FIBONACCI–CATALAN SERIES

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Abstract

We study certain series with Catalan numbers and reciprocal Catalan numbers, and provide presumably new closed form evaluations of these series with Fibonacci and Lucas entries. In addition, we state some combinatorial sums that can be inferred from the series.

1. Introduction and Motivation

The famous Catalan numbers C_n, $n \geq 0$, are defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$. The numbers are indexed as sequence A000108 in the On-Line Encyclopedia of Integer Sequences [25]. They have the generating function [27]

$$G(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}$$

and possess, among other fascinating properties, the integral representations [15, 22]

$$C_n = \frac{1}{2\pi}\int_0^4 z^n \sqrt{\frac{4-z}{z}}dz \qquad \text{and} \qquad C_n = \frac{1}{\pi}\int_0^2 z^{2n}\sqrt{4-z^2}dz.$$

[1]Statements and conclusions made in this paper by Robert Frontczak are entirely those of the author. They do not necessarily reflect the views of LBBW.

The reciprocals of Catalan numbers are generated by the function

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{C_n},\quad z\in[0,4),\tag{1}$$

which can be expressed as

$$f(z)=\frac{2(8+z)}{(4-z)^2}+\frac{24\sqrt{z}\arcsin\left(\frac{\sqrt{z}}{2}\right)}{\sqrt{(4-z)^5}}\,.\tag{2}$$

The function $f(z)$ was studied in detail recently by Amdeberhan et al. [3] and Koshy and Gao [19]. It also appears in the article by Yin and Qi [33] and is linked to an interesting problem proposed by Beckwith and Harbor in the American Mathematical Monthly [6].

Reciprocals of Catalan numbers possess the following integral representation derived by Dana-Picard [15]

$$\frac{1}{C_n}=\frac{(2n+3)(2n+2)(2n+1)}{2^{4n+4}}\int_0^2 z^{2n+1}\sqrt{4-z^2}\,dz.$$

Corresponding to $G(z)$ are the sub-series $G_1(z)$, $G_2(z)$, $G_3(z)$, and $G_4(z)$ (for $|z|\le 1$), namely,

$$\begin{aligned}
G_1(z)&=\sum_{n=1}^{\infty}\frac{C_{2n-1}}{4^{2n-1}}z^{2n-1}=\frac{2}{z}-\frac{\sqrt{1+z}+\sqrt{1-z}}{z}\,,\quad z\neq 0,\\
G_2(z)&=\sum_{n=0}^{\infty}\frac{C_{2n}}{4^{2n}}z^{2n}=\frac{\sqrt{1+z}-\sqrt{1-z}}{z}\,,\quad z\neq 0,\\
G_3(z)&=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{C_{2n-1}}{4^{2n-1}}z^n=\sqrt[4]{1+z}\cos\left(\frac{1}{2}\arctan\sqrt{z}\right)-1\,,\\
G_4(z)&=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{C_{2n}}{4^{2n}}z^n=\frac{\sqrt[4]{1+z}}{\sqrt{z}}\sin\Big(\frac{1}{2}\arctan\sqrt{z}\Big),\quad z\neq 0.
\end{aligned}$$

Since

$$\cos\left(\frac{1}{2}\arctan\sqrt{p}\right)=\sqrt{\frac{\sqrt{1+p}+1}{2\sqrt{1+p}}}\quad\text{and}\quad\sin\left(\frac{1}{2}\arctan\sqrt{p}\right)=\sqrt{\frac{\sqrt{1+p}-1}{2\sqrt{1+p}}}\,,$$

we have more compact formulas for functions $G_3(z)$ and $G_4(z)$ as follows:

$$G_3(z)=\sqrt{\frac{\sqrt{1+z}+1}{2}}-1\,,\qquad G_4(z)=\sqrt{\frac{\sqrt{1+z}-1}{2z}}\,.$$

