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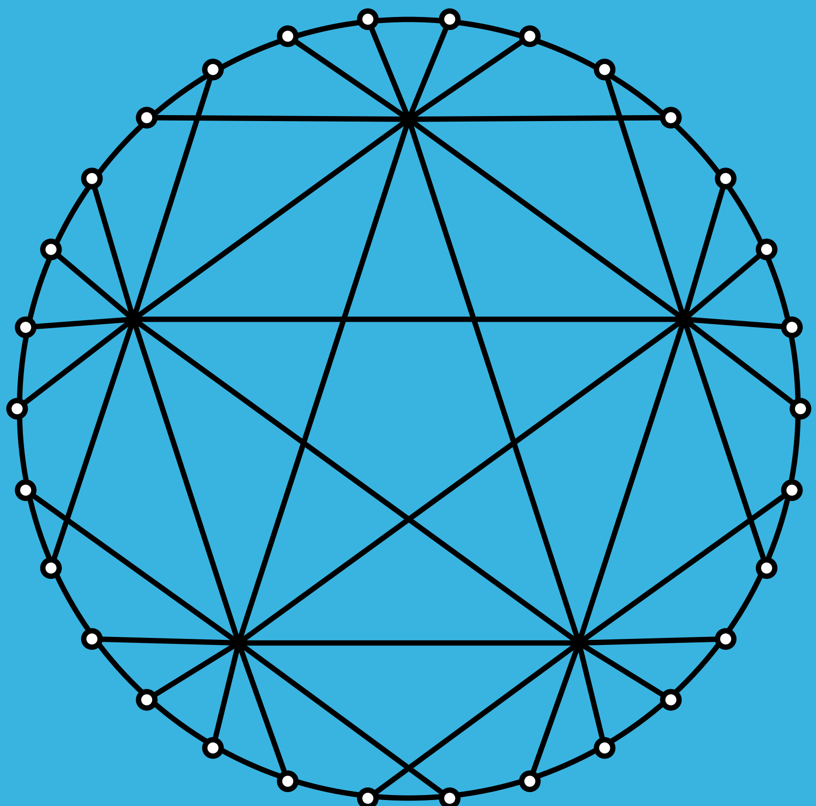

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Rainbow mean colorings of bipartite graphs

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Abstract: For an edge coloring c of a connected graph G with positive integers where adjacent edges may be colored the same, the chromatic mean of a vertex v of G is the average of the colors of the edges incident with v. Only those edge colorings c for which the chromatic mean of every vertex is a positive integer are considered. If distinct vertices have distinct chromatic means, then c is a rainbow mean coloring of G. The maximum vertex color in a rainbow mean coloring c of G is the rainbow mean index of c, while the rainbow mean index of G is the minimum rainbow mean index among all rainbow mean colorings of G. The rainbow mean index of several bipartite graphs are determined, namely prisms, hypercubes, and complete bipartite graphs.

1 Introduction

It is graph theory folklore that in every nontrivial graph, there are always two vertices having the same degree. Indeed, this fact is listed (indirectly) among the 24 theorems in an article by David Wells [7], asking which of 24 theorems is the most beautiful. A graph G was initially called *perfect*

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and later called *irregular* if the degrees of all vertices of G are distinct. Consequently, no nontrivial graph is perfect, that is, irregular.

Over the years, "irregular graphs" have been looked at in a variety of ways (see [1, 2, 3, 6], for example). While no nontrivial graph is irregular, there are irregular multigraphs of each order $n \geq 3$. A multigraph M can be looked at as a labeled graph G_M where each edge uv of G_M is labeled with the positive integer equal to the number of parallel edges joining u and v in M. The degree of v in M is then the sum of the labels of the edges in G_M that are incident with v. Later each edge label was considered as an edge color and the sum of the labels incident with a vertex was referred to as its chromatic sum which became the color of the vertex.

In 1986, at the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne (now called Purdue University Fort Wayne), the concept of "irregularity strength" was introduced by Gary Chartrand, defined as the smallest positive integer k for which a not necessarily proper edge coloring of a graph from the set $[k] = \{1, 2, \ldots, k\}$ exists giving rise to vertex colors (chromatic sums), all of which are distinct (see [5]). Consequently, the problem was to determine the smallest positive integer k such that each edge of a graph can be colored with an element of $[k]$ in such a way that the vertex colors are distinct. This then results in a vertex coloring of the graph, often called a *rainbow coloring* since all vertex colors are distinct. In many instances, such an edge coloring caused the largest vertex color to be significantly larger than the order of the graph. From this observation, the question occurred as to whether some edge coloring could be defined on a graph producing a vertex coloring in some natural way such that all vertex colors are integers, distinct vertices have distinct colors, and the largest vertex color is not as large as that produced when dealing with the irregularity strength.

While minimizing the positive integer k so that each edge color is an element of $[k]$ and requiring distinct vertices to have distinct vertex colors (chromatic sums) became a topic of study in many papers, a related topic is that of requiring distinct vertices to have distinct vertex colors, when the vertex color is defined as the integer chromatic average, and minimizing the largest vertex color. Here the emphasis is on the resulting vertex colors rather than the edge colors. Thus, such an edge coloring not only requires the average color of the edges incident with each vertex to be an integer and all resulting vertex colors to be distinct but that of minimizing the maximum vertex color. This concept was introduced and studied in [4].

Each connected graph G having an edge coloring c with positive integers, where adjacent edges may be colored the same, induces a vertex coloring cm defined as the positive integer $\mathrm{cm}(v) = \frac{\sum_{e \in E_v} c(e)}{\deg v}$, where E_v is the set of edges incident with a vertex v of G. Such edge colorings are called *mean colorings.* The vertex color $\mathrm{cm}(v)$ of v is called the *chromatic mean* of v. Consequently, only edge colorings c are considered for which $\mathrm{cm}(v)$ is a positive integer for every vertex v of G. If distinct vertices have distinct chromatic means, then the edge coloring c is called a *rainbow mean coloring* of G. The maximum vertex color in a rainbow mean coloring c is called the *rainbow mean index* $\mathrm{rm}(c)$ of c, while the minimum mean index among all rainbow mean colorings of G is the *rainbow mean index* $\mathrm{rm}(G)$ of G. For example, rainbow mean colorings of the trees F and H of order 6 are shown in Figure 1. The rainbow mean index of the edge coloring of F is 6, while the rainbow mean index of the edge coloring of H is 7. In fact, $\mathrm{rm}(F) = 6$ and $\mathrm{rm}(H) = 7$.

