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BALANCED n-COLOR COMPOSITIONS

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Abstract

We introduce the notion of balanced n-color compositions, and consider the related enumeration problems. We also establish bijections with four other combinatorial objects, namely two types of Motzkin lattice paths, pairs of regular compositions with restricted parts, and regular compositions with a centered maximum.

1. Introduction

A composition is an ordered list of positive integers $\upsilon = (p_1, \dots, p_k)$. Each of the numbers p_i in the list is called a *part of size* p_i of the composition. The sum of all the parts is called the *weight* of the composition, and the number of parts is called its *length*. We say that υ is a *composition of* ℓ *(with* k *parts)* if it has weight ℓ and length k.

Agarwal [1] introduced the concept of n-color compositions. An n-color composition is a composition in which a part of size s can be in one of s different colors, which are usually represented by numbered subscripts. For instance, $(3_2, 4_3, 1_1)$ is an example of an n-color composition of 8 with 3 parts. For any ℓ, we use $\mathcal{CC}_\ell$ to refer to the set of n-color compositions of ℓ.

A well known graphical representation of compositions is through *tilings*. The scheme of *spotted* tilings for n-color compositions was introduced by Hopkins in [4]. A part k_i is represented by a $1 \times k$ tile with a spot marked on the i-th square, as exemplified on Figure 1. This differs from the usual, spotless tiling that is used in the case of regular compositions.

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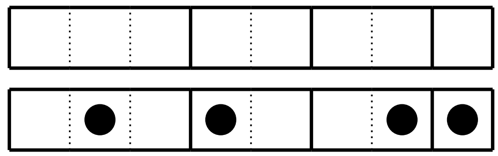


Figure 1: Tiling representation of the regular composition $(3, 2, 2, 1)$, and spotted tiling representation of the n-color composition $(3_2, 2_1, 2_2, 1_1)$.

In this pictorial view, we denote the barriers between parts as *cuts*, while the dotted lines between two squares in the same part are called *joins*. For example, in the tiling of Figure 1, the first two squares have a join between them, as opposed to the cut in between the third and fourth squares. We do not consider the outermost lines of the tiling as anything but the borders of the diagram.

Finally, we can describe the compositions through *boundary words* obtained from their tilings, by assigning the letter J to any join, and the letter C to any cut. For example, the tiling of the regular composition $(3, 2, 2, 1)$ (shown in Figure 1) corresponds to the word $JJCJCJC$. Note that the boundary word of the composition (1) is the empty word $\varnothing$.

We use these tilings throughout this paper to showcase and exemplify many of the results presented later. In Section 2, we establish an important link between n-color and regular compositions, which we refine to introduce the balanced compositions in Section 3. This is a subset of n-color compositions that has interesting connections to several combinatorial objects, which are shown in Section 4.

2. A Connection with Regular Compositions

In general, to any n-color composition σ we can associate a regular composition $E(\sigma)$ with an even number of parts (specifically, double the length of σ), which we call the *expanded form* of the composition. These expanded forms can be visualized through their corresponding tilings, as follows. For an n-color composition, think of the spotted squares as an additional type of barrier between parts of unmarked tiles (besides the usual cuts). Since there is the possibility that a spotted square is adjacent to a barrier (or to one of the ends of the tiling), we add an additional 1×1 square before and after each spotted square. This results in the tiling of a valid regular composition. An example is given in Figure 2. To reverse this process, simply replace every other cut with a spotted square, and then remove a square before and after each one of these spots.

Similar representations have been studied in previous work. This is a slight modification from the *c-blocks* and *tails* introduced by Hopkins and Wang in [5],

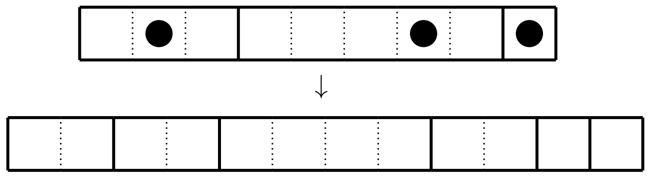


Figure 2: The expanded form of $(3_2, 5_4, 1_1)$ is $(2, 2, 4, 2, 1, 1)$.

but here we do not allow the tails to be empty. It is easy to see that the above map introduces the corresponding bijection between these compositions, formally stated in Proposition 1.