Our purpose in this paper is to study $G(z)$, $G_1(z)$, $G_2(z)$, $G_3(z)$, $G_4(z)$, $f(z)$, and the following similar series:

$$X(z) = \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)C_n}\,, \quad |z| < 4\,,$$
$$Y(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2(n+1)C_n}\,, \quad |z| < 4\,,$$
$$W(z) = \sum_{n=0}^{\infty} \frac{C_n}{2^{2n+1}} \frac{z^{2n+2}}{2n+1}\,, \quad |z| < 1\,,$$

focusing mainly on delivering new Fibonacci–Catalan relations. Similar series were studied by the authors [1], Qi and Guo [24], and Stewart [28]. Other recently published works on infinite sums with (reciprocal) Catalan numbers and central binomial coefficients $\binom{2n}{n}$ include the articles [7, 8, 9, 10, 13, 17, 26, 29].

We recall that Fibonacci numbers F_n and the companion sequence of Lucas numbers L_n are defined for $n \geq 0$ by the same recurrence $w_{n+2} = w_{n+1} + w_n$, but with initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$. These are sequences A000045 and A000032 in [25], respectively.

The Binet formulas, for any integer n, are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = -\frac{1}{\alpha} = \frac{1-\sqrt{5}}{2}$. See [18] for more details.

The next lemma will be used frequently.

Lemma 1. *We have*

$$\sin\Big(\frac{\pi}{10}\Big) = -\frac{\beta}{2}, \qquad \sin\Big(\frac{3\pi}{10}\Big) = \frac{\alpha}{2} = \alpha^2 \sin\Big(\frac{\pi}{10}\Big)\,,$$
$$\cos\Big(\frac{\pi}{10}\Big) = \frac{\sqrt{\alpha\sqrt{5}}}{2}, \qquad \cos\Big(\frac{3\pi}{10}\Big) = \frac{\sqrt{-\beta\sqrt{5}}}{2} = -\beta \cos\Big(\frac{\pi}{10}\Big)\,,$$
$$\cot\Big(\frac{2\pi}{5}\Big) = -\beta^3 \cot\Big(\frac{\pi}{5}\Big) = \frac{\beta^2}{\sqrt{5}}\sqrt{\alpha\sqrt{5}}\,. \tag{3}$$

2. Results from Functions $G(z)$, $G_1(z)$, $G_2(z)$, $G_3(z)$ and $G_4(z)$

Observe that $G(\frac{1}{5})$ and $G(-\frac{1}{5})$ give

$$\sum_{n=0}^{\infty} \frac{C_n}{5^n} = \frac{5-\sqrt{5}}{2}\,, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n C_n}{5^n} = \frac{3\sqrt{5}-5}{2}\,,$$

from which we also infer

$$\sum_{n=0}^{\infty}\frac{C_{2n}}{25^n}=\frac{\sqrt{5}}{2}\,,\qquad \sum_{n=1}^{\infty}\frac{C_{2n-1}}{25^n}=\frac{5-2\sqrt{5}}{10}\,.$$

The trigonometric version of the generating function $G(z)$ is

$$G_t(z)=\sum_{n=0}^{\infty}C_n\frac{\sin^{2n}z}{4^n}=\frac{1}{\cos^2\left(\frac{z}{2}\right)},\quad |z|\le\frac{\pi}{2}.$$

At $z=\frac{\pi}{2}$, $z=\frac{\pi}{3}$, $z=\frac{\pi}{4}$ and $z=\frac{\pi}{6}$, we have the following series:

$$\sum_{n=0}^{\infty}\frac{C_n}{4^n}=2\,,\qquad \sum_{n=0}^{\infty}\left(\frac{3}{16}\right)^n C_n=\frac{4}{3}\,,$$
$$\sum_{n=0}^{\infty}\frac{C_n}{8^n}=4-2\sqrt{2}\,,\qquad \sum_{n=0}^{\infty}\frac{C_n}{16^n}=8-4\sqrt{3}\,.$$

