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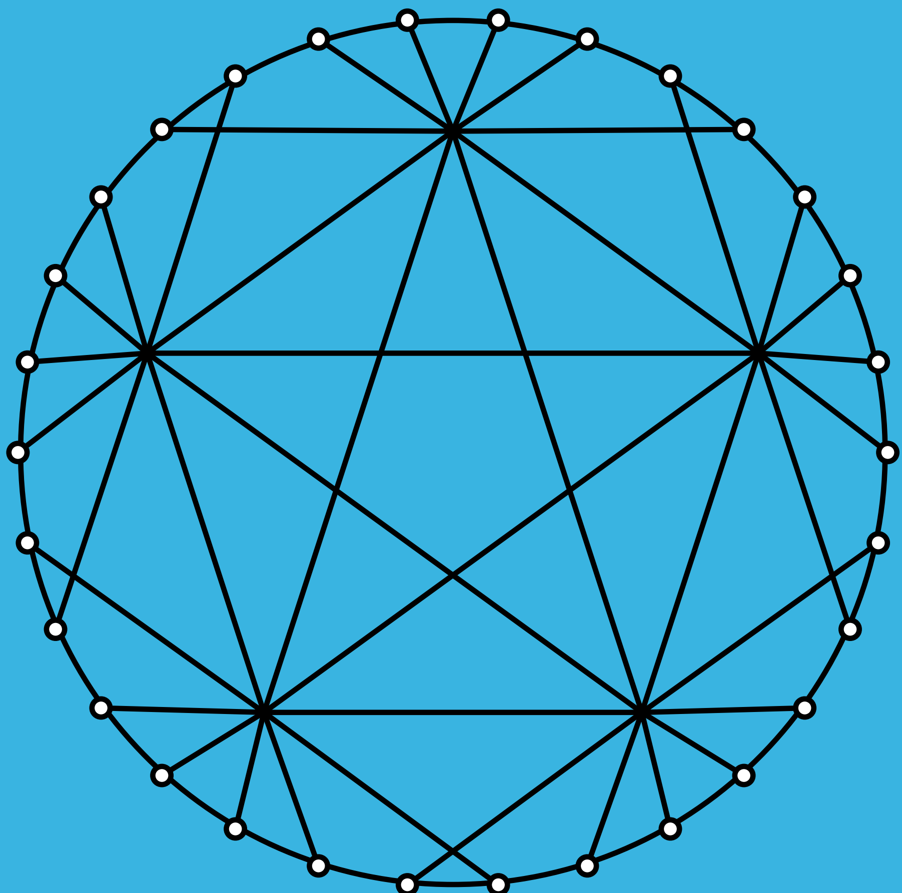

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A new graph coloring scheme

Ronald D. Dutton*and Robert C. Brigham

University of Central Florida, Orlando, Florida, U.S.A.
dutton@cs.ucf.edu and robert.brigham@ucf.edu

Abstract: Let G be a graph with vertices labeled $v_1, v_2, \ldots, v_n$. The Mycielski construction is the inspiration for a graph coloring scheme, called an IS coloring, in which the colors available for a vertex depend on the indices of the vertex and its neighbors. Specifically, a proper coloring is an IS coloring if, whenever the color of v_i is j, then $j \neq i$ and v_i is not adjacent to v_j. Such a coloring sometimes can force a color greater than n, although this paper concentrates on graphs for which that is not the case. Many properties of IS colorings are developed, and a relationship to a new type of marriage problem is shown.

1 Introduction

A proper k-coloring c of a graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots, k\}$ such that adjacent vertices have distinct images. A typical application is class scheduling. Over time several other coloring schemes have been developed due both to theoretical interest and demands of alternative applications. An introduction to many of these can be found in Chartrand and Zhang [1]. This paper describes a new method for vertex coloring a graph which was inspired by the Mycielski construction [3].

The symbols $N_G(v)$ and $N_G[v]$ represent the open and closed neighborhoods, respectively, of vertex v in graph G. If c is a coloring of G, $c(v)$ is

*Corresponding author.

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the color given vertex v, and c_i is the set of vertices of G colored i, $1 \leq i \leq n$, where $n = |V(G)|$. If no vertex is colored i, c_i is said to be an *empty* color class. The number of vertices in a largest clique of G is denoted $\omega(G)$; the vertex independence number of G, that is, the largest set of vertices such that no two are adjacent, is represented by $\beta_0(G)$; and $\delta(G)$ refers to the minimum degree of G. Unless otherwise noted, $|V(G)| = n$.