Proposition 1. *There is a one-to-one correspondence between the n-color compositions of ℓ with m parts and compositions of $\ell + m$ with $2m$ parts.*

3. Balanced n-Color Compositions

As the expanded form of an n-color composition has an even length, we can split it into an ordered pair of compositions of equal length: one formed by all parts in even positions, and the other one with the parts from odd positions. We say that an n-color composition is *balanced* if these two compositions have the same weight. For example, the n-color composition $(3_3, 2_1, 2_1, 1_1)$ is balanced, because its expanded form $(3, 1, 1, 2, 1, 2, 1, 1)$ splits into the pair of compositions $((3, 1, 1, 1), (1, 2, 2, 1))$, as shown in Figure 3. The n-color composition in Figure 2, on the other hand, is not balanced, because its expanded form splits into $((2, 4, 1), (2, 2, 1))$.

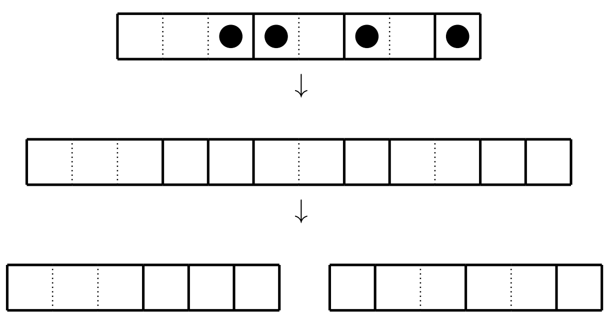


Figure 3: The expanded form of a balanced composition splits into two compositions of equal weight.

A more direct way to determine if an n-color composition is balanced is to count the number of non-spotted squares which, within their respective parts, are located to the left of the spotted 1×1 tile. If this equals the (analogously defined) number of squares to the right of spotted tiles, then the composition is balanced. Almost immediately from the definition, we have the following observation for balanced compositions.

Lemma 1. *There is a bijection between balanced n-color compositions of ℓ with m parts and ordered pairs of compositions of $\frac{\ell+m}{2}$ with m parts.*

In particular, there are no balanced n-color compositions of ℓ with m parts if $\ell+m$ is odd. More importantly, this lemma gives us a way of counting the number of balanced compositions for a given weight ℓ. If we let $q = \frac{\ell+m}{2}$ and write $m = 2q - \ell$ instead; notice that, because $1 \le m \le \ell$, then $\lfloor \frac{\ell}{2} \rfloor + 1 \le q \le \ell$. Finally, since the number of regular compositions of n with k parts is $\binom{n-1}{k-1}$, we have the following.

Theorem 1. *For any $\ell \in \mathbb{N}$, the number of balanced n-color compositions of ℓ is*

$$\left|\mathcal{CC}_\ell^B\right| = \sum_{q=\lfloor \ell/2 \rfloor+1}^{\ell} \binom{q-1}{2q-\ell-1}^2.$$

Proof. For each q, there are $\binom{q-1}{2q-\ell-1}^2$ ordered pairs of compositions of q with m parts. □

The sequence $\left\{\left|\mathcal{CC}_\ell^B\right|\right\}_{\ell\in\mathbb{N}}$, shown in Table 1, is listed as A051286 in the On-line Encyclopedia of Integer Sequences [6]. Next, we present explicit bijections to

ℓ	$\left\|\mathcal{CC}_\ell^B\right\|$	ℓ	$\left\|\mathcal{CC}_\ell^B\right\|$	ℓ	$\left\|\mathcal{CC}_\ell^B\right\|$
1	1	6	26	11	2317
2	1	7	63	12	5794
3	2	8	153	13	14545
4	5	9	376	14	36631
5	11	10	931	15	92512

Table 1: The sequence of the number of balanced n-color compositions.

three types of lattice paths and pairs of regular compositions also counted by this sequence.

4. Connections to Other Combinatorial Objects

A *Motzkin path* is a type of lattice path that begins at the origin and ends somewhere on the x axis. It consists of $U = (1, 1)$, $D = (1, -1)$, and $H = (1, 0)$ steps. The *length* of a path is the number of steps in the path, or equivalently, the x coordinate of its right endpoint. Conventionally, these paths are only allowed to go through the first quadrant, with the notation of *grand* reserved for the case in which paths are allowed to cross the x-axis. These paths can be uniquely represented by words using the letters that correspond to the steps, as shown in Figure 4.

