-elimination tournament and the winner than plays a home-away play-off against the fourth or fifth team from the top division, depending on the number of teams qualifying for the UEFA competitions in the given year. The winner than plays UEFA Europa League the following season.

The top and bottom divisions play a one-leg round-robin, with the top three teams in each division playing three home and two away games, while the bottom teams play two home and three away games. The points won in the post-season tournament are added to the points earned in the regular season.

3 Tournament properties

In early 2017, the Football League Association announced a press conference where the format and schedule of the top and bottom divisions would be announced, and only then asked the author to find a schedule satisfying a set of requirements, which we list below. Not too surprisingly, the Association contacted the author just two days before the press conference.

The requirements in decreasing order of importance were as follows:

1. One-leg round-robin tournament of 6 teams.
2. The top three teams play 3 home and 2 away games, the bottom three teams play 2 home and 3 away games.

3. No team plays three consecutive home or away games.
4. No team plays two home (or away) games in the last two rounds.

	1	2	3	4	5	6
1	--	*H*	*H*	*H*	*A*	*A*
2	*A*	--	*H*	*H*	*H*	*A*
3	*A*	*A*	--	*H*	*H*	*H*
4	*A*	*A*	*A*	--	*H*	*H*
5	*H*	*A*	*A*	*A*	--	*H*
6	*H*	*H*	*A*	*A*	*A*	--

Table 1: Required structure of home and away games

5. The structure of home and away games is given in Table 1. The entry H (or A) in row i and column j indicates that the game between teams i and j is a home (or away) game for team i.
6. The game between the top two teams is played in Round 2 (or Round 3 at worst).
7. No team plays both two home and two away games. That is, there is no home-away pattern $HHAAH$ or $AAHHA$.

One can see that the requirements are somewhat redundant. In particular, Requirement 2 follows from Requirement 5. There are two reasons for that. A common one is that the requirements are defined by non-mathematicians, who do not care about exactness and clarity as much as mathematicians do. The main reason in this case, however, is different. The FLA Director was well aware that it may happen that not all requirements could be satisfied, and considered Requirement 2 much more important than the structure of home and away games given in Requirement 5. Hence, the scheduler was given the freedom to potentially deviate from Requirement 5 if necessary, while Requirement 2 was considered an absolute must.

The reader may also wonder why the game between the top two teams is to be scheduled in Round 2 or 3. It seems more attractive to schedule their game for the last round to add suspense and have possibly the league winner decided by the "final clash." The scheduler was also surprised by this and asked the FLA Director about it. The league is obviously afraid that the point gap between the top two teams after the regular season can be too big (recall that the points won in the tournament are added to regular season points). For instance, if the gap is 4–6 points after Round 1, the game can be seen as the last chance to narrow it to reasonable 1–3 point difference (there are 3 points for a win, and 1 for a tie). On the other hand, the same game played in Round 5 is of any significance if the gap before that is at most 3 points.

4 Schedule

Due to limited time before the press conference, it was impossible to examine the problem in detail. Hence, between teaching classes and sitting on committee meetings the author cobbled together the schedule shown in Table 2 and relied on his intuition and experience to claim that Requirement 7 cannot be satisfied.

R1	R2	R3	R4	R5
1-3	1-2	2-5	2-3	1-4
2-4	3-6	3-4	4-6	3-5
5-6	4-5	6-1	5-1	6-2

Table 2: Tournament schedule

We formalize the existence of the schedule in the following.

Observation 4.1. *There is a schedule satisfying Requirements 1–6 stated above.*

Only several weeks after the schedule was announced at the press conference, the author assured the Association that he had proved the non-existence of schedules satisfying Requirements 1–7. While the Association took the assurance for granted, we present a proof below.

We call the sequence of entries H and A (standing for home and away games, respectively) the *home-away pattern*, or simply HAP. The subsequence HH or AA is called a *break* in the pattern.

Proposition 4.2. *There is no schedule satisfying Requirements 1–7 stated above.*

Proof. We begin by observing that no two teams can have the same home-way pattern, because they could never play each other. Also, there can be at most two teams without break in their HAP. Suppose there are at least three. Then WLOG, two of them start home and must have the same HAP, a nonsense.

Now we proceed by contradiction and assume that Requirements 1–7 are satisfied. Let the teams starting with a home game be T_1, T_2, and T_3. Then the only admissible home-away patterns for teams T_1, T_2, and T_3 are $P_1 = HHAHA, P_2 = HAAHA, P_3 = HAHHA$, and $P_4 = HAHAH$, because by Requirement 3 we cannot have the sub-sequence HHH, and by Requirement 4 the pattern cannot end with HH. Similarly, the only possible HAPs for teams T_4, T_5, and T_6 are $Q_1 = AAHAH, Q_2 = AHHAH, Q_3 = AHAAH$, and $Q_4 = AHAHA$. All eight HAPs are shown in Table 3.