It is convenient to define the Mycielski graph of $\overline{G}$, the complement of the graph G of interest below. Assume $\overline{G}$ has vertices $V = \{v_1, v_2, \ldots, v_n\}$. The Mycielski graph, denoted M, of $\overline{G}$ is constructed from it by adding new vertices $W = \{w_1, w_2, \ldots, w_n\}$ and z, with $N_M(w_i) = N_{\overline{G}}(v_i) \cup \{z\}$ for $1 \leq i \leq n$. In the complement, $\overline{M}$, V induces $G = (V, E)$, W induces K_n, z is adjacent to all vertices of G, and $N_{\overline{M}}[w_i] = N_G[v_i] \cup W$ for $1 \leq i \leq n$.

At most $2n$ colors are required for any proper coloring c' of $\overline{M}$. Without loss of generality we may assume that $c'(w_i) = i$ for $1 \leq i \leq n$. Then the restriction of c' to G produces a proper coloring c of G. However, there are constraints on c. Since $w_i v_i$ is an edge of $\overline{M}$, $c(v_i)$ is not equal to i. Furthermore, if $v_i v_j \in E(G)$, $v_i w_j$ is an edge of $\overline{M}$ so $c(v_i)$ is not equal to j. These limitations suggest the following definition for a new type of coloring for arbitrary graphs, called an Index Sensitive coloring.

Definition 1.1. *Let c be a proper coloring of the graph G with vertices $v_1, v_2, \ldots, v_n$. Then c is an* Index Sensitive (IS) *coloring of G if, when $c(v_i) = j$, $j \neq i$ and v_i is not adjacent to v_j.*

A consequence of Definition 1.1 is that the color of any vertex in an IS coloring must be different from its own index, from the index of all its neighbors, and from the color of all its neighbors. Thus a different indexing of the vertices likely will require a different coloring. An IS coloring of G using k colors is called an *IS k-coloring* of G. Note that an IS k-coloring of G does not mean the color classes $c_1, c_2, \ldots, c_k$ are the ones that are not empty since the colors are tied to the indices of the vertices. Figure 2 in Section 2.2 illustrates this, where the assigned colors are shown in circles.

A simple example of an IS coloring is given by C_5. Let the vertices be v_1, v_2, v_3, v_4, v_5 in order. Vertex v_1 is adjacent to v_2 and v_5 so it can be colored only 3 or 4 which are equivalent by symmetry. Color v_1 by 4. Now v_2 is adjacent to vertices v_1 and v_3 as well as a vertex colored 4 so it must be colored 5. Similarly the colors on v_3, v_4, and v_5 are forced to be 1, 2, and 3, respectively. See Figure 1. Since five colors are used, this is an IS 5-coloring of C_5.

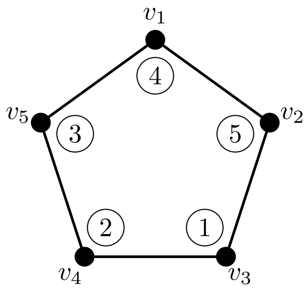

Figure 1: An IS 5-coloring of C_5

A further consequence of Definition 1.1 is the following useful result.

Proposition 1.2. *A proper coloring c of a graph G with no isolated vertex is an IS coloring if and only if, for every edge* v_iv_j, $|\{i, c(v_i), j, c(v_j)\}| = 4$.

Proof. Assume c is an IS coloring of G and v_iv_j is an arbitrary edge. Vertices v_i and v_j are distinct so $i \neq j$; c is a proper coloring meaning $c(v_i) \neq c(v_j)$; c is an IS coloring implying $c(v_i), c(v_j) \notin \{i, j\}$. Thus $|\{i, c(v_i), j, c(v_j)\}| = 4$.

Next assume c is not an IS coloring of G. Then either there is a vertex v_i such that $c(v_i) = i$, or there is an edge v_iv_j where $c(v_i) = j$ or $c(v_i) = c(v_j)$. In each possibility $|\{i, c(v_i), j, c(v_j)\}| \leq 3$. □

With traditional coloring, there never is a need for a color larger than $n = |V(G)|$. That is not the case for IS colorings, even when the total number of colors required is small. A simple example is $K_{1,n-1}$, $n \geq 3$, with leaves $v_1, v_2, \ldots, v_{n-1}$ and central vertex v_n. Let c be an IS coloring. It follows from Definition 1.1 that vertex v_n cannot be colored with any color less than or equal to n, so $c(v_n) \geq n + 1$. The rest of the IS coloring can be $c(v_1) = 2$ and $c(v_2) = c(v_3) = \ldots = c(v_{n-1}) = 1$. Thus only three colors are required, but one must be greater than n. If $G = K_n$, n distinct colors are necessary in an IS coloring of it, all of which are larger than n. In fact, if $\omega(G) > n/2$, at least $2\omega(G) - n$ colors are required that are greater than n. The remainder of this paper considers only graphs that have IS colorings that do not require a color greater than n.

