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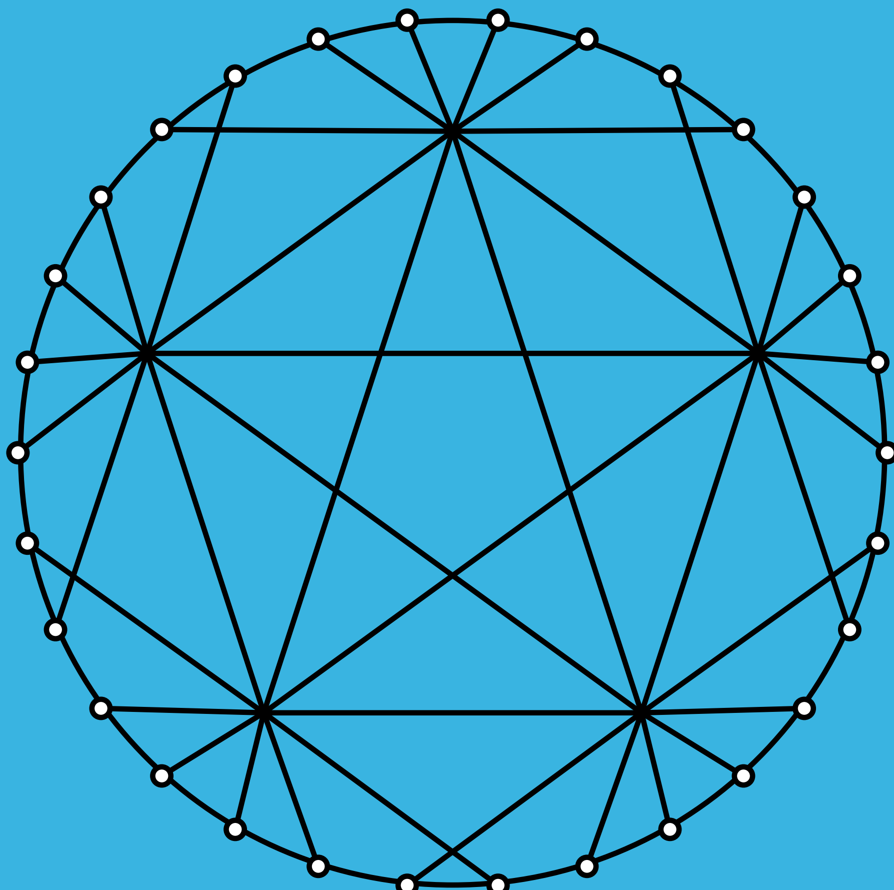

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Cyclic and rotational six-cycle systems

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Abstract: We obtain necessary and sufficient conditions for the existence of 2-rotational 6-cycle systems, and enumerate certain classes of 6-cycle systems of small orders with prescribed types of automorphisms.

1 2-rotational 6-cycle systems

A 6-cycle system of order n, denoted 6CS(n), is a pair (V, B) where V is an n-set and B is a collection of edge-disjoint cycles of length 6 such that each edge of the complete graph K_n on V is contained in exactly one of the 6-cycles of B. In other words, a 6-cycle system of order n is a decomposition of the complete graph K_n into 6-cycles.

There has been recently some interest in 6-cycle systems, mainly because of their connections to algebra and to Steiner triple systems. It is well known that a 6CS(n) exists if and only if $n \equiv 1$ or $9 \pmod{12}$, see, e.g., [4].

It was shown recently [3] that 6-cycle systems are universal in the sense that every abstract group is the full automorphism group of a 6-cycle system. Therefore the next natural question is: Which permutations of degree n are automorphisms of a 6CS(n)?

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A 6CS(n) is *cyclic* if it admits an automorphism permuting the vertices in a single cycle of length n. A 6CS(n) is *k-rotational* ($k \geq 1$) if it admits as an automorphism a permutation containing exactly one fixed point and k cycles of length $\frac{n-1}{k}$ each. Note that we are using the term "cycle" to describe both, a sequence in a graph (or a graph itself) as well as a part of a permutation; no confusion should arise by this, however.

It was shown in [5], [6] that a cyclic 6CS(n) exists if and only if $n \equiv 1$ or 9 (mod 12), $n \geq 13$.

In [1], it was shown that there exists no 1-rotational 6CS(n) for any n. It is not difficult to show that, in fact, a k-rotational 6CS(n) cannot exist for any odd k.

The main purpose of this note is to settle the existence question for 2-rotational 6CS(n), i.e. those that admit an automorphism having exactly one fixed point and two cycles of length $\frac{n-1}{2}$ each. In [7] which deals more generally with k-cycle systems with an even k, it was established that there exists a 2-rotational 6CS(9), and the same can be gleaned also from [2].

