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Research Note on

THE MINIMUM DISTANCE NUMBER OF TREES

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ABSTRACT: In addition to a known theorem [3] it is shown that those vertices which form the centroids of trees have minimum distance numbers.

The basic information about the metric properties of a connected simple graph G with n vertices is stored in its distance matrix $\underline{D}(G)$: The entries of $\underline{D}(G)$ are the distances d(uv) between any pair of vertices $u,v \in G$. The sum of the entries of any column $d(u|G)$ is called the distance number of the vertex u and represents the sum of the distances between u and all the other vertices of G. Finally, the sum over the upper or lower triangle of $\underline{D}(G)$ is known as the Wiener number W(G) of the graph; obviously, W(G) equals the half sum of all distance numbers of G.

The distance numbers of adjacent vertices, u adj v, have the following property [1,2]:

$$d(u|G) = d(v|G) - |V_u| + |V_v| \quad (1)$$

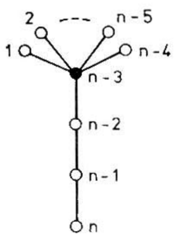
where V_u (V_v) denote that vertex subset which collects those vertices of G which have shorter distance to u(v) than to v(u). Note that $n = |V_u| + |V_v|$ only holds if the adjacent vertices u and v do not belong to an odd membered cycle.

Equation (1) gives rise to a variation of the distance numbers within a graph. For acyclic graphs one has the following

Theorem [3]: A tree has exactly one vertex with minimum distance number or exactly two vertices with minimum distance numbers and they are adjacent.

With regard to some other work [4] this statement is of some interest. It is verified by the distance numbers of the centers of path graphs P_n, star graphs $K_{1,n-1}$, and some other acyclic graphs. But in P_n, $k_{1,n-1}$, and some other trees, the center of these graphs coincides with the respective centroid which are defined [5] with respect to minimum eccentricity and minimum weight, respectively. Thus it is not obvious to which kind of vertices the above theorem refers.

In order to clarify that point we consider first a tree, the center of which does not coincide with its centroid. Such a tree T with n > 6 vertices is shown below. The center of T is represented by vertex (n-2), its centroid by vertex (n-3).



$$d(j|T) = 2n \ , \qquad 1 \leq j \leq n-4 \ ,$$
$$d(n-3|T) = n + 2 \ ,$$
$$d(n-2|T) = 2n - 4 \ ,$$
$$d(n-1|T) = 3n - 3 \ ,$$
$$d(n|T) = 4n - 10 \ .$$

Because $n > 6$, from the distance numbers of the vertices of this tree it can be seen that in T the minimal distance number is associated with the centroid vertex (n-3). Therefore, the above theorem may refer to the centroid vertices of trees and, hence, we have to prove generally the following

Proposition: The centroid vertex (vertices) in a tree possess the minimal distance number.

Proof: a) Let T be an arbitrary tree with k edges and $n = k+1$ vertices; further let $z \in T$ be a centroid vertex of degree g. Let vertex z be incident with g branches $\{B_j | 1 \leqq j \leqq g\}$ each one having k_j edges. Let g vertices adjacent to z be denoted by y_j, where $y_j \in B_j$ holds for all j. Edge (zy_j) is thus a terminal edge of B_j. b) The application of eq. (1) implies the following: the k_j-1 vertices of B_j, different from z and y_j, have a shorter distance to y_j than to z while the remaining $n-k_j = k+1-k_j$ vertices of the other branches $\{B_k | k \neq j\}$ are closer to z than to y_j. Making use of eq. (1) one obtains: $d(y_j|T) - d(z|T) = k+1-2k_j$. c) The proposition is proved when $(k+1-2k_j) \geqq 0$ holds for each j, since according to eq. (1) the distance numbers of those vertices which lie between y_j and any terminal vertex of B_j must, be larger than $d(y_j|T)$. d) The order $k_1 \geqq k_2 \geqq \ldots$ produces $(k+1-2k_1) = \min\{k+1-2k_j\}$, hence it suffices to prove that $(k+1-2k_1) \geqq 0$. e) Note that k_1 is the weight at the vertex z. According to a theorem given by C. Jordan [6] $k_1 \leqq (k+1)/2$, where equality holds for a centroid consisting of two adjacent vertices. f) From this one finds that, indeed, $(k+1-2k_1) \geqq 0$, which proves the proposition. ◻

References:

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