The following definition is a natural extension of the traditional chromatic number of a graph.

Definition 1.3. *The* IS chromatic number $\chi_{IS}(G)$ *of graph G is the smallest k such that there is an IS k-coloring of G.*

Since the IS coloring of C_5 shown above is forced, five colors are necessary. Thus $\chi_{IS}(C_5) = 5$. In Section 2 it is shown that any minimum IS coloring of G, no matter what the indexing of the vertices, uses $\chi_{IS}(G)$ colors, and furthermore there is a vertex indexing for which $c_1, c_2, \ldots, c_k$ are all nonempty, where $k = \chi_{IS}(G)$. Section 2 also develops several other properties of IS colorings. Section 3 introduces a new marriage problem whose solution depends on IS colorings and Section 4 discusses results related to when $\chi_{IS}(G) = n$.

2 Properties

2.1 Elementary properties

The properties of traditional colorings are well known. Because the indices of vertices influence choices in IS colorings, it is not obvious that those properties still hold for them. This section shows several do remain valid. The following theorem demonstrates how a given IS coloring associated with a specific indexing can be transformed to another IS coloring if the indexing of the vertices is altered.

Theorem 2.1. (Reindexing Theorem) *Let c be an IS coloring of graph G with vertices labeled $v_1, v_2, \ldots, v_n$, none of which are isolated. Let π be a permutation of the integers $1, 2, \ldots, n$. Then the coloring c' defined by $c'(v_{\pi(i)}) = \pi(c(v_i))$ is an IS coloring of G using the same number of colors as c.*

Proof. Suppose $v_i v_j$ is an edge of G. By Proposition 1.2,

$$|\{i, c(v_i), j, c(v_j)\}| = 4.$$

After the reindexing specified by π, the same edge is indicated by $v_{\pi(i)} v_{\pi(j)}$. Then

$$|\{\pi(i), c'(v_{\pi(i)}), \pi(j), c'(v_{\pi(j)})\}| = |\{\pi(i), \pi(c(v_i)), \pi(j), \pi(c(v_j))\}| = 4$$

since the four integers in the last set result from a permutation of four distinct integers. Again appealing to Proposition 1.2, c' is an IS coloring.

Since every color of c' is a permutation of a color of c, the number of colors in c' is the same as the number of colors in c. $\square$

It follows from Theorem 2.1 that $\chi_{IS}(G)$ can be determined from any indexing of the vertices. In a standard coloring of a graph the colors of two color classes can be interchanged and the result is another proper coloring of the graph. A similar result holds for IS colorings.

Corollary 2.2. *Let c be an IS k-coloring of graph G. Then, for any two integers r and s such that $1 \leq r < s \leq n$, a new IS k-coloring of G is obtained by interchanging both the indices of vertices v_r and v_s and the vertices in classes c_r and c_s.*

Proof. Employ Theorem 2.1 using the permutation π on $\{1, 2, \ldots, n\}$ defined by $\pi(i) = i$ if $i \notin \{r, s\}$, $\pi(r) = s$, and $\pi(s) = r$. The indices of v_r and v_s are interchanged. Since $c'(v_{\pi(i)}) = \pi(c(v_i))$, any vertex originally colored s is now colored r and vice versa. $\square$

Any application of Corollary 2.2 keeps all color classes the same except for c_r and c_s for which the vertices are interchanged. Thus, while the colors may differ, the partition of vertices into color classes remains the same. Corollary 2.2 is valid even if one or both of c_r and c_s are empty. This observation allows the conclusion that, if G has an IS k-coloring with $k \leq n$, there is an indexing of the vertices such that only colors 1 through k are used.

Theorem 2.3. *Let c be an IS k-coloring of G with $k \leq n$. Then there is an IS k-coloring c' of G in which color classes $c'_1, c'_2, \ldots, c'_k$ are not empty.*

Proof. Let c_r be an empty color class with $r \leq k$ and c_s be a nonempty color class with $s > k$. Then by Corollary 2.2 there is an IS k-coloring of G in which color class r is nonempty and color class s is empty. Repeating as necessary yields the result. $\square$

If there is a standard coloring of a graph with $k < n$ colors, there is one with $k + 1$ colors. All that is necessary is replace the color of any vertex whose color class contains at least two vertices by $k+1$. A similar property is also true for IS colorings, but showing it is more difficult.