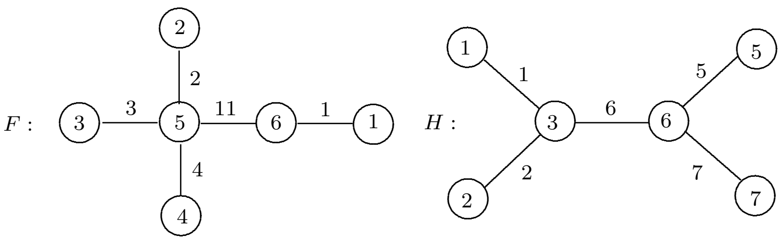

Figure 1: Rainbow mean colorings of two graphs

There are several observations made in [4] that will be useful to us here.

Observation 1.1. *Every connected graph G of order $n \geq 3$ has a rainbow mean coloring and* $\mathrm{rm}(G) \geq n$.

Let c be a rainbow mean coloring of a connected graph G. For a vertex v of G, the *chromatic sum* $\mathrm{cs}(v)$ of v is defined as the sum of the colors of the edges incident with v. Hence, $\mathrm{cs}(v) = \sum_{e \in E_v} c(e) = \deg v \cdot \mathrm{cm}(v)$.

Observation 1.2. *If c is a rainbow mean coloring of a connected graph G, then*

$$\sum_{v \in V(G)} \mathrm{cs}(v) = 2 \sum_{e \in E(G)} c(e).$$

Furthermore, if the order of G is n and $\mathrm{rm}(c) = n$*, then*

$$\sum_{v \in V(G)} \mathrm{cm}(v) = \binom{n+1}{2}.$$

A connected graph of order 3 or more with a rainbow mean coloring is referred to as a *mean-colored graph*. A vertex v in a mean-colored graph G is *chromatically odd* if $\mathrm{cs}(v) = \deg v \cdot \mathrm{cm}(v)$ is an odd integer; otherwise, v is *chromatically even*. The following is an immediate consequence of Observation 1.2.

Corollary 1.3. *Every mean-colored graph contains an even number of chromatically odd vertices.*

Corollary 1.4. *If G is a connected graph of order $n \geq 6$ with $n \equiv 2 \pmod 4$ all of whose vertices are odd, then $\mathrm{rm}(G) \geq n+1$.*

Proof. Assume, to the contrary, that $\mathrm{rm}(G) = n$. Consequently, there exists a rainbow mean coloring $c : E(G) \to \mathbb{N}$ of G such that $\{\mathrm{cm}(v) : v \in V(G)\} = [n] = \{1, 2, \ldots, n\}$. Since $n \equiv 2 \pmod 4$, it follows that $n = 4k+2$ for some positive integer k. Thus, the set $[n]$ contains $2k+1$ odd integers, namely $1, 3, \ldots, 4k+1$. Suppose that $u_1, u_2, \ldots, u_{2k+1}$ are the vertices of G such that $\mathrm{cm}(u_i) = 2i-1$ for $i = 1, 2, \ldots, 2k+1$. Since every vertex of G has odd degree, the vertices $u_1, u_2, \ldots, u_{2k+1}$ are the only chromatically odd vertices. This contradicts Corollary 1.3. $\square$

Our primary interest here will be dealing with connected bipartite graphs G of order 3 or more having partite sets U and W. Because each of $\sum_{u \in U} \mathrm{cs}(u)$ and $\sum_{w \in W} \mathrm{cs}(w)$ counts the sum of the colors of the edges of G, we have the following fact.

Observation 1.5. *Let G be a connected bipartite graph with partite sets U and W. If c is an edge coloring of G, then $\sum_{u \in U} \mathrm{cs}(u) = \sum_{w \in W} \mathrm{cs}(w)$.*

Among the results obtained in [4], the rainbow mean index was determined for each complete graph and cycle.

Theorem 1.6. *For an integer $n \geq 3$,*

$$\mathrm{rm}(K_n) = \begin{cases} n & \text{if } n \geq 4 \text{ and } n \equiv 0, 1, 3 \pmod 4 \\ n+1 & \text{if } n = 3 \text{ or } n \equiv 2 \pmod 4 \end{cases}$$

$$\mathrm{rm}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 1 \pmod 4 \\ n+1 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

A connected graph G of order $n \geq 3$ with $\mathrm{rm}(G) = n$ is called *Type* 1. If $\mathrm{rm}(G) = n+1$, then G is *Type* 2; while if $\mathrm{rm}(G) = n+2$, then G is *Type* 3. With this terminology, a conjecture made in [4] can be rephrased as follows.

Conjecture 1.7. *Every connected graph of order* 3 *or more is Type* 1, 2 *or* 3.

In this paper, we verify Conjecture 1.7 for bipartite graphs belonging to some well known classes.

2 Prisms and hypercubes

In this section, we will be interested in determining the rainbow mean indexes of graphs belonging to one of two classes, each expressed as Cartesian products of graphs. For this purpose, the following result will be useful to us.

Theorem 2.1. *Let G be a connected regular graph of order $n \geq 4$ with $\mathrm{rm}(G) = n$ containing a 1-factor and let H be a connected graph of order p. Then $\mathrm{rm}(G \,\square\, H) = np$.*

Proof. Let G be r-regular for some integer $r \geq 2$ and let F be a 1-factor of G. Furthermore, let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{w_1, \ldots, w_p\}$. In the construction of $G \,\square\, H$, each vertex v_i $(1 \leq i \leq n)$ of G is replaced by a copy of H_i of H, where $V(H_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,p}\}$ in $G \,\square\, H$. If $v_iv_j \in E(G)$, then $u_{i,k}u_{j,k} \in E(G \,\square\, H)$ for $k = 1, 2, \ldots, p$. The 1-factor F in G gives rise to p 1-factors $F_1, F_2, \ldots, F_p$ in $G \,\square\, H$ such that if $v_iv_j \in E(F)$, then $u_{i,k}u_{j,k} \in E(F_k)$ for $k = 1, 2, \ldots, p$.