	R1	R2	R3	R4	R5
P_1	H	H	A	H	A
P_2	H	A	A	H	A
P_3	H	A	H	H	A
P_4	H	A	H	A	H
Q_1	A	A	H	A	H
Q_2	A	H	H	A	H
Q_3	A	H	A	A	H
Q_4	A	H	A	H	A

Table 3: Admissible home-away patterns

By $T(P_k)$ or $T(Q_k)$ we denote the team with HAP P_k or Q_k, respectively. Hence, $T(P_k) = i$ means that team i has the home-away pattern P_k.

First we observe that we cannot have teams with patterns P_1, P_2, P_3, as shown in Table 4. The only rounds in which they can play their three mutual games are Rounds 2 and 3, since in the remaining rounds they either play all home, or all away. However, we cannot schedule these three games in two rounds, as one team would have to play two games in the same round. Reasoning in the same way we cannot have three teams with patterns Q_1, Q_2, Q_3.

	R1	R2	R3	R4	R5
P_1	*H*	*H*	*A*	*H*	*A*
P_2	*H*	*A*	*A*	*H*	*A*
P_3	*H*	*A*	*H*	*H*	*A*

Table 4: Teams with patterns P_1, P_2, P_3

We cannot have HAPs P_1, P_2 and P_4 either. Suppose there are teams with these patterns. Then two of them play an away game in Round 3, and only one of patterns Q_3, Q_4 can be present, otherwise we have four teams playing away in Round 3 (see Table 5).

	R1	R2	R3	R4	R5
P_1	*H*	*H*	*A*	*H*	*A*
P_2	*H*	*A*	*A*	*H*	*A*
P_4	*H*	*A*	*H*	*A*	*H*
Q_3	*A*	*H*	*A*	*A*	*H*
Q_4	*A*	*H*	*A*	*H*	*A*

Table 5: Teams with patterns P_1, P_2, P_4, Q_3, Q_4

This forces both Q_1 and Q_2 to be used. Then teams $T(Q_1)$ and $T(Q_2)$ can play their mutual game only in Round 2 as their HAPs are identical in all other rounds. See Table 6.

	R1	R2	R3	R4	R5
P_4	<u>*H*</u>	*A*	*H*	*A*	*H*
Q_1	<u>*A*</u>	***A***	*H*	*A*	*H*
Q_2	<u>*A*</u>	***H***	*H*	*A*	*H*

Table 6: Teams with patterns P_4, Q_1, Q_2

But then $T(P_4)$ can play each of $T(Q_1)$ and $T(Q_2)$ only in Round 1, since in Round 2 $T(Q_1)$ and $T(Q_2)$ play each other and their HAPs are identical in the remaining three rounds. By similar reasoning, we can show that the patterns P_2, P_3 and P_4 are not compatible. This leaves us with the only remaining combination, namely P_1, P_3 and P_4. At the same time, by symmetry it follows that the only combination of the HAPs starting with A is Q_1, Q_3 and Q_4.

We recall that according Requirement 6, the game between teams 1 and 2 is played in Round 2 or 3. First we consider the former and assume that the game is played in Round 2. Each of the top three teams must have one of the patterns P_1, P_3 and P_4, since these are the only ones with three home games. The only top team playing a home game in Round 2 is the team with pattern P_1, so we must have $T(P_1) = 1$. Assume that $T(P_3) = 2$ and $T(P_4) = 3$.

	R1	R2	R3	R4	R5
1	*H*	***H***	*A*	*H*	*A*
2	*H*	***A***	*H*	*H*	*A*
3	*H*	*A*	*H*	*A*	*H*

Table 7: Teams $1, 2, 3$ with patterns P_1, P_3, P_4

Team 3 is now scheduled to play away games in Rounds 2 and 4. But, by Requirement 5, the only two games team 3 plays away are against teams 1 and 2, and these teams are already scheduled to play each other in Round 2. Obviously, team 3 can play an away game in Round 4 against either 1 or 2, but the other game cannot be scheduled and we have a contradiction.

The argument is similar when team 2 has pattern P_4 and team 3 has P_3. That is, $T(P_4) = 2$ and $T(P_3) = 3$ (see Table 8).

	R1	R2	R3	R4	R5
1	*H*	***H***	*A*	*H*	*A*
3	*H*	*A*	*H*	*H*	*A*
2	*H*	***A***	*H*	*A*	*H*

Table 8: Teams $1, 3, 2$ with patterns P_1, P_3, P_4

The only available round for team 3 to play either team 1 or 2 is Round 5, in which team 3 can play only one of 1 and 2 and the other game cannot be scheduled.

If the game between teams 1 and 2 is played in Round 3, then we must have $T(P_1) = 2$, as 2 is the only top team playing away in that round. Now we have $\{T(P_3), T(P_4)\} = \{1, 3\}$. We thus distinguish two cases.