Definition 2.4. *Let c be an IS coloring of a graph. Two color classes c_i and c_j are an* excluded pair *if $|c_i| = |c_j| = 1$, $v_j \in c_i$ and $v_i \in c_j$.*

The reason for the term "excluded" in Definition 2.4 will become evident in the proof of the next theorem because such a pair will not play a role in the development.

Theorem 2.5. *Let graph G have an IS k-coloring where $k < n$. Then there is an IS $(k+1)$-coloring of G.*

Proof. Since $k < n$, there are three color classes c_i, c_j and c_r such that $c_i = \emptyset$, $v_i \in c_j$ and $v_j \in c_r$. None of these color classes is in an excluded pair.

Suppose c_j contains a second vertex v_s, where $s = r$ is a possibility. Then move v_s from c_j to c_i. If c_r contains a second vertex v_s, move v_j to c_i. In either case the result is an IS $(k+1)$-coloring of G.

If $|c_j| = |c_r| = 1$, move v_j from c_r to c_i to create a new IS k-coloring where c_i is no longer empty, but c_r now is. The process can be repeated. However, c_i and c_j will not be involved in any repetition because they form an excluded pair and no color classes in an excluded pair play a role in the process. Thus the number of color classes that still can play a role is decreased by two and iteration must eventually lead to one of the previous cases that create an IS $(k+1)$-coloring. $\square$

Appealing to Theorem 2.5 repeatedly shows that, if $\chi_{IS}(G) = k$, there is an IS j-coloring for $k \leq j \leq n$. The concept of an excluded pair can be extended to an excluded set of color classes, that is, a collection of classes where each contains exactly one vertex and the set of colors is identical to the set of indices of the vertices in those classes. No color class in any such excluded set is involved in the proof of Theorem 2.5. Excluded pairs are sufficient in that proof.

2.2 Graphs with given number of vertices and χ_{IS} value

This section demonstrates the existence of connected graphs for most values of χ_{IS} and any value of n. The following definition is key to showing this.

Definition 2.6. *Let n and p be positive integers such that $n \geq 2p$. Then $Q(p,n)$ is a complete p-partite graph with partite sets S_i such that $|S_i| \geq 2$ for $1 \leq i \leq p$.*

Lemma 2.7. $\chi_{IS}(Q(p,n)) = 2p$.

Proof. A vertex $v_j \in S_i$ can be colored k only if $v_k \in S_i$. Similarly v_k can be colored m only if $v_m \in S_i$. Thus at least two colors are required to color the vertices of each S_i, implying $\chi_{IS}(Q(p,n)) \geq 2p$. The following is an IS coloring with $2p$ colors. For any set S_i select two distinct contained vertices v_j and v_k. Color v_j with color k and all remaining vertices of S_i, including v_k, with color j. □

The *sum* $G + H$ of graphs G and H is the graph obtained from G and H by adding an edge between every vertex of G and every vertex of H.

Lemma 2.8. *Let G and H be two graphs. Then $\chi_{IS}(G + H) = \chi_{IS}(G) + \chi_{IS}(H)$.*

Proof. Let the vertices corresponding to G in $G + H$ be $v_1, v_2, \ldots, v_n$ and those corresponding to H be $v_{n+1}, v_{n+2}, \ldots, v_{n+s}$. In $G + H$ no vertex of either graph can be colored with the index of a vertex in the other graph. Therefore, any IS coloring of G in $G+H$ must employ only colors 1 through n while only colors $n + 1$ to $n + s$ can be used for H. This establishes the result as an upper bound. However, no IS coloring of G or H can use fewer than $\chi_{IS}(G)$ or $\chi_{IS}(H)$ colors, respectively, and the lower bound is shown. □

Theorem 2.9. *Let $m = 3$ and $n \geq 5$ or $4 \leq m \leq n$. Then there is a connected graph G on n vertices for which $\chi_{IS}(G) = m$.*

Proof. By Lemma 2.7, $G = Q(m/2, n)$ demonstrates the result when m is even. When m is odd and at least 7, let $G = C_5 + Q((m-5)/2, n-5)$. Then Lemmas 2.7 and 2.8 show $\chi_{IS}(G) = 5 + 2[(m-5)/2] = m$. If $n = m = 5$, the C_5 indexed and colored as in Figure 1 illustrates the result. When $n \geq 6$, add to that C_5 an independent set of $n - 5$ vertices $v_6, v_7, \ldots, v_n$, each adjacent to v_1, v_3 and v_4, and color all of them with 5. There are no connected graphs G on four or fewer vertices with $\chi_{IS}(G) = 3$. Figure 2 shows such a graph for any $n \geq 5$. □