Since $\mathrm{rm}(G) = n$, there is a rainbow mean coloring $c_G : E(G) \to \mathbb{N}$ with $\mathrm{rm}(c_G) = n$ and so $\{\mathrm{cm}_{c_G}(v_i) : 1 \leq i \leq n\} = [n]$. Consequently, we may assume that $\mathrm{cm}_{c_G}(v_i) = i$ for $i \in [n]$. Since the order of $G \,\square\, H$ is np, it follows by Observation 1.1 that $\mathrm{rm}(G \,\square\, H) \geq np$. Hence, it suffices to show that there is a rainbow mean coloring $c : E(G \,\square\, H) \to \mathbb{N}$ of $G \,\square\, H$ with $\mathrm{rm}(c) = np$. We next define an edge coloring c of $G \,\square\, H$ such that the resulting rainbow mean vertex coloring cm gives

$$\mathrm{cm}(u_{i,j}) = \underset{c_G}{\mathrm{cm}}(v_i) + n(j-1)$$

for each pair i, j of integers with $1 \leq i \leq n$ and $1 \leq j \leq p$, that is, $\{\mathrm{cm}(u_{1,j}), \mathrm{cm}(u_{2,j}), \ldots, \mathrm{cm}(u_{n,j})\} = [n(j-1)+1,\ nj]$ for $1 \leq j \leq p$.

Let the edge coloring $c : E(G \,\square\, H) \to \mathbb{N}$ of $G \,\square\, H$ be defined as follows:

$$c(e) = \begin{cases} \mathrm{cm}_{c_G}(v_i) & \text{if } e \in E(H_i),\ 1 \le i \le n \\ c_G(v_s v_t) & \text{if } e = u_{s,j}u_{t,j},\ 1 \le s \ne t \le n,\ 1 \le j \le p \\ & \quad \text{and } e \notin E(F_1) \cup E(F_2) \cup \cdots \cup E(F_p) \\ c_G(v_s v_t) & \\ +n(j-1)(r + \deg_H w_j) & \text{if } e = u_{s,j}u_{t,j} \in E(F_j),\ 1 \le j \le p. \end{cases}$$

Consequently, if an edge e of $G \,\square\, H$ belongs to the copy H_i $(1 \le i \le n)$ of H, then the color $c(e)$ is defined as the chromatic mean $\mathrm{cm}_{c_G}(v_i)$ of the vertex v_i of G produced from the rainbow mean coloring c_G of G. If $e \in E(G \,\square\, H)$ such that $e = u_{s,j}u_{t,j}$ and e belongs to neither a graph H_i for $1 \le i \le n$ nor a 1-factor F_j for $1 \le j \le p$, then $c(e)$ is the color $c_G(v_s v_t)$ assigned to the edge $v_s v_t \in E(G)$ by the rainbow mean coloring c_G of G. If $e \in E(G \,\square\, H)$ belongs to a 1-factor F_j $(1 \le j \le p)$, where say $e = u_{s,j}u_{t,j} \in E(F_j)$, then we add to the color $c_G(v_s v_t)$ of the edge $v_s v_t \in E(G)$ the number $n(j-1)(r + \deg_H w_j)$. Therefore, the sum of the colors of the edges incident with a vertex $u_{i,j}$ $(1 \le i \le n$ and $1 \le j \le p)$ of $G \,\square\, H$ is

$$\begin{aligned} \mathrm{cm}(u_{i,j}) &= (\deg_H w_j)\mathrm{cm}_{c_G}(v_i) + r\mathrm{cm}_{c_G}(v_i) + n(j-1)(r + \deg_H w_j) \\ &= (r + \deg_H w_j)\left[\mathrm{cm}_{c_G}(v_i) + n(j-1)\right]. \end{aligned}$$

Since $\deg_{G \,\square\, H} u_{i,j} = r + \deg_H w_j$, it follows that

$$\mathrm{cm}(u_{i,j}) = \underset{c_G}{\mathrm{cm}}(v_i) + n(j-1),$$

giving the desired result. $\square$

Theorem 2.1 therefore states that if G is Type 1 regular graph containing a 1-factor, then for every connected graph H, the graph $G \,\square\, H$ is Type 1 as well. The *prism* $C_n \,\square\, K_2$, $n \ge 3$, is the Cartesian product of the n-cycle C_n and K_2. Of course, $C_n \,\square\, K_2$ is bipartite if and only if n is even. We now determine the rainbow mean index of every prism.

Theorem 2.2. *For each integer $n \ge 3$,*

$$\mathrm{rm}(C_n \,\square\, K_2) = \begin{cases} 2n & \text{if } n \text{ is even} \\ 2n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If $n \ge 4$ is even, the n-cycle C_n is 2-regular and has a 1-factor. Therefore, $\mathrm{rm}(C_n \,\square\, K_2) = 2n$ by Theorem 2.1. Hence, we may assume that $n \geq 3$ and n is odd. Let $G = C_n \,\square\, K_2$ be constructed from the two n-cycles $(u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ and $(v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ and the edges $u_i v_i$ for $1 \leq i \leq n$. Thus, G is a cubic graph of order $2n$. By Corollary 1.4, $\mathrm{rm}(G) \geq 2n+1$. Thus, it remains to show that there is a rainbow mean coloring c of G with $\mathrm{rm}(c) = 2n+1$. Define the edge coloring $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} 1 & \text{if } e = u_i u_{i+1} \text{ where } i \text{ is odd and } 1 \leq i \leq n \\ 4 & \text{if } e = u_i u_{i+1} \text{ where } i \text{ is even and } 2 \leq i \leq n-1 \\ 3i-2 & \text{if } e = u_i v_i \text{ for } 1 \leq i \leq n \\ \frac{3n+5}{2} & \text{if } e = v_i v_{i+1} \text{ where } 1 \leq i \leq n. \end{cases}$$

Since $\mathrm{cm}(u_1) = 1$, $\mathrm{cm}(u_i) = i+1$ for $2 \leq i \leq n$, and $\mathrm{cm}(v_i) = n+1+i$ for $1 \leq i \leq n$, it follows that c is a rainbow mean coloring of G with $\mathrm{rm}(c) = 2n+1$. $\square$

Another well-known class of bipartite graphs defined by means of Cartesian products is that of the hypercubes. The *hypercube* Q_n is K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as the Cartesian product $Q_{n-1} \,\square\, K_2$ of Q_{n-1} and K_2. For each integer $n \geq 2$, the hypercubes Q_n is an n-regular bipartite graph of order 2^n.

Theorem 2.3. *For each integer $n \geq 2$, $\mathrm{rm}(Q_n) = 2^n$.*

Proof. We proceed by induction on $n \geq 2$. Since $\mathrm{rm}(Q_2) = \mathrm{rm}(C_4) = 4$ by Theorem 1.6, it follows that the statement is true for $n = 2$. Assume that $\mathrm{rm}(Q_n) = 2^n$ for an integer $n \geq 2$. Since $Q_{n+1} = Q_n \,\square\, K_2$, Q_n is n-regular, and has a 1-factor, it follows by Theorem 1.6 that $\mathrm{rm}(Q_{n+1}) = 2 \cdot 2^n = 2^{n+1}$. The desired result follows by induction. $\square$

3 Complete bipartite graphs

In this section, we turn our attention to another familiar class of bipartite graphs, namely the complete bipartite graphs $K_{s,t}$ where $s + t \geq 3$. The rainbow mean index of all stars $K_{1,t}$, $t \geq 2$, was determined in [4], showing that every star of order 3 or more is either Type 1 or Type 3.