The identities $G_1(\frac{1}{\sqrt{5}})$ and $G_2(\frac{1}{\sqrt{5}})$ give

$$\sum_{n=1}^{\infty}\frac{C_{2n-1}}{80^n}=\frac{1}{2}-\frac{\sqrt{5}}{20}\sqrt{10+4\sqrt{5}}\,,\qquad \sum_{n=0}^{\infty}\frac{C_{2n}}{80^n}=\sqrt{10-4\sqrt{5}}\,.$$

The trigonometric versions of $G_1(z)$ and $G_2(z)$, for $|z|\le\frac{\pi}{2}$, are

$$G_{1t}(z)=\sum_{n=1}^{\infty}C_{2n-1}\frac{\sin^{2n-1}z}{4^{2n-1}}=\frac{4\sin^2(\frac{z}{4})}{\sin z}\,,$$
$$G_{2t}(z)=\sum_{n=0}^{\infty}C_{2n}\frac{\sin^{2n}z}{4^{2n}}=\frac{1}{\cos(\frac{z}{2})}\,.$$

Example 1. Evaluating $G_{1t}(z)$ and $G_{2t}(z)$ at appropriate arguments yield

$$\sum_{n=1}^{\infty}\frac{C_{2n-1}}{16^n}=\frac{2-\sqrt{2}}{4}\,,$$
$$\sum_{n=1}^{\infty}\left(\frac{3}{64}\right)^n C_{2n-1}=\frac{2-\sqrt{3}}{4}\,,$$
$$\sum_{n=0}^{\infty}\frac{C_{2n}}{16^n}=\sqrt{2}\,,$$
$$\sum_{n=0}^{\infty}\left(\frac{3}{64}\right)^n C_{2n}=\frac{2\sqrt{3}}{3}\,,$$
$$\sum_{n=0}^{\infty}\frac{C_{2n}}{64^n}=\sqrt{6}-\sqrt{2}\,.$$

Lemma 2. *We have*

$$\sin^2\Big(\frac{3\pi}{20}\Big) = \frac{1}{4}\Big(2 - \sqrt{-\beta\sqrt{5}}\Big)\,, \tag{4}$$

$$\sin^2\Big(\frac{\pi}{20}\Big) = \frac{1}{4}\Big(2 - \sqrt{\alpha\sqrt{5}}\Big)\,, \tag{5}$$

$$\sin^2\Big(\frac{3\pi}{20}\Big) - \sin^2\Big(\frac{\pi}{20}\Big) = \frac{-\beta\sqrt{-\beta\sqrt{5}}}{4}\,, \tag{6}$$

$$\sin^2\Big(\frac{3\pi}{20}\Big) = \Big(1 + \sqrt{\alpha\sqrt{5}}\Big)^2 \sin^2\Big(\frac{\pi}{20}\Big)\,, \tag{7}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = -\frac{1}{\alpha}$.

Proof. Identities (4) and (5) are straightforward consequences of $\sin^2\big(\frac{x}{2}\big) = \frac{1-\cos x}{2}$. Identity (6) comes from $\sin^2 3x - \sin^2 x = \sin 2x \sin 4x$. □

Theorem 1. *For any integer s,*

$$\sum_{n=0}^{\infty} \frac{F_{2n+s}C_n}{16^n} = 4\Big(2 - \sqrt{\alpha\sqrt{5}}\Big)F_{s-2} + \frac{4\sqrt{5}\,\alpha^{s-4}}{5}\sqrt{\alpha\sqrt{5}}, \tag{8}$$

$$\sum_{n=0}^{\infty} \frac{L_{2n+s}C_n}{16^n} = 4\Big(2 - \sqrt{\alpha\sqrt{5}}\Big)L_{s-2} + 4\,\alpha^{s-4}\sqrt{\alpha\sqrt{5}}\,. \tag{9}$$