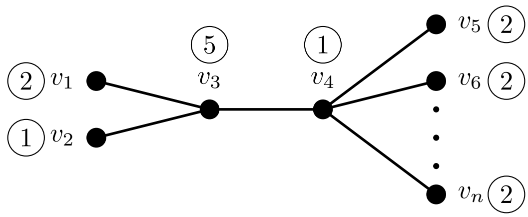

Figure 2: A graph G on $n \geq 5$ vertices and $\chi_{IS}(G) = 3$

2.3 Relationship with vertex independence number

Let G be a graph with indexed vertices $\{v_1, v_2, \ldots, v_n\}$ and IS coloring c. From Proposition 1.2, if v_iv_j is an edge, then $|\{i, c(v_i), j, c(v_j)\}| = 4$. However, that condition can be satisfied even if v_iv_j is not an edge. In such an instance the edge can be added and c remains an IS coloring for the revised graph. If all possible such edges are added, the resultant graph is said to be *edge maximal with respect to c*. It follows that, if G is edge maximal with respect to c, v_iv_j is an edge if and only if $|\{i, c(v_i), j, c(v_j)\}| = 4$. Conversely, if v_i is not adjacent to v_j in an edge maximal graph, then at least one of $c(v_i) = c(v_j)$, $i = c(v_j)$, or $j = c(v_i)$ must hold. One might expect, if a graph has an IS coloring c, adding all edges as above would create a graph G for which $\chi_{IS}(G)$ is equal to the number of colors used in c. This is not necessarily the case. As an example, Figure 3(a) shows an IS 5-coloring c of P_5. It is easy to check that $\chi_{IS}(P_5) = 4$. Using c, the edge maximal graph with respect to it is obtained by adding edges v_1v_4 and v_2v_5 as shown in Figure 3(b). But the result can be IS-colored with four colors, that is, adding all the edges possible did not raise the IS chromatic number to the number of colors of c.

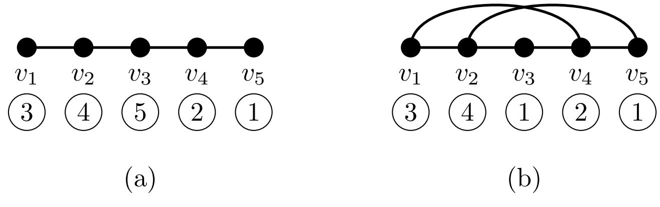

Figure 3: Edge maximal graph with respect to a 5-coloring c using fewer than five colors

The main result of this subsection is an upper bound on $\beta_0(G)$ for edge maximal graphs. A series of preliminary results is required.

Lemma 2.10. *Let G be a graph with $\chi_{IS}(G) = n$ and an IS n-coloring c. Then for every vertex v_i, $N_G[v_i]$ contains at least one of v_j and the unique vertex colored j, for $1 \leq j \leq n$.*

Proof. Let j be an integer such that $N_G[v_i]$ contains neither v_j nor the vertex colored j. Then v_i can be recolored j, resulting in an IS $(n-1)$-coloring of G, a contradiction. $\square$

Lemma 2.11. *For graph G, $\chi_{IS}(G) \geq n/(\beta_0(G) - 1)$.*

Proof. Since G is not complete, $\beta_0(G) \geq 2$. Consider any χ_{IS}-coloring c of G. For any nonempty color class c_i, $v_i \notin c_i$ and has no neighbors in c_i, so $c_i \cup \{v_i\}$ is an independent set of vertices. Thus $|c_i| \leq \beta_0(G) - 1$. Hence $n \leq \chi_{IS}(G)(\beta_0(G) - 1)$. $\square$

The remainder of this section deals with edge maximal graphs.

Lemma 2.12. *Let G be an edge maximal graph with respect to an IS coloring c. If a pair of vertices in an independent set X are colored the same, say r, then every vertex in X, except v_r, is colored r. Thus $|X| \leq |c_r| + 1 \leq \beta_0(G)$.*

Proof. Consider any two vertices v_i and v_j in c_r. Let v_t be any other vertex in X and suppose it is not in c_r. Since G is edge maximal,

$$|\{i, c(v_i), t, c(v_t)\}| = |\{i, r, t, c(v_t)\}| < 4.$$

Since $i \neq t$ and $c(v_i) \neq c(v_t)$, either $i = c(v_t)$ or $t = c(v_i) = r$. Similarly either $j = c(v_t)$ or $t = c(v_j) = r$. It follows that $t = r$ and this can happen for only one vertex. $\square$

Another possibility is if every vertex in X is in a different color class and the next lemma deals with that.