Case 1. $T(P_3) = 1, T(P_4) = 3$

	R1	R2	R3	R4	R5
2	*H*	*H*	*A*	*H*	*A*
1	*H*	*A*	*H*	*H*	*A*
3	*H*	*A*	*H*	*A*	*H*

Table 9: Teams $2, 1, 3$ with patterns P_1, P_3, P_4

The only round when team 1 is scheduled home and team 3 away is Round 4, therefore the game 1–3 must be in Round 4 by Requirement 5. For similar reasons the game 2–3 must be either in Round 2 or in Round 4. Anyway we have just said that in Round 4 team 3 is scheduled against team 1. Thus the game 2–3 is necessarily played in Round 2.

According to Requirement 5, the only lower ranked team that team 2 plays away is 6. Hence, we must have the game 6–2 in Round 5. Team 1 plays away in Round 5. By Requirement 5, team 1 plays away teams 5 and 6, so we are forced to schedule 5–1 in Round 5. Thus the remaining game of Round 5 is 3–4 and hence $T(Q_4) = 4$, since both Q_1 and Q_3 end with H.

R1	R2	R3	R4	R5
	2-3	**1-2**	1-3	6-2
				5-1
				3-4

Table 10: Partial schedule for Case 1

Now $T(Q_4) = 4$ implies that in Round 3 we have either the game 5–4 or 6–4, which contradicts Requirement 5. Hence, this case is impossible.

Case 2. $T(P_3) = 3, T(P_4) = 1$ (This implies that $T(P_1) = 2$.)

	R1	R2	R3	R4	R5
2	*H*	*H*	*A*	*H*	*A*
3	*H*	*A*	*H*	*H*	*A*
1	*H*	*A*	*H*	*A*	*H*

Table 11: Teams $2, 3, 1$ with patterns P_1, P_3, P_4

Again the only round when team 2 is scheduled home and team 3 away is Round 2, and we have the game 2–3 in that round. Similarly, we must have the game 1–3 in Round 5. According to Requirement 5, the only lower ranked team that team 2 plays away is 6. Hence, we must have the game 6–2 in Round 5. By Requirement 5, the remaining game is 4–5, and $T(Q_4) = 5$.

Obviously, in rounds 1 and 4 we can only schedule games between the upper three and the lower three teams.

R1	R2	R3	R4	R5
	2-3	1-2		1-3
				6-2
				4-5

Table 12: Partial schedule for Case 2

Partial assignment of HAPs is shown in Table 13.

	R1	R2	R3	R4	R5
2	*H*	*H*	*A*	*H*	*A*
3	*H*	*A*	*H*	*H*	*A*
1	*H*	*A*	*H*	*A*	*H*
Q_1	*A*	*A*	*H*	*A*	*H*
Q_3	*A*	*H*	*A*	*A*	*H*
5	*A*	*H*	*A*	*H*	*A*

Table 13: Partial HAPs assignment for Case 2

From the HAPs in Table 13 we see that in Round 4 team 1 can only play team 5. So 5–1 is in Round 4 which necessarily implies that 1–4 is in Round 1 and that the remaining game of Round 4 is 3–6. In Round 1 we also have the game 2–5 since 4 is already scheduled in this round and the games 1–2, 2–3 and 6–2 have been already scheduled elsewhere. But now the last game in Round 1 should be 3–6 which we already scheduled in Round 4. Therefore, Case 2 cannot completed either.

R1	R2	R3	R4	R5
1-4	2-3	1-2	5-1	1-3
2-5			2-4	6-2
3-6			3-6	4-5

Table 14: Partial schedule for Case 2

We have exhausted all possibilities and the proof is now complete. □

Finally, in Table 15 we present the home-away patterns induced by the schedule shown in Table 2 and summarize our findings.

	R1	R2	R3	R4	R5
1	*H*	*H*	*A*	*A*	*H*
2	*H*	*A*	*H*	*H*	*A*
3	*A*	*H*	*H*	*A*	*H*
4	*A*	*H*	*A*	*H*	*A*
5	*H*	*A*	*A*	*H*	*A*
6	*A*	*A*	*H*	*A*	*H*

Table 15: Tournament schedule

Proposition 4.3. *There is no schedule satisfying Requirements 1–7 stated above. However, there are schedules satisfying Requirements 1–6 for all teams and Requirement 7 for all teams but one.*

Acknowledgement

The author thanks the anonymous referees for their valuable suggestions, which significantly helped to improve the paper.

References

[1] I. Anderson, *Combinatorial Designs and Tournaments*, Clarendon Press, Oxford, 1997.

[2] D. de Werra, Scheduling in sports, *Ann. Discrete Math.*, **11** (1981), 381–395.

[3] J. Dinitz and D. Froncek, Scheduling the XFL, *Congr. Numer.*, **147** (2000), 5–15.

[4] D. Froncek, Scheduling Czech National Basketball League, *Congr. Numer.*, **153** (2001), 5–24.

[5] T. Griggs and A. Rosa, A tour of European soccer schedules, or Testing the popularity of GK_{2n}, *Bull. Inst. Combin. Appl.*, **18** (1996), 65–68.

[6] T. P. Kirkman, On a problem in combinatorics, *Cambridge and Dublin Math. J.*, **2** (1847), 191–204.