Proof. Determine $\alpha^s G_t(\frac{3\pi}{10}) \mp \beta^s G_t(\frac{\pi}{10})$, where s is an arbitrary integer, using the Binet formulas and Lemma 2. □

Example 2. We have

$$\sum_{n=0}^{\infty} \frac{F_{2n}C_n}{16^n} = -8 - \frac{10 - 14\sqrt{5}}{5}\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=0}^{\infty} \frac{L_{2n}C_n}{16^n} = 24 + (2 - 6\sqrt{5})\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}C_n}{16^n} = 8 - \frac{8\sqrt{5}}{5}\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=0}^{\infty} \frac{L_{2n+1}C_n}{16^n} = -8 - (4 - 4\sqrt{5})\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=0}^{\infty} \frac{F_{2n+2}C_n}{16^n} = \frac{6\sqrt{5} - 10}{5}\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=0}^{\infty} \frac{L_{2n+2}C_n}{16^n} = 16 - 2(1 + \sqrt{5})\sqrt{\alpha\sqrt{5}}\,.$$

Now we consider functions $G_3(z)$ and $G_4(z)$.
At $z=1$, $z=\frac{1}{3}$, and $z=3$, $G_3(z)$ gives

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}C_{2n-1}}{16^n}=\frac{\sqrt{2+2\sqrt{2}}}{4}-\frac{1}{2}\,,$$
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}C_{2n-1}}{48^n}=\frac{\sqrt{18+12\sqrt{3}}}{12}-\frac{1}{2}\,,$$
$$\sum_{n=1}^{\infty}\Big(-\frac{3}{16}\Big)^nC_{2n-1}=\frac{1}{2}-\frac{\sqrt{6}}{4}\,.$$

Similarly, $G_4(1)$, $G_4(\frac{1}{3})$, and $G_4(3)$, respectively, give

$$\sum_{n=0}^{\infty}\frac{(-1)^{n}C_{2n}}{16^n}=\sqrt{2\sqrt{2}-2}\,,$$
$$\sum_{n=0}^{\infty}\frac{(-1)^{n}C_{2n}}{48^n}=\sqrt[4]{27}-\sqrt[4]{3}\,,$$
$$\sum_{n=0}^{\infty}\left(-\frac{3}{16}\right)^nC_{2n}=\frac{\sqrt{6}}{3}\,.$$

Lemma 3. *We have*

$$\sqrt{\alpha}=\alpha\sqrt{-\beta},\quad \sqrt{\alpha\sqrt{5}}=\alpha\sqrt{-\beta\sqrt{5}}\,.$$

Lemma 4. *For any integer r,*

$$\alpha^r+\beta^{r-1}=\alpha F_{r-2}+F_{r+1},\qquad \alpha^r-\beta^{r-1}=\alpha F_{r+1}-F_{r-2}\,.$$

Theorem 2. *For any integer s,*

$$\sum_{n=0}^{\infty}\frac{F_{2n+s}C_{2n}}{64^n}=\frac{\sqrt{10}}{5}\Big(\big(\beta F_{s-2}+F_{s+1}\big)\sqrt{\alpha\sqrt{5}}-L_{s-2}\Big),\tag{10}$$
$$\sum_{n=0}^{\infty}\frac{L_{2n+s}C_{2n}}{64^n}=\sqrt{2}\,\Big(\big(F_{s-2}-\beta F_{s+1}\big)\sqrt{\alpha\sqrt{5}}-\sqrt{5}F_{s-2}\Big),\tag{11}$$
$$\sum_{n=1}^{\infty}\frac{F_{2n+s}C_{2n-1}}{64^n}=\frac{\sqrt{10}}{40}\Big(2\sqrt{10}F_s-L_{s-1}-(F_{s+2}+\beta F_{s-1})\sqrt{\alpha\sqrt{5}}\Big),\tag{12}$$
$$\sum_{n=1}^{\infty}\frac{L_{2n+s}C_{2n-1}}{64^n}=\frac{1}{8}\Big(4L_s-\sqrt{10}F_{s-1}-\sqrt{2}(F_{s-1}-\beta F_{s+2})\sqrt{\alpha\sqrt{5}}\Big).\tag{13}$$