Lemma 2.13. *Let G be an edge maximal graph with respect to an IS coloring c. If no pair of vertices in an independent set X are colored the same, then $|X| \leq 3$.*

Proof. Suppose $|X| \geq 4$. Assume v_i, v_j, and v_t are in X. Since all colors are different, we must have either $i = c(v_j)$ or $j = c(v_i)$. Without loss of generality assume $i = c(v_j)$. Then we need $i = c(v_t)$ or $t = c(v_i)$. The first is not possible, so $t = c(v_i)$. Finally one of $t = c(v_j)$ or $j = c(v_t)$ must hold. Again the first is not possible, so $j = c(v_t)$. Now consider a fourth vertex v_s. It is necessary that either $i = c(v_s)$ or $s = c(v_i)$, neither of which is possible since $i = c(v_j) \neq c(v_s)$ and $s \neq t = c(v_i)$. □

Theorem 2.14. *Let G be an edge maximal graph with respect to a $\chi_{IS}(G)$-coloring c. Then $\chi_{IS}(G) = n$ if and only if $\beta_0(G) = 2$.*

Proof. Assume $\beta_0(G) = 2$. From Lemma 2.11 and the fact $\chi_{IS}(G) \leq n$, $\chi_{IS}(G) = n$. Suppose next that $\chi_{IS}(G) = n$ and G has three independent vertices v_i, v_j, and v_t. Each is the only vertex in its color class so by the proof to Lemma 2.13 we may assume $i = c(v_j)$. Then, by Lemma 2.10, v_t must have a neighbor indexed by i, that is, v_i, or one colored i, that is, v_j. But v_t is independent of v_i and v_j, so this is impossible. □

Edge maximal is not required for the first part of the above proof, so, for any graph G, $\beta_0(G) = 2$ implies $\chi_{IS}(G) = n$.

Theorem 2.15. *Let G be an edge maximal graph with respect to a $\chi_{IS}(G)$-coloring. For graph G, $\beta_0(G) \leq n - \chi_{IS}(G) + 2$.*

Proof. Let $k = n - \chi_{IS}(G)$ and X be a maximum independent set. Exactly k color classes are empty which means k vertices are in the same color classes as other vertices. It follows that no color class can contain more than $k + 1$ vertices. Theorem 2.14 covers the case when $k = 0$, so assume $k \geq 1$. Suppose X is composed of vertices from different color classes. Then Lemma 2.13 indicates $|X| = \beta_0(G) \leq 3 \leq k + 2$ since $k \geq 1$. If X contains two vertices in the same color class, Lemma 2.12 shows only one vertex not in that class is independent of all the vertices there. Thus $|X| \leq (k + 1) + 1$. □

3 Sibling marriage problem

In the Classical Marriage Problem, n men and n women seek matrimony with harmonious man/woman pairs specified. The goal is to determine if there is a pairing such that everyone is happily married. This can be thought of as a bipartite graph with the partite sets being the men and women and an edge signifying a compatible couple. A solution is then a perfect matching of the graph, that is, a set of n independent edges. One can think of the problem as one defined on an arbitrary bipartite graph with equal sized partite sets and asking if there is a solution on that graph, that is, does the graph have a perfect matching?

Here a new marriage problem, the Sibling Marriage Problem, is discussed. It involves n brother/sister pairs $\{b_i, s_i\}$, $1 \leq i \leq n$, and all wish to marry happily. Of course, no brother should marry his sister. Furthermore, if two families F_i anf F_j do not get along, no marriage should join them. This can be modeled as an n-vertex graph G where the vertices represent the families F_i, $1 \leq i \leq n$, and an edge joins two families if and only if they are incompatible. It will be seen that a solution which allows everyone to be happily married is equivalent to an IS n-coloring on G. As with the classical problem, we can start with an arbitrary graph G and ask if there is a solution on that graph.

The description of graph G seems a bit counterintuitive as, unlike the classical problem, an edge means no marriage is allowed, the opposite of the classical problem. However, if one considers the complement $\overline{G}$ of G, edges refer to allowable pairings. Notice that if F_iF_j is an edge of $\overline{G}$, marriage between b_i and s_j is possible as is one between b_j and s_i. A solution might include none, one or both of those marriages. Unlike the classical problem, a solution is not necessarily a perfect matching in $\overline{G}$. Indeed, solutions are possible even when n is odd. A condition on $\overline{G}$ that does give a solution is discussed in Section 4.