Theorem 3.1. *For an integer $t \geq 2$,* $\mathrm{rm}(K_{1,t}) = \begin{cases} t+1 & \textit{if } t \textit{ is even} \\ t+3 & \textit{if } t \textit{ is odd.} \end{cases}$

First, we show that if s and t are both odd with $s, t \geq 3$, then $K_{s,t}$ is not Type 1.

Proposition 3.2. *If s and t are odd integers with $s, t \geq 3$, then*

$$\mathrm{rm}(K_{s,t}) \geq s + t + 1.$$

Proof. Let $G = K_{s,t}$ where s and t are odd integers with $s, t \geq 3$. Therefore, $s = 2a + 1$ and $t = 2b + 1$ for some positive integers a and b. If $s \equiv t \pmod 4$, then the statement follows by Corollary 1.4. Nevertheless, we verify the statement without any additional assumption. Assume, to the contrary, that there is a rainbow mean coloring $c : E(G) \to \mathbb{N}$ of G with $\mathrm{rm}(c) = s + t$. Thus,

$$\sum_{v \in V(G)} \mathrm{cm}(v) = \binom{s+t+1}{2} = \frac{(s+t+1)(s+t)}{2} = (2a+2b+3)(a+b+1).$$

Let $\{X, Y\}$ be a partition of the set $[s+t]$ where $|X| = t$ and $|Y| = s$ such that the sum of the elements in X is x and the sum of the elements in Y is y. Since $x + y = (2a + 2b + 3)(a + b + 1)$ and $sx = ty$, it follows that $sx = t[(2a+2b+3)(a+b+1) - x]$ from which we have $2x = t(2a+2b+3)$. Since $t(2a+2b+3)$ is an odd integer, this is a contradiction and so $\mathrm{rm}(G) \geq s + t + 1$. $\square$

We now determine the rainbow mean indexes of all complete bipartite graphs.

Theorem 3.3. *If s and t are positive integers with $\min\{s, t\} \geq 2$, then*

$$\mathrm{rm}(K_{s,t}) = \begin{cases} s+t & \text{if } st \text{ is even} \\ s+t+1 & \text{if } st \text{ is odd} \end{cases}$$

Proof. Let $G = K_{s,t}$ with partite sets $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. We consider two cases, according to whether st is even or odd.

Case 1. *st is even.* By Observation 1.1, it suffices to show that there is a rainbow mean coloring of $K_{s,t}$ with rainbow mean index $s + t$. We proceed with the following three steps:

(1) Partition the set $[s+t]$ into the two subsets $X = \{x_1, x_2, \ldots, x_t\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ where $x_1 < x_2 < \cdots < x_t$ and $y_1 < y_2 < \cdots < y_s$ such that $s \sum_{i=1}^{t} x_i = t \sum_{j=1}^{s} y_j$.

(2) Construct a $t \times s$ matrix $M = [a_{ij}]$ such that sx_i is the sum of the entries in row i for $1 \leq i \leq t$ and ty_j is the sum of the entries in column j for $1 \leq j \leq s$.

(3) Use the matrix $M = [a_{ij}]$ to construct a rainbow mean coloring c of $K_{s,t}$. For each vertex u_j of $K_{s,t}$ where $1 \leq j \leq s$, we define a t-vector $\vec{u}_j = (a_{1j}, a_{2j}, \ldots, a_{tj})$ to be column j in M. This in turn gives rise to the corresponding s-vectors $\vec{w}_i = (a_{i1}, a_{i2}, \ldots, a_{is})$ to be row i in M for each vertex w_i where $1 \leq i \leq t$. The edge coloring $c : E(K_{s,t}) \to \mathbb{N}$ is defined by $c(w_i u_j) = a_{ij}$ for each pair i, j of integers with $1 \leq i \leq t$ and $1 \leq j \leq s$. Since the chromatic means of the vertices of $K_{s,t}$ are given by $\mathrm{cm}(u_j) = x_j$ for $1 \leq j \leq s$ and $\mathrm{cm}(w_i) = y_i$ for $1 \leq i \leq t$, it follows that $\{\mathrm{cm}(v) : v \in V(K_{s,t})\} = [s+t]$ and so $\mathrm{rm}(c) = s + t$.

There are two subcases, depending on whether s and t are both even or exactly one of s and t is even.

Subcase 1.1. *Both s and t are even.* We may assume that $s \leq t$. Since s and t are positive even integers, it follows that $s = 2a$ and $t = 2b$ for some integers a and b with $1 \leq a \leq b$. First, we partition the $(2a+2b)$-element set $[2a+2b]$ into the two subsets

$$X = [b] \cup [b + 2a + 1, 2a + 2b]$$

and

$$Y = [b + 1, b + 2a],$$

where then $|X| = 2b$ and $|Y| = 2a$. Let $X = \{x_1, x_2, \ldots, x_{2b}\}$ where $x_1 < x_2 < \cdots < x_{2b}$ and $Y = \{y_1, y_2, \ldots, y_{2a}\}$ where $y_1 < y_2 < \cdots < y_{2a}$. Since

$$x = \sum_{i=1}^{2b} x_i = 2b^2 + 2ab + bs$$

and

$$y = \sum_{i=1}^{2a} y_i = 2a^2 + 2ab + a,$$

it follows that $2ax = 2by = 4a^2b + 4ab^2 + 2ab$.

Next, we define a $(2b) \times (2a)$ matrix $M = [a_{i,j}]$ such that

- $2ax_i$ is the sum of the entries in row i for $1 \leq i \leq 2b$ and
- $2by_j$ is the sum of the entries in column j for $1 \leq j \leq 2a$.