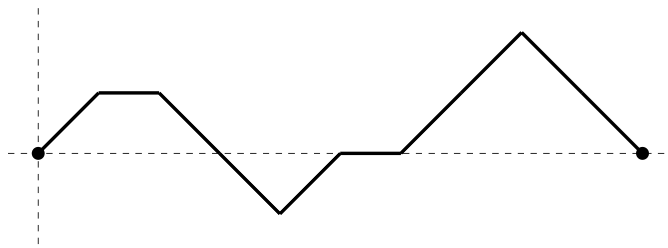


Figure 4: The grand Motzkin path $UHDDUHUUDD$.

In [3], Elizalde found a bijection between a modified type of Motzkin path and unimodal bargraphs with a centered maximum. Both sets are also counted by the OEIS Sequence A051286. A *unimodal bargraph* [7] is a composition $(a_1, \ldots, a_k)$ that satisfies $a_1 \leq \cdots \leq a_j \geq \cdots \geq a_k$ for some index j. A *centered maximum* occurs when j is one of $\left\{\left\lfloor \frac{k+1}{2} \right\rfloor, \left\lceil \frac{k+1}{2} \right\rceil\right\}$. From Figure 5, observe that we can also describe bargraphs as lattice paths, using $N = (0, 1)$, $E = (1, 0)$ and $S = (0, -1)$ steps. In this case, the corresponding path is $NENNENESSEESES$.

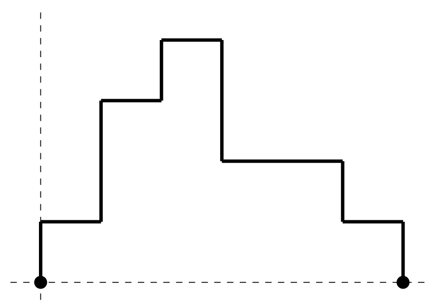


Figure 5: The composition $(1, 3, 4, 2, 2, 1)$ has a centered maximum because it 'peaks' as close to the middle as possible.

The lattice paths for unimodal bargraphs give an accessible way of analyzing these objects. But unlike the Motzkin paths, we need to deal with the issue of parity: notice there is only one possible index for the peak of the bargraph when it has an odd number of parts, as opposed to two when it is even. In addition, observe that the lattice paths of bargraphs always begin and end with vertical steps.

4.1. Peakless Grand Motzkin Paths

Extending the definition given earlier, a *peakless* Motzkin path is a path which does not contain the digraph UD anywhere in its word representation. We use the word representations and the spotted tilings of balanced compositions to prove the following result. Note that an empty path is a valid grand Motzkin path of length 0.

Theorem 2. *Let $\ell \geq 1$. There is a bijection between the set of balanced compositions of ℓ and the set of peakless grand Motzkin paths of length $\ell - 1$.*

Proof. The case for $\ell = 1$ is trivial since there is only one balanced composition of 1, namely (1_1), and the empty path is the only peakless grand Motzkin path of length 0. For positive ℓ, we provide one-to-one mappings from a set to another and vice versa. Thus, a bijection is established.

To find the balanced n-color composition associated with a given path of length $\ell - 1$, we begin by appending a D and an H at the beginning and end of its word, respectively. Next, we split the modified word into an ordered list of ℓ digraphs, which we then transform into a list of 1×1 squares, following the rules of Table 2. Finally, we stick the squares together in the same order (the only 'exception' to these rules occurs when transforming the first digraph of the modified word, which will begin with a D, for which the left boundary is necessarily a border). The resulting diagram will be the spotted tiling of a balanced n-color composition of length ℓ.

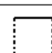
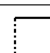
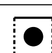
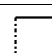
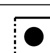
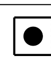
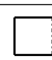
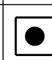
Begins \ Ends	U	D	H
U		N/A	
D			
H			

Table 2: Digraph transformation rules.

To prove the validity of the generated diagram, it is necessary to confirm the following:

1. The squares fit consistently. For any two consecutive squares, the right boundary of the first one must be of the same type as the left boundary of the second one.

2. The tiling corresponds indeed to a balanced n-color composition of $\ell + 1$, meaning that each part of the tiling has exactly one spotted 1×1 square, and that the left and right weights equal each other.