Theorem 3.1. *Given graph G, the Sibling Marriage Problem has a solution if and only if there is an IS n-coloring of G.*

Proof. Suppose there is a solution. The index i of vertex F_i is interpreted as representing b_i. A color j represents s_j. Since there is a solution, color i is not assigned to F_i since no sister marries her brother. If color j is assigned to v_i, then families F_i and F_j are compatible and this means there is no edge between F_i and F_j. Thus this is an IS n-coloring of G. Now let there be an IS n-coloring of G. Then no sister s_i marries her brother b_i and anyone she does marry must correspond to no edge between the corresponding families. Thus there is a solution. □

In view of the comment following Theorem 2.5, there is a solution to the Sibling Marriage Problem on graph G if and only if $\chi_{IS}(G) \leq n$.

There is a straightforward Classical Marriage Problem equivalent to the Sibling Marriage Problem. Given graph G and employing the notation above, construct a bipartite graph B on $2n$ vertices with partite sets $\{b_1, b_2, \ldots, b_n\}$ and $\{s_1, s_2, \ldots, s_n\}$. Place no edge in B between s_i and b_i for $1 \leq i \leq n$, and place both edges $s_i b_j$ and $s_j b_i$ if and only if there is no edge joining F_i and F_j in G, that is, if and only if that edge is present in $\overline{G}$. Then there is a solution on G to the Sibling Marriage Problem if and only if there is a solution on B to the Classical Marriage Problem, that is, if and only if B contains a perfect matching. Since the transformation from G to B is polynomial as is the finding of a maximum matching on B [2], it follows that determining if there is an IS n-coloring on G is polynomial.

4 $\chi_{IS}(G) = n$

Lemma 2.10 deals with the special siituation of $\chi_{IS}(G) = n$. This is an important case because it lies at the border between graphs with IS coloring numbers at most n and those requiring a color greater than n, graphs not considered in this paper. A number of relevant results are presented in this section and together provide information about such graphs. The first is straightforward.

Observation 4.1. *For graph G, $\chi_{IS}(G) = n$ if and only if, for every IS n-coloring c of G, $|c_i| = 1$ for $1 \leq i \leq n$.*

Not surprisingly, $\chi_{IS}(G) = n$ requires a fairly large minimum degree.

Proposition 4.2. *For graph G, if $\chi_{IS}(G) = n$, then $\delta(G) \geq (n-2)/2$.*

Proof. Let v_i be a vertex of minimum degree. Then, for any IS n-coloring of G, $N_G[v_i]$ contains vertices with $\delta(G)+1$ different indices having $\delta(G)+1$ different colors. By Lemma 2.10 this collection of indices and colors must include every i for $1 \leq i \leq n$, that is, $2(\delta(G) + 1) \geq n$, yielding the result. □

Theorem 4.3. *For graph G, $\chi_{IS}(G) = n$ if and only if, for every IS n-coloring c and every vertex v_i, there is a vertex v_j where $j = c(v_i)$ such that $\{v_i, v_j\}$ forms an independent dominating set of G.*

Proof. Suppose $\chi_{IS}(G) = n$ and let v_i be an arbitrary vertex and v_j be the unique vertex such that $j = c(v_i)$. Vertices v_i and v_j are independent since

v_i cannot have a neighbor whose index is $c(v_i)$. Consider any vertex v_t, $t \notin \{i, j\}$. By Lemma 2.10, $N_G[v_t]$ possesses at least one of a vertex with index j or a vertex with color j. Since v_j is the only vertex indexed j and v_i the only vertex colored j, v_t must be adjacent to at least one of them. Thus $\{v_i, v_j\}$ is an independent dominating set of G.

Next suppose for every IS n-coloring c of G and any vertices v_i and v_j where $j = c(v_i)$, $\{v_i, v_j\}$ is an independent dominating set of G. Since c is an IS n-coloring of G, $\chi_{IS}(G) = k \leq n$. Assume $k < n$. Then there is a color class c_j containing at least two vertices, say v_i and v_r. Thus neither v_i nor v_j is adjacent to v_r so $\{v_i, v_j\}$ is not a dominating set, a contradiction implying $\chi_{IS}(G) = n$. □

From Theorem 2.14, $\chi_{IS}(G) = n$ for edge maximal graph G implies $\beta_0(G) = 2$. However, when G is not edge maximal, $\beta_0(G)$ can be significantly larger.