Proof. With s an arbitrary integer and noting Lemma 3, $\alpha^sG_2(\frac{\alpha}{2})\mp\beta^sG_2(\frac{\beta}{2})$ means

$$\sum_{n=0}^{\infty}C_{2n}\frac{\alpha^{2n+s}\mp\beta^{2n+s}}{4^{3n}}=\sqrt{2}(\alpha^s\mp\beta^{s-1})\sqrt{-\beta\sqrt{5}}-\sqrt{2}(\alpha^{s-2}\pm\beta^{s-2}),$$

and hence, using Lemma 4 and the Binet formulas, we obtain identities (10) and (11). The proof of (12) and (13) is similar. Use $\alpha^s G_1(\frac{\alpha}{2}) \mp \beta^s G_1(\frac{\beta}{2})$. □

Example 3. Theorem 2 yields

$$\sum_{n=0}^{\infty} \frac{F_{2n}C_{2n}}{64^n} = \frac{\sqrt{10}}{5}\Big(\alpha\sqrt{\alpha\sqrt{5}} - 3\Big),$$
$$\sum_{n=0}^{\infty} \frac{L_{2n}C_{2n}}{64^n} = \sqrt{2}\Big(\sqrt{5} - \beta^2\sqrt{\alpha\sqrt{5}}\Big),$$
$$\sum_{n=1}^{\infty} \frac{F_{2n}C_{2n-1}}{64^n} = \frac{\sqrt{10}}{40}\Big(1 - \beta^2\sqrt{\alpha\sqrt{5}}\Big),$$
$$\sum_{n=1}^{\infty} \frac{L_{2n}C_{2n-1}}{64^n} = 1 - \frac{\sqrt{2}}{8}\Big(\alpha\sqrt{\alpha\sqrt{5}} + \sqrt{5}\Big),$$
$$\sum_{n=0}^{\infty} \frac{F_{2n+1}C_{2n}}{64^n} = \frac{\sqrt{10}}{5}\Big(1 - \beta^2\sqrt{\alpha\sqrt{5}}\Big),$$
$$\sum_{n=0}^{\infty} \frac{L_{2n+1}C_{2n}}{64^n} = \sqrt{2}\Big(\alpha\sqrt{\alpha\sqrt{5}} - \sqrt{5}\Big).$$

3. Results from $f(z)$

It is convenient to write the function $f(z)$ from (1) as

$$f(z) = \frac{2(8+z)}{z^2}\cot^4\Big(\arccos\Big(\frac{\sqrt{z}}{2}\Big)\Big) + \frac{24}{z^2}\cot^5\Big(\arccos\Big(\frac{\sqrt{z}}{2}\Big)\Big)\arcsin\Big(\frac{\sqrt{z}}{2}\Big).$$

Lemma 5. *For any integer r,*

$$3\alpha^r - \beta^{r+3} = L_{r+1}\sqrt{5} - L_{r-1}, \qquad 3\alpha^r + \beta^{r+3} = \sqrt{5}(F_{r+1}\sqrt{5} - F_{r-1}).$$

Theorem 3. *For any integer s,*

$$\sum_{n=0}^{\infty} \frac{F_{2n+s}}{C_n} = \frac{2}{5}(F_{s+4} + 8F_{s+2}) + \big(\sqrt{5}F_{s+3} - F_{s+1}\big)\frac{12\sqrt{5}\pi\alpha}{125}\sqrt{\alpha\sqrt{5}}\,, \tag{14}$$
$$\sum_{n=0}^{\infty} \frac{L_{2n+s}}{C_n} = \frac{2}{5}(L_{s+4} + 8L_{s+2}) + \big(\sqrt{5}L_{s+3} - L_{s+1}\big)\frac{12\sqrt{5}\pi\alpha}{125}\sqrt{\alpha\sqrt{5}}\,. \tag{15}$$