Observing (1) is straightforward. In Table 2, observe that the right boundary of two squares in the same column is the same. Similarly for the left boundaries in the rows. Now, notice that the relevant boundary is the same for the D column and the D row (in this case, a dotted delimiter). The same thing occurs for the U column and the U row, and again for the H column and the H row. Thus any two consecutive digraphs (for which the ending letter of the first one is the beginning letter of the second one) will transform into two squares for which the right boundary of the first one will be of the same type as the left boundary of the second one.

For (2), notice that the only cuts in the tiling arise from digraphs containing the letter H. Thus, the 'parts' in the tiling correspond to subwords beginning and ending with an H, with no other H in between. In each of these subwords, since there are no digraphs UD, there must be exactly one digraph from the set $\{DU, DH, HU, HH\}$ (corresponding to the cases in which there are both D's and U's, only D's, only U's, or nothing in between the two H's, respectively). The part corresponding to this subword will thus have only 1 spotted tile. The property that the resulting diagram corresponds to a balanced composition is due to the fact that the Motzkin path necessarily has the same number of D and U steps, since it must end in the x-axis.

To see that the map is one-to-one, note that most digraphs in Table 2 generate different types of squares. The only exceptions are the digraphs DD and UU, which generate the same spotless square. If two distinct Motzkin words generated the same composition, the only possibility would be for one of them to have a DD and the other a UU (both in a same position). Call such event an *awkward pair*, and consider its rightmost occurrence within the Motzkin words. We appended an H at the end of each word, so this cannot be at the very last digraph of the list. In particular, the digraph immediately following must be different in both words (one begins with a D, and the other with a U), and because it can no longer be an awkward pair, the corresponding generated tiles are different as well. So the hypothesis that both words generated the same diagram would yield a contradiction.

Delimiter	Step
Cut	H
Join before spot	D
Join after spot	U

Table 3: Mapping from spotted tilings to words of Motzkin paths.

It is easy to describe the inverse map from spotted tilings to Motzkin paths.

This time, we focus on the joins and cuts of the diagram. We say that a join is before (or after) a spot if it is to the left (or right) of the spotted square within the join's part. The map described in Table 3 is one-to-one and sends balanced n-color compositions of ℓ to peakless Motzkin paths of length $\ell - 1$, because that is the number of inner delimiters in the spotted tilings. Moreover, the resulting path is always peakless, because there is always a cut (an H step) in between the joins that map to U steps and those which map to D steps. □

Example 1. Consider the peakless Motzkin path $DUHUHDH$ of length 7. We construct the digraph list $(DD, DU, UH, HU, UH, HD, DH, HH)$ after appending the corresponding D and H. The spotted tiling generated from Table 2 corresponds to $(3_2, 2_1, 2_2, 1_1)$, a balanced n-color composition of 8.

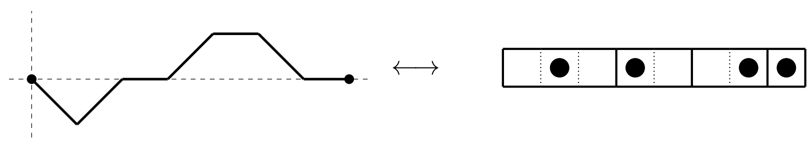


Figure 6: The Motzkin path on the left is bijected with the balanced composition on the right.

Note that turning this composition back into a Motzkin path using the mapping in Table 3 gives the original path $DUHUHDH$.

4.2. Pairs of Regular Compositions with Restricted Parts

This bijection is inspired by the work of Bóna and Knopfmacher, who showed in [2] that pairs of compositions of a shared length and weight and with parts equal to only 1 or 2 are also enumerated by Sequence A051286.

Theorem 3. *Let $\ell \geq 2$. There is a bijection between the set of balanced compositions of ℓ and the set of ordered pairs of compositions of $\ell-1$ with the same length (number of parts), with the restriction that these parts can only equal 1 or 2.*

We ignore the case for $\ell = 1$ because there may not be compositions of 0, unless we consider an empty list () to be a valid composition. We note that in this case, the pair $((), ())$ is the only pair of restricted compositions of 0, so the result would still hold.

Proof. Recall from Lemma 1 that a balanced n-color composition of ℓ with m parts can be split into an ordered pair of compositions of $\frac{\ell+m}{2}$ with m parts. We use the boundary words of these compositions to build a new pair of compositions of $\ell - 1$ with parts equal to 1 or 2 by assigning the value 2 to any join, and the value 1 to any cut. Since $\ell \geq 2$, then $\frac{\ell+m}{2} > 1$, so the boundary words are nonempty.