For the first b rows in M, we define each entry in row i to be i where $1 \leq i \leq b$. That is,

$$M = \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 2 & 2 & 2 & \cdots & \cdots & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & \cdots & \cdots & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{1}$$

Thus, the sum of the entries in row i is $2ax_i = 2ai$ for $1 \leq i \leq b$. Next, we determine the remaining entries in the last $b-1$ rows of M. Since we want the sum of the entries in column 1 to be $2by_1 = 2b(b+1)$ and $\sum_{i=1}^{b} a_{i1} = \frac{b(b+1)}{2}$, it follows that

$$\sum_{i=b+1}^{2b} a_{i1} = 2b(b+1) - \frac{b(b+1)}{2} = \frac{b(3b+3)}{2}.$$

We now choose each of the remaining b entries $a_{b+1,1}, a_{b+2,1}, \ldots, a_{2b,1}$ in column 1 as either

$$\left\lfloor \frac{b(3b+3)}{2b} \right\rfloor = \left\lfloor \frac{3b+3}{2} \right\rfloor \quad \text{or} \quad \left\lceil \frac{b(3b+3)}{2b} \right\rceil = \left\lceil \frac{3b+3}{2} \right\rceil$$

so that the sequence $a_{b+1,1}, a_{b+2,1}, \ldots, a_{2b,1}$ is nondecreasing and the column sum is $2by_1 = 2b(b+1)$. Furthermore, since we want the sum of the entrees in row $b+1$ to be $2ax_{b+1} = 2a(b+2a+1)$, we choose each of the entries $a_{b+1,2}, a_{b+1,3}, \ldots, a_{b+1,2a}$ as

$$\left\lfloor \frac{2a(b+2a+1) - a_{b+1,1}}{2a-1} \right\rfloor \quad \text{or} \quad \left\lceil \frac{2a(b+2a+1) - a_{b+1,1}}{2a-1} \right\rceil$$

so that the sequence $a_{b+1,2}, a_{b+1,3}, \ldots, a_{b+1,2a}$ is nondecreasing and the row sum is $2ax_{b+1} = 2a(b+2a+1)$. We now proceed in this manner to determine the remaining entries in M.

Finally, we define the edge coloring $c : E(G) \to \mathbb{N}$ by $c(w_i u_j) = a_{ij}$ for every pair i, j of integers where $1 \leq j \leq 2a$ and $1 \leq i \leq 2b$. By the defining property of the matrix M, it follows that $\mathrm{cm}(u_j) = x_j$ for $1 \le j \le 2a$ and $\mathrm{cm}(w_i) = y_i$ for $1 \le i \le 2b$. Thus, $\{\mathrm{cm}(v) : v \in V(G)\} = X \cup Y = [2a+2b]$ and so $\mathrm{rm}(c) = 2a + 2b = s + t$. As an illustration, we construct a rainbow mean coloring c of $K_{4,6}$ with $\mathrm{rm}(c) = 10$. In this case, $a = 2$ and $b = 3$. We partition the set $[10]$ into the two sets $X = [3] \cup [8, 10] = \{1, 2, 3, 8, 9, 10\}$ and $Y = \{4, 5, 6, 7\}$. Thus, $x = 33$ and $y = 22$ and so $4x = 6y = 132$. Using the technique described above, we obtain the 6×4 matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 6 & 8 & 9 & 9 \\ 6 & 8 & 11 & 11 \\ 6 & 8 & 10 & 16 \end{bmatrix}.$$

The resulting rainbow mean coloring c of $K_{4,6}$ with $\mathrm{rm}(c) = 10$ is shown in Figure 2, where the four vertices u_1, u_2, u_3, u_4 of the partite set U of $K_{4,6}$ are drawn in bold.

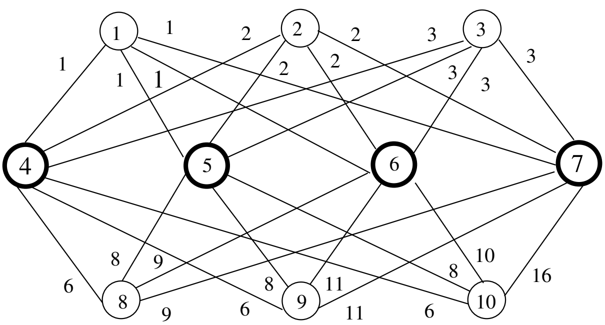

Figure 2: Constructing a rainbow mean coloring of $K_{4,6}$

Subcase 1.2. *Exactly one of s and t is even.* We may assume that $s \ge 2$ is even and $t \ge 3$ is odd (since the argument for the case when $s \ge 3$ is odd and $t \ge 2$ is even is similar). Thus $s = 2a$ and $t = 2b + 1$ for some positive integers a and b. First, we partition the $(2a + 2b + 1)$-element set $[2a + 2b + 1]$ into the two subsets

$$\begin{aligned} X &= [b] \cup \{a + b + 1\} \cup [2a + b + 2, 2a + 2b + 1] \\ Y &= [b + 1, b + a] \cup [a + b + 2, 2a + b + 1], \end{aligned}$$

where then $|X| = 2b + 1$ and $|Y| = 2a$. Let $X = \{x_1, x_2, \ldots, x_{2b+1}\}$ where $x_1 < x_2 < \cdots < x_{2b+1}$ and $Y = \{y_1, y_2, \ldots, y_{2a}\}$ where $y_1 < y_2 < \cdots < y_{2a}$.

Since

$$x = \sum_{i=1}^{2b+1} x_i = 2b^2 + 2ab + 3b + a + 1$$

and

$$y = \sum_{i=1}^{2a} y_i = 2a^2 + 2ab + 2a,$$

it follows that $2ax = (2b+1)y = 4a^2b + 4ab^2 + 6ab + 2a^2 + 2a$.

Next, we define a $(2b+1) \times (2a)$ matrix $M = [a_{i,j}]$ such that

- $2ax_i$ is the sum of the entries in row i for $1 \le i \le 2b+1$ and
- $(2b+1)y_j$ is the sum of the entries in column j for $1 \le j \le 2a$.