Theorem 1.1. *A 2-rotational* 6CS(n) *exists if an only if* $n \equiv 1$ *or* 9 (mod 12), $n \geq 9$.

Proof. Our proof is by direct construction. In what follows we use standard notation x_i to denote (x, i).

I. Let $n \equiv 1 \pmod{12}$, $n = 12k+1$. The set of elements $V = \mathbb{Z}_{6k} \times \{1,2\} \cup \{\infty\}$. Each base cycle below is to be developed modulo $(6k, -)$.

I.1. k is odd, $k \geq 1$

i) $(0_1, k_1, (2k)_1, (3k)_1, (4k)_1, (5k)_1)$ (orbit of length k)

ii) $(0_1, 0_2, (3k-1)_2, (6k-1)_2, (3k)_2, (3k)_1)$ (orbit of length $3k$)

iii) $(0_1, (2k)_2, 1_1, (3k)_1, (5k)_2, (3k+1)_1)$ (orbit of length $3k$)

iv) $(\infty, 0_2, k_2, (k+1)_1, (k-2)_2, k_1)$

v) $(0_2, (4i+2)_1, (5i+2)_1, (i+2)_2, (5i+3)_1, i_2)$, $i = 1, 2, \ldots, k-1$

vi) $(0_2, (4k+4i)_1, (5k+8i-3)_1, (4k+4i-1)_1, 1_2, (k+4i-2)_2)$, $i = 1, 2, \ldots, (k-1)/2$

vii) $(0_2, (4k+4i+1)_1, (5k+8i)_1, (4k+4i)_1, 1_2, (k+4i)_2)$, $i = 1, 2, \ldots, (k-1)/2$

I.2. k is even, $k \geq 2$.

i) $(0_1, k_1, (2k)_1, (3k)_1, (4k)_1, (5k)_1)$ (orbit of length k)

ii) $(0_1, 1_2, (3k)_2, 0_2, (3k+1)_2, (3k)_1)$, (orbit of length $3k$)

iii) $(0_1, (2k)_2, 1_1, (3k)_1, (5k)_2, (3k+1)_1)$ (orbit of length $3k$)

iv) $(\infty, 0_2, k_2, (k+2)_1, (k-1)_2, k_1)$

v) $(0_2, (4i+2)_1, (5i+2)_1, (i+2)_2, (5i+3)_1, 1_2)$,
$i = 1, 2, \ldots, k-1$

vi) $(0_2, (4k+4i)_1, (5k+8i-3)_1, (4k+4i-1)_1, 1_2, (k+4i-2)_2)$,
$i = 1, 2, \ldots, (k-2)/2$

vii) $(0_2, (4k+4i+1)_1, (5k+8i)_1, (4k+4i)_1, 1_2, (k+4i)_2)$,
$i = 1, 2, \ldots, (k-2)/2$

viii) $(0_1, 2_2, (3k-1)_2, 1_2, 1_1, (3k-2)_1)$

II. Let $n \equiv 9 \pmod{12}$, $n = 12k+9$. The set of elements $V = \mathbb{Z}_{6k+4} \times \{1,2\} \cup \{\infty\}$. Each base block cycle is to be developed modulo $(6k+4, -)$.

II.1. k is even, $k \geq 0$.

i) $(0_1, 0_2, 1_2, (3k+3)_2, (3k+2)_2, (3k+2)_1)$ (orbit of length $3k+2$))

ii) $(\infty, 0_2, 3_1, 1_2, 2_1, 1_1)$

iii) $(0_2, (4i+2)_1, (5i+3)_1, (i+3)_2, (5i+4)_1, (i+1)_2)$,
$i = 1, 2, \ldots, k$

iv) $(0_2, (4k+4i+2)_1, (5k+8i)_1, (4k+4i+1)_1, 1_2, (k+4i-1)_2)$,
$i = 1, 2, \ldots, k/2$

v) $(0_2, (4k+4i+3)_1, (5k+8i+3)_1, (4k+4i+2)_1, 1_2, (k+4i+1)_2)$,
$i = 1, 2, \ldots, k/2$

II.2 k is odd, $k \geq 1$.

i) $(0_2, 1_1, (3k+3)_1, (3k+2)_2, (3k+3)_2, 1_2)$ (orbit of length $3k+2$)

ii) $(\infty, 0_2, 5_1, 1_2, 4_1, 3_1)$

iii) $(0_2, (4i+4)_1, (5i+5)_1, (i+3)_2, (5i+6)_1, (i+1)_2)$,
$i = 1, 2, \ldots, k$

iv) $(0_2, 2_1, (k+4)_1, 1_1, 1_2, (k+3)_2)$

v) $(0_2, (4k+4i+4)_1, (5k+8i+4)_1, (4k+4i+3)_1, 1_2, (k+4i+1)_2)$,
$i = 1, 2, \ldots, (k-1)/2$

vi) $(0_2, (4k+4i+5)_1, (5k+8i+7)_1, (4k+4i+4)_1, 1_2, (k+4i+3)_2)$,
$i = 1, 2, \ldots, (k-1)/2$