Proof. Considering $\alpha^s f(\alpha^2)$ and $\beta^s f(\beta^2)$, where s is an arbitrary integer, we have

$$\sum_{n=0}^{\infty}\frac{\alpha^{2n+s}}{C_n}=2(\alpha^{s-2}+8\alpha^{s-4})\cot^4\Big(\frac{\pi}{5}\Big)+\frac{36\pi}{5}\alpha^{s-4}\cot^5\Big(\frac{\pi}{5}\Big),$$
$$\sum_{n=0}^{\infty}\frac{\beta^{2n+s}}{C_n}=2(\beta^{s-2}+8\beta^{s-4})\cot^4\Big(\frac{2\pi}{5}\Big)+\frac{12\pi}{5}\beta^{s-4}\cot^5\Big(\frac{2\pi}{5}\Big),$$

from which, using the Binet formulas and relevant identities from Lemma 1, we get

$$\sum_{n=0}^{\infty}\frac{F_{2n+s}}{C_n}$$
$$=\frac{2\sqrt{5}}{25}(\alpha^{s+4}-\beta^{s+4})+\frac{16\sqrt{5}}{25}(\alpha^{s+2}-\beta^{s+2})+(3\alpha^{s+2}+\beta^{s+5})\frac{12\pi\alpha}{125}\sqrt{\alpha\sqrt{5}}\,,$$
$$\sum_{n=0}^{\infty}\frac{L_{2n+s}}{C_n}$$
$$=\frac{2}{5}(\alpha^{s+4}+\beta^{s+4})+\frac{16}{5}(\alpha^{s+2}+\beta^{s+2})+(3\alpha^{s+2}-\beta^{s+5})\frac{12\sqrt{5}\pi\alpha}{125}\sqrt{\alpha\sqrt{5}}\,.$$

The stated identities in the theorem now follow when we use the Binet formulas and invoke Lemma 5 with $r=s+2$. □

Example 4. Formulas (14) and (15) yield

$$\sum_{n=1}^{\infty}\frac{F_{2n-1}}{C_n}=3+\frac{12\alpha\pi}{25}\sqrt{\alpha\sqrt{5}},$$
$$\sum_{n=1}^{\infty}\frac{L_{2n-1}}{C_n}=\frac{29}{5}+\frac{6(5+13\sqrt{5})\pi}{125}\sqrt{\alpha\sqrt{5}}\,,$$
$$\sum_{n=1}^{\infty}\frac{F_{2n}}{C_n}=\frac{22}{5}+\frac{6(5+9\sqrt{5})\pi}{125}\sqrt{\alpha\sqrt{5}}, \qquad (16)$$
$$\sum_{n=0}^{\infty}\frac{L_{2n}}{C_n}=\frac{62}{5}+\frac{6(15+19\sqrt{5})\pi}{125}\sqrt{\alpha\sqrt{5}}\,.$$

The identity (16) was also obtained by Stewart [28, Identity (38a)].

Since $L_{2n}=5F_n^2+2(-1)^n$, $L_{2n}=L_n^2-2(-1)^n$, and

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{C_n}=\frac{14}{25}-\frac{24\sqrt{5}}{125}\log\alpha,$$

we have also the following series:

$$\sum_{n=1}^{\infty} \frac{F_n^2}{C_n} = \frac{282}{125} + \frac{6(15+19\sqrt{5})\pi}{625}\sqrt{\sqrt{5}\alpha} + \frac{48\sqrt{5}}{625}\log\alpha\,,$$

$$\sum_{n=0}^{\infty} \frac{L_n^2}{C_n} = \frac{338}{25} + \frac{6(15+19\sqrt{5})\pi}{125}\sqrt{\sqrt{5}\alpha} - \frac{48\sqrt{5}}{125}\log\alpha\,.$$