For the first b rows in M, we define each entry in row i to be i where $1 \le i \le b$. Hence, M has the form shown in (1). Thus, the sum of the entries in row i is $2ax_i = 2ai$ for $1 \le i \le b$. Next, we determine the remaining entries in the last $b+1$ rows of M. Since we want the sum of the entries in column 1 to be $(2b+1)y_1 = (2b+1)(b+1)$ and $\sum_{i=1}^{b} a_{i1} = \frac{b(b+1)}{2}$, it follows that

$$\sum_{i=b+1}^{2b+1} a_{i1} = (2b+1)(b+1) - \frac{b(b+1)}{2} = \frac{(b+1)(3b+2)}{2}.$$

We now choose each of the remaining $b+1$ entries $a_{b+1,1}, a_{b+2,1}, \ldots, a_{2b+1,1}$ in column 1 as either

$$\left\lfloor \frac{(b+1)(3b+2)}{2(b+1)} \right\rfloor = \left\lfloor \frac{3b+2}{2} \right\rfloor \quad \text{or} \quad \left\lceil \frac{(b+1)(3b+2)}{2(b+1)} \right\rceil = \left\lceil \frac{3b+2}{2} \right\rceil$$

so that the sequence $a_{b+1,1}, a_{b+2,1}, \ldots, a_{2b+1,1}$ is nondecreasing and the column sum is $(2b+1)y_1 = (2b+1)(b+1)$. Furthermore, since we want the sum of the entrees in row $b+1$ to be $2ax_{b+1} = 2a(b+a+1)$, we choose each of the entries $a_{b+1,2}, a_{b+1,3}, \ldots, a_{b+1,2a}$ as either

$$\left\lfloor \frac{2a(b+a+1) - a_{b+1,1}}{2a-1} \right\rfloor \quad \text{or} \quad \left\lceil \frac{2a(b+a+1) - a_{b+1,1}}{2a-1} \right\rceil$$

so that the sequence $a_{b+1,2}$, $a_{b+1,3}$, $\ldots$, $a_{b+1,2a}$ is nondecreasing and the row sum is $2ax_{b+1} = 2a(b + a + 1)$. We now proceed in this manner to determine the remaining entries in M.

Finally, we define the edge coloring $c : E(G) \to \mathbb{N}$ by $c(w_i u_j) = a_{ij}$ for every pair i, j of integers where $1 \le i \le 2b + 1$ and $1 \le j \le 2a$. By the defining property of the matrix M, it follows that $\mathrm{cm}(u_j) = x_j$ for $1 \le j \le 2a$ and $\mathrm{cm}(w_i) = y_i$ for $1 \le i \le 2b + 1$. Thus, $\{\mathrm{cm}(v) : v \in V(G)\} = X \cup Y = [2a + 2b + 1]$ and so $\mathrm{rm}(c) = 2a + 2b + 1 = s + t$. As an illustration, we construct a rainbow mean coloring c of $K_{4,7}$ with $\mathrm{rm}(c) = 11$. In this case, $a = 2$ and $b = 3$. We partition the set $[11]$ into the two sets $X = [3] \cup \{6\} \cup [9, 11] = \{1, 2, 3, 6, 9, 10, 11\}$ and $Y = [4, 5] \cup [7, 8] = \{4, 5, 7, 8\}$. Thus, $x = 42$ and $y = 24$ and so $4x = 7y = 168$. Using the technique described above, we obtain the 7×4 matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 5 & 6 & 6 & 7 \\ 5 & 7 & 12 & 12 \\ 6 & 8 & 12 & 14 \\ 6 & 8 & 13 & 17 \end{bmatrix}.$$

The resulting rainbow mean coloring c of $K_{4,7}$ with $\mathrm{rm}(c) = 11$ is shown in Figure 3, where the four vertices u_1, u_2, u_3, u_4 in the partite set U of $K_{4,7}$ are drawn in bold.

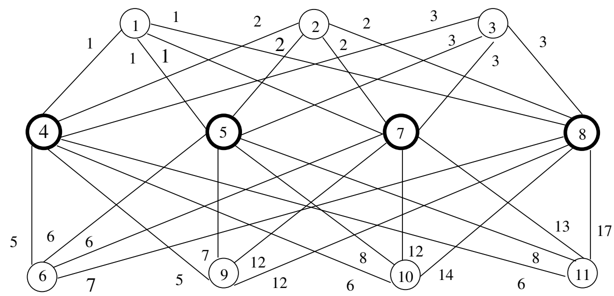

Figure 3: Constructing a rainbow mean coloring of $K_{4,7}$

Case 2. *st is odd.* We may assume that $s \le t$. By Proposition 3.2, it suffices to show that there is a rainbow mean coloring c of $K_{s,t}$ with $\mathrm{rm}(c) = s+t+1$. We proceed with the following three steps:

(1) Choose an appropriate element $p \in [s+t+1]$ and then partition the $(s+t)$-element set $[s+t+1]-\{p\}$ into the two subsets $X = \{x_1, x_2, \ldots, x_t\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ where $x_1 < x_2 < \cdots < x_t$ and $y_1 < y_2 < \cdots < y_s$ such that $s \sum_{i=1}^{t} x_i = t \sum_{j=1}^{s} y_j$.

(2) Construct a $t \times s$ matrix $M = [a_{ij}]$ such that sx_i is the sum of the entries in row i for $1 \le i \le t$ and ty_j is the sum of the entries in column j for $1 \le j \le s$.

(3) Use the matrix $M = [a_{ij}]$ to construct a rainbow mean coloring c of $K_{s,t}$. For each vertex u_j of $K_{s,t}$ where $1 \le j \le s$, we define a t-vector $\vec{u}_j = (a_{1j}, a_{2j}, \ldots, a_{tj})$ to be column j in M. This in turn gives rise to the corresponding s-vectors $\vec{w}_i = (a_{i1}, a_{i2}, \ldots, a_{is})$ to be row i in M for each vertex w_i where $1 \le i \le t$. The edge coloring $c : E(K_{s,t}) \to \mathbb{N}$ is defined by $c(w_i u_j) = a_{ij}$ for each pair i, j of integers with $1 \le i \le t$ and $1 \le j \le s$. Since the chromatic means of the vertices of $K_{s,t}$ are given by $\mathrm{cm}(u_j) = x_j$ for $1 \le j \le s$ and $\mathrm{cm}(w_i) = y_i$ for $1 \le i \le t$, it follows that $\{\mathrm{cm}(v) : v \in V(K_{s,t})\} = [s+t+1] - \{p\}$ and so $\mathrm{rm}(c) = s+t+1$.

Since s and t are both odd integers, it follows that $s = 2a+1$ and $t = 2b+1$ for some positive integers a and b with $a \le b$. We consider two subcases, according to whether $a = b$ or $a < b$.

Subcase 2.1. $a = b$. Figure 4 shows a rainbow mean coloring c of $K_{3,3}$ with $\mathrm{rm}(c) = 7$. Thus, we may assume that $a = b \ge 2$ and show that $\mathrm{rm}(K_{2a+1,2a+1}) = 4a+3$.