Now, because both tilings have m parts, then there are exactly $m-1$ cuts in each, and because both have the same size as well, then there must be $\frac{\ell+m}{2} - 1 - (m-1)$ joins. Thus the weight of these new compositions will be

$$2\left(\frac{\ell+m}{2} - 1 - (m-1)\right) + m - 1 = (\ell+m-2) - (m-1) = \ell - 1.$$

This one-to-one transformation has a straightforward inverse (turn any 2 back into a join and a 1 into a cut), so it establishes the desired bijection. □

4.3. Uneven Bicolored Grand Motzkin Paths

An *uneven bicolored* grand Motzkin path is a modified type of Motzkin path, in which the U steps have a weight of 2, and there is an additional type of horizontal step $H' = (1, 0)$ with weight 2 (hence the 'bicolored'). An alternative (but equivalent) way of thinking about these paths is by considering the steps $U_2 = (2, 1)$, $D_1 = (1, -1)$, $H_1 = (1, 0)$, and $H_2 = (2, 0)$. The advantage of this representation is that the weight of a path matches the ending x-coordinate of the path, as occurs with the regular Motzkin paths.

Theorem 4. *Let $\ell \geq 1$. There is a bijection between the set of balanced n-color compositions of ℓ and the set of uneven bicolored grand Motzkin paths of weight $\ell - 1$.*

Proof. Again, here if $\ell = 1$, then the proposition is trivial because the empty path is the only valid path of weight 0. In the rest of the cases, consider again the boundary words of the two compositions of $\frac{\ell+m}{2}$ that arise from its expanded form. To describe the bijection, first construct an ordered list of the ordered pairs of letters in corresponding positions. For example, for the pair of words (JJC, JCC), we would have the pairs $(J, J), (J, C)$ and (C, C), in that order.

We build an uneven bicolored grand Motzkin path from that list, by turning each pair into a step, as described in Table 4. Once again, this is a one-to-one, reversible process (starting with a path, construct a pair of compositions using the mappings from the table and rearranging the elements of the generated pairs to form boundary words).

Pair type	Step
(J, J)	H_2
(J, C)	U_2
(C, J)	D_1
(C, C)	H_1

Table 4: Mapping between spotted tilings and modified Motzkin paths.

To see that the result is indeed a path of weight n, recall that both compositions of $\frac{\ell+m}{2}$ have the same weight and length, so they also have the same number of cuts and joins. From here, the number of U_2 steps must be the same as the number of D_1 steps (and thus the path is a valid one). To obtain the weight, notice that a step has weight 2 if and only if the first element of the pair that generated it is a join. Thus the total weight of the path will be two times the number of joins plus the number of cuts of the first composition which, as shown in the previous section, equals $\ell - 1$. □

The choice of assigning a weight 2 to the U step is independent of this result. If we had instead put the weight on the D step, we would have obtained an analogous statement.

Example 2. The results in Theorems 3 and 4 are fairly similar. Consider the balanced composition $(3_3, 2_1, 2_1, 1_1)$, which splits into the pair of compositions $((3, 1, 1, 1), (1, 2, 2, 1))$, as illustrated in Figure 3.

Their corresponding boundary words are $(JJCCC, CJCJC)$. From the weights indicated in the proof of Theorem 3, we construct the pair $((2, 2, 1, 1, 1), (1, 2, 1, 2, 1))$ of equally long compositions of weight 7, with parts all equal to 1 or 2. We also construct the list of corresponding delimiters $((J, C), (J, J), (C, C), (C, J), (C, C))$, which yields the uneven bicolored grand Motzkin path $U_2H_2H_1D_1H_1$ of weight 7.

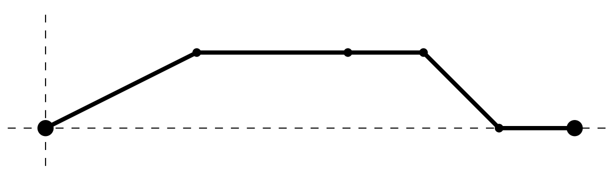


Figure 7: The uneven bicolored Motzkin path corresponding to the composition $(3_3, 2_1, 2_1, 1_1)$.

4.4. Unimodal Bargraphs with a Centered Maximum

In this case, we look at bargraphs not by their weight, but by their *semiperimeter*. The semiperimeter of an unimodal bargraph is the number of N (or S) and E steps in its lattice path. An equivalent definition using compositions involves adding the length of the composition and the size of the largest part. Either way, we end up with the semiperimeter of the smallest rectangle that contains the bargraph, as shown in Figure 8.