$\square$

Corollary 1.2. *A* 4*-rotational* 6CS(n) *exists if and only if* $n \equiv 1$ *or* 9 (mod 12)*.*

Proof. Let α be an automorphism of a 2-rotational 6CS(n) such that it consists of one fixed point and two cycles of length $l = \frac{n-1}{2}$. Since l is even, α^2 is an automorphism that contains one fixed point and four cycles of length $\frac{n-1}{4}$. □

By analogy with Steiner triple systems, let us call a 6-cycle system *reverse* if it admits as an automorphism an involution with exactly one fixed point.

Corollary 1.3. *A reverse* 6CS(n) *exists if and only if* $n \equiv 1$ *or* 9 (mod 12)*.*

Proof. If α is an automorphism of a 2-rotational 6CS(n) that contains one fixed point and two cycles of length $\frac{n-1}{2}$, then $\alpha^{\frac{n-1}{4}}$ is a required involution. □

2 Some enumeration results

The 6-cycle systems appear to be very numerous. But the only enumeration result on 6CS(n) that we are aware of is in [2] where it is established that there are exactly 640 nonisomorphic 6CS(9). We have enumerated *cyclic* 6CS(n) for $n = 13$ and $n = 21$. They number 16, and 378, respectively. In Tables 1 and 2, we list the base cycle of all 16 cyclic 6CS(13) and the three base cycles of the 10 lexicographically smallest 6CS(21); the remaining cyclic 6CS(21) can be found in

http://home.agh.edu.pl/~meszka/2r6cs_13.html

We have also enumerated the 2-rotational 6CS(13); there are 384 of them. In Table 3 we list the three base cycles of the 10 lexicographically smallest 2-rotational 6CS(13); the remaining such systems can be found in

http://home.agh.edu.pl/~meszka/2r6cs_13.html

Table 1 Cyclic 6-cycle systems of order 13.

$V = \mathbb{Z}_{13}$.

1. $(0,1,3,6,2,7)$
2. $(0,1,3,6,2,8)$
3. $(0,1,3,6,11,7)$
4. $(0,1,3,6,11,4)$
5. $(0,1,3,6,12,8)$
6. $(0,1,3,6,12,4)$
7. $(0,1,3,12,2,7)$
8. $(0,1,3,12,2,8)$
9. $(0,1,3,12,4,7)$
10. $(0,1,3,12,4,10)$
11. $(0,1,3,12,5,10)$
12. $(0,1,3,9,5,8)$
13. $(0,1,4,6,2,7)$
14. $(0,1,4,6,2,8)$
15. $(0,1,4,9,11,7)$
16. $(0,1,6,9,11,7)$

The full automorphism group of each of the above systems has order 13, except for systems No. 5, 9, and 16 whose group has order 39.

Table 2 Cyclic 6-cycle systems of order 21 (the first ten out of 378).

$V = \mathbb{Z}_{21}$.

1. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,14,1,10)$
2. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,14,1,12)$
3. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,14,2,10)$
4. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,14,2,13)$
5. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,14,4,12)$
6. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,14,4,13)$
7. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,15,1,10)$
8. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,15,1,12)$
9. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,15,5,12)$
10. $(0,1,7,8,14,15), (0,2,7,9,14,16), (0,3,7,15,5,14)$

Each of the 378 cyclic 6CS(21) has $\mathbb{Z}_{21}$ as its full automorphism group.

Table 3 2-rotational 6-cycle systems of order 13.

$V = \mathbb{Z}_6 \times \{1,2\} \cup \{\infty\}$.

1. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,1_2,2_1,4_2,0_2)$
2. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,1_2,2_1,4_2,2_2)$
3. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,1_2,5_1,4_2,0_2)$
4. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,1_2,5_1,4_2,2_2)$
5. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,1_2,3_2,1_1,0_2)$
6. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,1_2,3_2,4_1,0_2)$
7. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,1_2,5_2,3_1,2_2)$
8. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,2_2,1_1,0_2,4_2)$
9. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,2_2,3_1,4_2,0_2)$
10. $(0_1,2_1,5_1,3_1,0_2,3_2),(0_1,0_2,1_2,3_1,3_2,4_2),(\infty,0_1,2_2,0_2,5_1,4_2)$

Each of the 384 nonisomorphic 2-rotational 6CS(13) has $\mathbb{Z}_6$ as its full automorphism group.

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