4. Results from $W(z)$

The trigonometric version of the function

$$W(z) = \sum_{n=0}^{\infty} \frac{C_n}{2^{2n+1}} \frac{z^{2n+2}}{2n+1} = z\arcsin z + \sqrt{1-z^2} - 1$$

is

$$W_t(z) = \sum_{n=0}^{\infty} \frac{C_n}{2^{2n+1}} \frac{\sin^{2n+2} z}{2n+1} = z\sin z - 2\sin^2\left(\frac{z}{2}\right).$$

At $z = \frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$, respectively, $W_t(z)$ gives

$$\sum_{n=0}^{\infty} \frac{C_n}{4^n(2n+1)} = \pi - 2\,,$$

$$\sum_{n=0}^{\infty} \left(\frac{3}{16}\right)^n \frac{C_n}{2n+1} = \frac{4(\pi\sqrt{3}-3)}{9}\,,$$

$$\sum_{n=0}^{\infty} \frac{C_n}{8^n(2n+1)} = \frac{(\pi+4)\sqrt{2}}{2} - 4\,,$$

$$\sum_{n=0}^{\infty} \frac{C_n}{16^n(2n+1)} = \frac{2(\pi+6\sqrt{3}-12)}{3}\,.$$

Lemma 6. *For any integer r,*

$$3\alpha^r + \beta^r = 2L_r + F_r\sqrt{5}\,, \qquad 3\alpha^r - \beta^r = L_r + 2F_r\sqrt{5}\,.$$

Theorem 4. *For any integer s,*

$$\sum_{n=0}^{\infty} \frac{F_{2n+s}C_n}{16^n(2n+1)}$$
$$= \frac{2\sqrt{5}\pi}{25}\left(2L_{s-1} + \sqrt{5}F_{s-1}\right) - 8F_{s-2} + \frac{4\sqrt{5}}{5}\left(\beta^2 F_{s-1} - 2F_{s-2}\right)\sqrt{\alpha\sqrt{5}},$$

$$\sum_{n=0}^{\infty}\frac{L_{2n+s}C_n}{16^n(2n+1)}$$
$$= \frac{2\pi}{5}\left(L_{s-1}+2\sqrt{5}F_{s-1}\right)-8L_{s-2}-\frac{4\sqrt{5}}{5}\left(\beta^2L_{s-1}-2L_{s-2}\right)\sqrt{\alpha\sqrt{5}}.$$

Proof. Determining $\alpha^{s-2}W_t(\frac{3\pi}{10})\mp\beta^{s-2}W_t(\frac{\pi}{10})$, where s is an arbitrary integer, and employing identity (6), we have

$$\sum_{n=0}^{\infty}\frac{C_n}{16^n}\frac{\alpha^{2n+s}\mp\beta^{2n+s}}{2n+1} = \frac{2\pi}{5}(3\alpha^{s-1}\pm\beta^{s-1})$$
$$-16(\alpha^{s-2}\mp\beta^{s-2})\sin^2\left(\frac{\pi}{20}\right)-4\sqrt{-\beta^3\sqrt{5}}\alpha^{s-2},$$

from which the stated identities follow in view of the Binet formulas and Lemmas 2 and 6. □

Example 5. For $s=0$ and $s=1$, from Theorem 4 we obtain

$$\sum_{n=1}^{\infty}\frac{F_{2n}C_n}{16^n(2n+1)} = 8+\frac{2(5-2\sqrt{5})\pi}{25}+\frac{2(5-7\sqrt{5})}{5}\sqrt{\alpha\sqrt{5}}\,,$$
$$\sum_{n=0}^{\infty}\frac{L_{2n}C_n}{16^n(2n+1)} = -24-\frac{2(1-2\sqrt{5})\pi}{5}-2(1-3\sqrt{5})\sqrt{\alpha\sqrt{5}}\,,$$
$$\sum_{n=0}^{\infty}\frac{F_{2n+1}C_n}{16^n(2n+1)} = -8+\frac{8\sqrt{5}\pi}{25}+\frac{8\sqrt{5}}{5}\sqrt{\alpha\sqrt{5}}\,,$$
$$\sum_{n=0}^{\infty}\frac{L_{2n+1}C_n}{16^n(2n+1)} = 8+\frac{4\pi}{5}+4(1-\sqrt{5})\sqrt{\alpha\sqrt{5}}\,.$$