Theorem 5. *Let $\ell \geq 2$. There exists a bijection between the balanced n-color compositions of ℓ and the number of unimodal bargraphs with a centered maximum whose semiperimeter is also ℓ.*

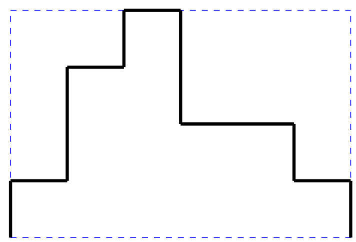


Figure 8: The semiperimeter of a bargraph.

Notice that we ignore the case $\ell = 1$ since the minimal semiperimeter of any bargraph is 2. The strategy for proving this result is the following. From the pair of regular compositions that we obtain from a balanced n-color composition, we build a pair of lattice paths with common endpoints using horizontal (E) and vertical (either S or N, but not both) steps. We then reflect one of the two and assemble a valid bargraph.

Proof. We look again at the joins and cuts of the compositions generated when splitting the expanded form, and build a pair of lattice paths. We describe this bijection geometrically, so we use the words *left* and *right* to describe, respectively, the paths corresponding to the first and second boundary words in the ordered pair of compositions. For example, in the pair of words $(JJCJ, CJJJ)$, the left word (and the corresponding path) is $JJCJ$. The assignment of steps is as shown below in Table 5.

Delimiter	Step
Join	E
Cut	N or S

Table 5: Mapping between balanced compositions and bargraph lattice paths.

The algorithm for this bijection consists of two parts, namely:

1. If both of the first boundaries in the compositions are cuts (i.e. if the pair of the first letters in both words is of the form (C, C)), then we use N steps. Otherwise, we use S steps.

2. If we used N steps, we append an E at the end of both paths. Then, we reflect the right path vertically (i.e. change all N's for S's, and read it backwards), and we merge both paths so that they overlap on the added E steps. If we instead used S steps, then we append a S step at the end of both paths. Then, we reflect the left path vertically, and merge both.

As expected, the left path will become the left part of the bargraph, and similarly for the right case. The resulting path will be unimodal because, each time we transform compositions into bargraphs, we only use one type of vertical step. In addition, both paths have the same number of horizontal steps (again because both boundary words have the same number of joins), so when merging them we will end up with a centered maximum.

Now, in the final bargraph, the number of E steps will be two times the number of joins in any of the boundary words that are generated from the expanded form, since each path contributes an equal number of E steps. The number of N steps, on the other hand, will only equal the number of cuts of the composition that was reflected. Finally, the added steps will either increase the number of E steps by 1 (because we overlap both of them) or increase the number of N steps by 1. In both cases, the semiperimeter is $(\ell - 1) + 1 = \ell$, again using the identity from the previous two sections.

This map is one-to-one because the cases in which we used N steps will yield bargraphs with an odd number of parts, while those in which we used S steps will yield bargraphs of even length, taking care of the parity issue in the process, and ensuring that the two sets of generated bargraphs are disjoint. Now, within each case, the map is one-to-one, because the map that sends balanced compositions to the pair of boundary words is one-to-one.

Inverting this mapping is straightforward, but we need to be careful when describing how to do so. Informally, we break the bargraph right at the middle, and then build a composition from the resulting halves. Here is the algorithm that reverts the previous process.

1. If the length is odd, then we can decompose the bargraph as $B_L E B_R$, where E is the middle E step. Consider the pair of paths (B_L, B'_R), where B'_R is the vertical reflection of B_R. If it is even, then decompose it as $N B_L B_R S$, where B_L and B_R have the same number of steps (we can do this because the first and last steps in a bargraph are always vertical, and the length is even). In this case, consider the pair of paths (B'_L, B_R), where again B'_L is the reflection of B_L.

2. From the pair of paths, construct boundary words using Table 5. Then, reassemble a balanced n-color composition as usual.