Theorem 4.4. *For graph G having at least four vertices, if $\chi_{IS}(G) = n$, then $\beta_0(G) \leq n/2$.*

Proof. The result holds for $\beta_0(G) = 2$ so assume c is an IS n-coloring of G and X is a set of $\beta_0(G) \geq 3$ independent vertices. Suppose there are two vertices in X, say v_r and v_t, such that $r = c(v_t)$. Since $\chi_{IS}(G) = n$, r is not the index or color assigned to any vertex in $V(G) - X$. Thus, for any third vertex v_i in X, $N_G[v_i]$ does not contain a vertex indexed r or colored r, a contradiction to Lemma 2.10. Therefore, this situation can not occur. It follows that each of the $\beta_0(G)$ colors assigned to the vertices of X match an index of a vertex in $V(G) - X$, that is, $n - \beta_0(G) = |V(G) - X| \geq |X| = \beta_0(G)$, establishing the result. □

Theorem 4.4 is sharp. Let G be the graph on an even number n vertices $\{v_1, v_2, \ldots, v_n\}$ where $V_1 = \{v_1, v_2, \ldots, v_{n/2}\}$ induces a complete graph and $V_2 = \{v_{n/2+1}, v_{n/2+2}, \ldots, v_n\}$ forms an independent set of $n/2$ vertices. Vertex $v_i \in V_2$ is adjacent to $V_1 - \{v_{i-n/2}\}$. It is straightforward to show the only IS coloring has $c(v_i) = i + n/2$ for $v_i \in V_1$ and $c(v_i) = i - n/2$ for $v_i \in V_2$. Thus $\chi_{IS}(G) = n$.

We now show the complement $\overline{G}$ of a graph G can yield information about the existence of an IS n-coloring of G.

Definition 4.5. *A graph is* IS-covered *if there is a partition of its vertices such that each set of the partition induces either a K_2 or a chordless odd cycle.*

Theorem 4.6. *Let G be an n-vertex graph. Then G has an IS n-coloring if and only if $\overline{G}$ is IS-covered.*

Proof. Assume $\overline{G}$ is IS-covered. Define a coloring c of G as follows. For every set of the partition containing two vertices v_i and v_j which induce a K_2, let $c_i = \{v_j\}$ and $c_j = \{v_i\}$. For every set of the partition containing vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\}$ which induce a chordless odd cycle, let $c_{i_1} = \{v_{i_t}\}$ and $c_{i_j} = \{v_{i_{j-1}}\}$ for $2 \leq j \leq t$. Every color class contains a single vertex so this defines an n-coloring of G. Furthermore, by the way c is defined, if $c_j = \{v_i\}$, $i \neq j$ and v_i and v_j are adjacent in $\overline{G}$ and hence are not joined by an edge in G. It follows that the coloring is an IS n-coloring of G.

Now assume G has an IS n-coloring c. Create a graph H having the same vertex set as G as follows. For $1 \leq i \leq n$, if $c_i = \{v_j\}$, include edge v_iv_j which also is an edge of $\overline{G}$. Consider $c_j = \{v_k\}$. If $k = i$, this creates the same edge v_iv_j and it is included only once. In this case v_i and v_j have degree one in H and induce a K_2 there. If $k \neq i$, edges v_iv_j and v_jv_k both appear in H and v_j has degree two. Thus degree one vertices occur in pairs and induce K_2's and all other vertices are degree two, and these latter must induce cycles. If a cycle is even on t vertices, alternating edges can be removed leaving $t/2$ vertex disjoint K_2's. If an odd cycle $\{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\}$ has a chord in $\overline{G}$, say $v_{i_1}v_{i_r}$, one of $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ and $\{v_{i_1}, v_{i_r}, v_{i_{r+1}}, \ldots, v_{i_t}\}$ defines an odd cycle and the other an even cycle. Without loss of generality assume $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ gives an even cycle. Then removing alternative edges along the path $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$, beginning with $v_{i_1}v_{i_2}$, leaves K_2's that include all the vertices of the even cycle not on the odd cycle. The process can be repeated if the resulting odd cycle still has a chord in $\overline{G}$. The final result shows $\overline{G}$, the complement of G is IS-covered. □

Theorem 4.6 represents the condition on $\overline{G}$ mentioned prior to Theorem 3.1 necessary for graph G to have a solution to the Sibling Marriage Problem. There also is an interesting comparison between solutions to the two marriage problems. The Sibling Marriage Problem has a solution for a graph G if and only if the vertices of $\overline{G}$ can be partitioned into sets inducing K_2's and chordless odd cycles. The Classical Marriage Problem has a solution for a bipartite graph B if and only if the vertices of B can be partitioned into sets inducing K_2's.

References

[1] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, CRC Press, 2009.

[2] J. Hopcroft and R.M. Karp, An $O(n^{2.5})$ algorithm for maximum matching in bipartite graphs, *SIAM J. Comput.*, **2** (1973), 225–231.

[3] J. Mycielski, Sur le coloriage des graphs, *Colloq. Math.*, **3** (1955), 161–162.