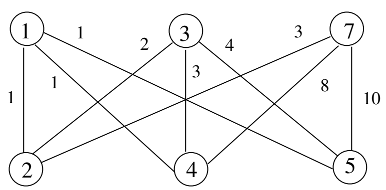

Figure 4: A rainbow mean coloring of $K_{3,3}$

First, we partition the $(4a+2)$-element set $[4a+3] - \{2a+2\}$ into the two subsets

$$\begin{aligned} X &= [2, a+2] \cup [3a+4, 4a+3] \\ Y &= \{1\} \cup [a+3, 2a+1] \cup [2a+3, 3a+3], \end{aligned}$$

where then $|X| = |Y| = 2a + 1$. Let $X = \{x_1, x_2, \dots, x_{2a+1}\}$ where $x_1 < x_2 < \dots < x_{2a+1}$ and $Y = \{y_1, y_2, \dots, y_{2a+1}\}$ where $y_1 < y_2 < \dots < y_{2a+1}$. Since

$$x = \sum_{i=1}^{2b+1} x_i = 4a^2 + 6a + 2 = \sum_{i=1}^{2a+1} y_i = y,$$

it follows that $(2a + 1)x = (2a + 1)y$.

Next, we define a $(2a + 1) \times (2a + 1)$ square matrix $M = [a_{i,j}]$ such that

- $(2a + 1)x_i$ is the sum of the entries in row i for $1 \le i \le 2a + 1$ and
- $(2a + 1)y_j$ is the sum of the entries in column j for $1 \le j \le 2a + 1$.

The first column and the first $a + 1$ rows of M are defined as follows:

⋆ Every entry in column 1 is 1;

⋆ For $1 \le i \le a + 1$, $a_{ij} = i + 1$ for $2 \le j \le 2a$ and $a_{i,2a+1} = 2i + 1$.

Thus,

$$M = \begin{bmatrix} 1 & 2 & 2 & 2 & \cdots & 2 & 3 \\ 1 & 3 & 3 & 3 & \cdots & 3 & 5 \\ 1 & 4 & 4 & 4 & \cdots & 4 & 7 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a+2 & a+2 & a+2 & \cdots & a+2 & 2a+3 \\ \vdots & & & & & & \\ \vdots & & & & & & \\ 1 & & & & & & \end{bmatrix}.$$

Hence, for $1 \le i \le a + 1$, the sum of the entries in row i is $(2a + 1)x_i = (2a + 1)(i + 1)$ and the sum of the entries in column 1 is $(2a + 1)y_1 = (2a + 1) \cdot 1 = 2a + 1$. Next, we define the remaining entries in M. Since we want the sum of the entries in column 2 to be $(2a + 1)y_2 = (2a + 1)(a + 3)$ and $\sum_{i=1}^{a+1} a_{i2} = \frac{(a+1)(a+4)}{2}$, it follows that

$$\sum_{i=a+2}^{2a+1} a_{i2} = (2a + 1)(a + 3) - \frac{(a + 1)(a + 4)}{2} = \frac{a^2 + 2a - 1}{2}.$$

We now choose each of the a entries $a_{a+2,2}, a_{a+3,2}, \ldots, a_{2a+1,2}$ in column 2 as

$$\left\lfloor \frac{a^2+2a-1}{2a} \right\rfloor \quad \text{or} \quad \left\lceil \frac{a^2+2a-1}{2a} \right\rceil$$

so that the sequence $a_{a+2,2}, a_{a+3,2}, \ldots, a_{2a+1,2}$ is nondecreasing and the column sum is $(2a+1)y_2 = (2a+1)(a+3)$. Furthermore, since we want the sum of the entrees in row $(a+2)$ to be $(2a+1)x_{a+2} = (2a+1)(3a+4)$, we choose each of the $2a-1$ entries $a_{a+2,3}, a_{a+2,4}, \ldots, a_{a+2,2a+1}$ in row $(a+2)$ as either

$$\left\lfloor \frac{(2a+1)(3a+4)-1-a_{a+2,2}}{2a-1} \right\rfloor \quad \text{or} \quad \left\lceil \frac{(2a+1)(3a+4)-1-a_{a+2,2}}{2a-1} \right\rceil$$

so that the sequence $a_{a+2,3}, a_{a+2,4}, \ldots, a_{a+2,2a+1}$ is nondecreasing and the row sum is $(2a+1)x_{a+2} = (2a+1)(3a+4)$. We now proceed in this manner to determine the remaining entries in M.

The matrix M then gives rise to a rainbow mean coloring c of $K_{2a+1,2a+1}$ with the desired properties. As an illustration, we construct a rainbow mean coloring c of $K_{5,5}$ with $\mathrm{rm}(c) = 11$. In this case, $a = b = 2$. We partition the set $[11] - \{6\}$ into the two sets $X = \{2, 3, 4, 10, 11\}$ and $Y = \{1, 5, 7, 8, 9\}$. Thus, $x = y = 30$. Using the technique described above, we obtain the 5×5 matrix

$$M = \begin{bmatrix} 1 & 2 & 2 & 2 & 3 \\ 1 & 3 & 3 & 3 & 5 \\ 1 & 4 & 4 & 4 & 7 \\ 1 & 8 & 13 & 14 & 14 \\ 1 & 8 & 13 & 17 & 16 \end{bmatrix}.$$

The matrix M gives rise to a rainbow mean coloring c of $K_{5,5}$ with $\mathrm{rm}(c) = 11$. To describe the coloring c, it is convenient to introduce additional notation. For each vertex u_i of $K_{5,5}$ where $1 \leq i \leq 5$, we define a 5-vector $\vec{u}_i = (c(u_i w_1), c(u_i w_2), \ldots, c(u_i w_5))$, which is column i in M. This, in turn, gives rise to the corresponding 5-vectors $\vec{w}_i$ for the vertices w_i $(1 \leq i \leq 5)$, which is row i in M. The coloring c is shown in Figure 5, where the vertices in U are drawn in bold.