The resulting composition will be balanced because the number of horizontal and vertical steps in B_L and B_R are the same, respectively (recall that we split the bargraph in the middle). Its weight will be ℓ again because of the identity arising from the number of joins and cuts in the boundary words. □

Example 3. Consider the balanced composition $(2_2, 2_2, 3_1, 3_2)$. From the splitting of the expanded form, we get the boundary words $JCJCCJ$ and $CCJJCJ$. Since the pair of first letters is (J, C), we use S steps, so we end up with the pair of paths $(ESESSE, SSEESE)$. After adding the additional S steps and reflecting the path on the left, we end up with $(NENNENE, SSEESES)$, which we assemble into the bargraph corresponding to the composition $(1, 3, 4, 2, 2, 1)$ as shown in Figure 9.

Now, for the composition $(1_1, 4_3, 2_1, 3_2)$, which is also balanced, we obtain the pair of boundary words $(CJJCCJ, CJCJCJ)$. Because both of them begin with a C, we use N steps instead, resulting in the paths $(NEENNE, NENENE)$, which we then turn into $(NEENNEE, EESESES)$ after adding an E and reflecting the right path. We merge them to obtain the bargraph associated with the composition $(1, 1, 3, 3, 3, 2, 1)$, also shown in Figure 9.

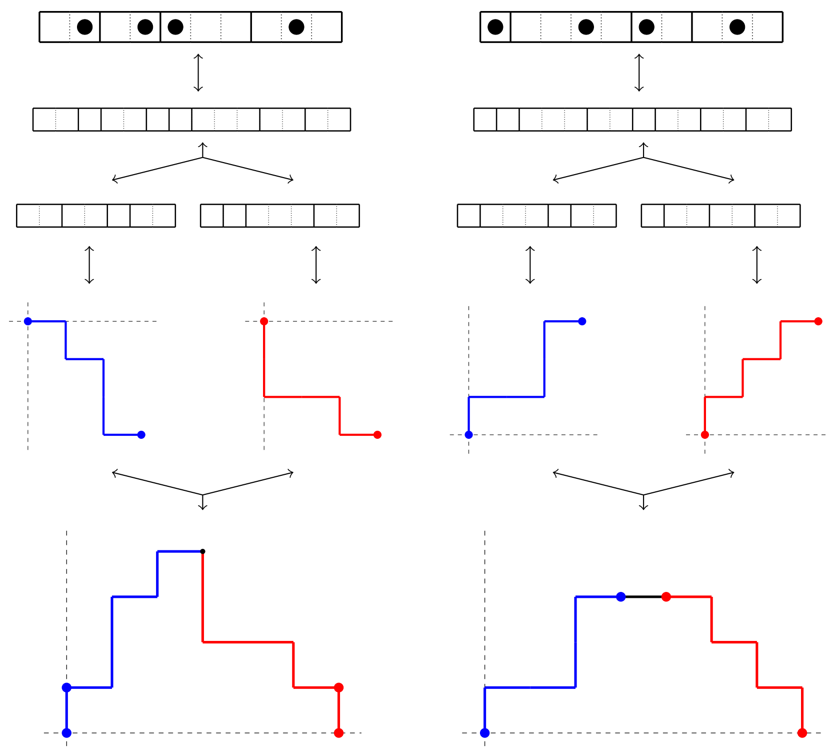


Figure 9: The bijection between balanced compositions and unimodal bargraphs.

5. Conclusion

In this paper, we introduced the concept of balanced n-color compositions. The study of such compositions started with a simple but interesting connection to pairs of regular compositions, from which a straightforward sum arises as the total number of such compositions. Evaluating the sum leads to an OEIS sequence that also enumerates a number of interesting combinatorial objects including two types of Motzkin lattice paths, pairs of regular compositions with restricted parts, and regular compositions with a centered maximum. Bijections are found between the balanced n-color compositions and these other objects through spotted tiling representations of n-color compositions.

Through the combinatorial proofs, we gain more insights on the objects under consideration. For instance, from the algorithm that transforms compositions into bargraphs, we note that all bargraphs with an odd length are in bijection with balanced compositions for which the first part is 1_1, because that is the only way in which the first delimiter in each boundary word is a C. Now, taking a 1_1 part away from a balanced composition leaves another balanced composition. Thus the bargraphs of odd length are also implicitly bijected with balanced compositions of a smaller weight, one less than the original. In other words, if we only count bargraphs that have an odd number of columns, we again obtain the same sequence, but shifted by one position.

Corollary 1. *Let $\ell \geq 2$. The number of unimodal bargraphs with a centered maximum and with an odd number of parts whose semiperimeter is ℓ equals the number of balanced n-color compositions of $\ell - 1$.*

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