Subase 2.2. $a < b$. Here, we show that $\mathrm{rm}(K_{2a+1,2b+1}) = 2a+2b+3$. First, we partition the $(2a+2b+2)$-element set $[2a+2b+3] - \{a+b+2\}$ into the two subsets

$$\begin{aligned} X &= [b-a] \cup [b+1, a+b+1] \cup [a+b+3, 2a+b+1] \\ &\qquad \cup [3a+b+3, 2a+2b+3] \\ Y &= [b-a+1, b] \cup [2a+b+2, 3a+b+2], \end{aligned}$$

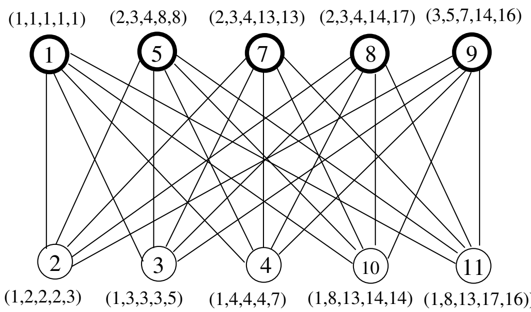

Figure 5: A rainbow mean coloring of $K_{5,5}$

where then $|X| = 2b+1$ and $|Y| = 2a+1$. Let $X = \{x_1, x_2, \ldots, x_{2b+1}\}$ where $x_1 < x_2 < \cdots < x_{2b+1}$ and $Y = \{y_1, y_2, \ldots, y_{2a+1}\}$ where $y_1 < y_2 < \cdots < y_{2a+1}$. Since

$$x = \sum_{i=1}^{2b+1} x_i = 2b^2 + 2ab + 5b + a + 2$$

and

$$y = \sum_{i=1}^{2a+1} y_i = 2a^2 + 2ab + 5a + b + 2,$$

it follows that

$$(2a+1)x = (2b+1)y = 4a^2b + 4ab^2 + 12ab + 2a^2 + 2b^2 + 5a + 5b + 2.$$

Next, we define a $(2b+1) \times (2a+1)$ matrix $M = [a_{i,j}]$ such that

- $\star$ $(2a+1)x_i$ is the sum of the entries in row i for $1 \le i \le 2b+1$ and
- $\star$ $(2b+1)y_j$ is the sum of the entries in column j for $1 \le j \le 2a+1$.

For the first $b-a$ rows in M, we define each entry in row i to be i where $1 \le i \le b-1$. That is,

$$M = \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 2 & 2 & 2 & \cdots & \cdots & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b-a & b-a & b-a & \cdots & \cdots & \cdots & b-a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Thus, the sum of the entries in row i is $(2a+1)x_i = 2ai$ for $1 \le i \le b-a$. Next, we determine the remaining entries in M. Since we want the sum of the entries in column 1 to be $(2b+1)y_1 = (2b+1)(b-a+1)$ and $\sum_{i=1}^{b} a_{i1} = \frac{(b-a)(b-a+1)}{2}$, it follows that

$$\sum_{i=b+1}^{2b+1} a_{i1} = (2b+1)(b-a+1) - \frac{(b-a)(b-a+1)}{2} = \frac{(b-a+1)(3b+a+2)}{2}.$$

We now choose each of the $b+a+1$ entries $a_{b-a+1,1}, a_{b-a+2,1}, \ldots, a_{2b+1,1}$ as either

$$\left\lfloor \frac{(b-a+1)(3b+a+2)}{2(b+a+1)} \right\rfloor \quad \text{or} \quad \left\lceil \frac{(b-a+1)(3b+a+2)}{2(b+a+1)} \right\rceil$$

so that the sequence $a_{b-a+1,1}$, $a_{b-a+2,1}$, $\ldots$, $a_{2b+1,1}$ is nondecreasing and the column sum is $(2b+1)y_1 = (2b+1)(b-a+1)$. Furthermore, since we want the sum of the entrees in row $(b-a+1)$ to be $(2a+1)x_{b-a+1} = (2a+1)(b+1)$, we choose each of the $2a$ entries $a_{b-a+1,2}, a_{b-a+1,3}, \ldots, a_{b-a+1,2a+1}$ as either

$$\left\lfloor \frac{(2a+1)(b+1) - a_{b-a+1,1}}{2a} \right\rfloor \quad \text{or} \quad \left\lceil \frac{(2a+1)(b+1) - a_{b-a+1,1}}{2a} \right\rceil$$

so that the sequence $a_{b-a+1,2}$, $a_{b-a+1,3}$, $\ldots$, $a_{b-a+1,2a+1}$ is nondecreasing and the row sum is $(2a+1)x_{b-a+1} = (2a+1)(b+1)$. We now proceed in this manner to determine the remaining entries in M. The matrix M then gives rise to a rainbow mean coloring c of $K_{2a+1,2b+1}$ with the desired properties. As an illustration, we construct a rainbow mean coloring c of $K_{5,9}$ with $\text{rm}(c) = 15$. In this case, $a = 2$ and $b = 4$. We partition the set $[15] - \{8\}$ into the two sets $X = \{1,2,5,6,7,9,13,14,15\}$ and $Y = \{3,4,10,11,12\}$. Thus, $x = 72$ and $y = 40$. Thus, $5x = 9y = 360$. Using the technique described above, we obtain the 5×9 matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 5 & 5 & 6 & 6 \\ 3 & 4 & 7 & 8 & 8 \\ 3 & 4 & 15 & 6 & 7 \\ 3 & 5 & 15 & 19 & 3 \\ 4 & 5 & 15 & 19 & 22 \\ 4 & 5 & 15 & 19 & 27 \\ 4 & 5 & 15 & 19 & 32 \end{bmatrix}.$$

The matrix M gives rise to a rainbow mean coloring c of $K_{5,9}$ with $\text{rm}(c) = 15$. As indicated in Subcase 2.1, each vertex u_j of $K_{5,9}$ where $1 \le j \le 5$ is associated with a 9-vector $\vec{u}_j = (c(u_jw_1), c(u_jw_2), \ldots, c(u_jw_9))$ (which is columm j in M). This, in turn, gives rise to the corresponding 5-vectors $\vec{w}_i$ for each vertex w_i (which is row i in M) for $1 \leq i \leq 9$. The coloring c is shown in Figure 6, where the vertices in U are drawn in bold. $\square$

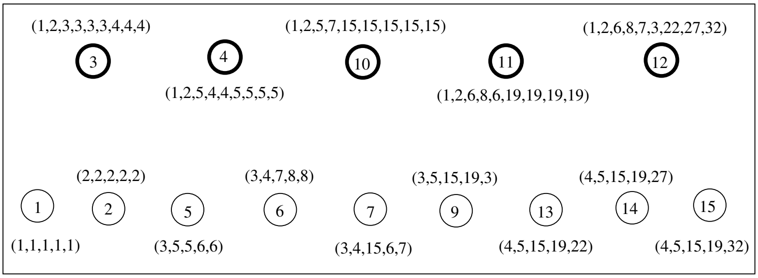

Figure 6: A rainbow mean coloring of $K_{5,9}$

All of the graphs considered in this article thereby substantiate Conjecture 1.7. Indeed, the only Type 3 graphs found thus far are stars of even order. Consequently, not only may Conjecture 1.7 be true but if the stars of even order are excluded, all other connected graphs may be Type 1 or Type 2.

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