5. Results from $Y(z)$

The identity

$$Y(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2(n+1)C_n}=2\arcsin^2\left(\frac{\sqrt{z}}{2}\right)$$

immediately yields the following summation formulas:

$$\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)C_n}=\frac{\pi^2}{18}\,,\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2(n+1)C_n}=2\log^2\alpha\,, \tag{17}$$
$$\sum_{n=1}^{\infty}\frac{4^n}{n^2(n+1)C_n}=\frac{\pi^2}{2}\,,\qquad \sum_{n=1}^{\infty}\frac{3^n}{n^2(n+1)C_n}=\frac{2\pi^2}{9}\,. \tag{18}$$

Most likely all these summations are known. The first one is a classical result due to Euler. Both sums in (18) can be found in [28].

From (17) we also obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)C_{2n}} = \frac{\pi^2}{9} - 4\log^2\alpha\,,$$
$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)^2C_{2n-1}} = \frac{\pi^2}{18} + 2\log^2\alpha\,.$$

Theorem 5. *For any integer s,*

$$\sum_{n=1}^{\infty} \frac{F_{2n+s}}{n^2(n+1)C_n} = \frac{\pi^2}{50\sqrt{5}}(9\alpha^s - \beta^s)\,,$$
$$\sum_{n=1}^{\infty} \frac{L_{2n+s}}{n^2(n+1)C_n} = \frac{\pi^2}{50}(9\alpha^s + \beta^s)\,.$$

Proof. Determine $\alpha^s Y(\alpha^2) \pm \beta^s Y(\beta^2)$ and use the Binet formulas. □

Example 6. We have

$$\sum_{n=1}^{\infty} \frac{F_{2n-1}}{n^2(n+1)C_n} = \frac{(25-4\sqrt{5})\pi^2}{250}\,,$$
$$\sum_{n=1}^{\infty} \frac{L_{2n-1}}{n^2(n+1)C_n} = \frac{(4\sqrt{5}-5)\pi^2}{50}\,,$$
$$\sum_{n=1}^{\infty} \frac{F_{2n}}{n^2(n+1)C_n} = \frac{4\sqrt{5}\pi^2}{125}\,, \tag{19}$$
$$\sum_{n=1}^{\infty} \frac{L_{2n}}{n^2(n+1)C_n} = \frac{\pi^2}{5}\,,$$
$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{n^2(n+1)C_n} = \frac{(25+4\sqrt{5})\pi^2}{250}\,,$$
$$\sum_{n=1}^{\infty} \frac{L_{2n+1}}{n^2(n+1)C_n} = \frac{(5+4\sqrt{5})\pi^2}{50}\,.$$

Identity (19) was also obtained by Stewart [28, Identity (37c)].

6. Results from $X(z)$

The function

$$X(z) = \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)C_n} = \frac{2\sqrt{z}\arcsin(\sqrt{z}/2)}{\sqrt{4-z}},$$

can also be written as

$$X(z) = 2\cot\left(\arccos\left(\frac{\sqrt{z}}{2}\right)\right)\arcsin\left(\frac{\sqrt{z}}{2}\right).$$

At $z=1$, $z=-1$, $z=2$, and $z=3$, we have the following series:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)C_n} = \frac{\sqrt{3}\pi}{9}\,, \tag{20}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)C_n} = \frac{2\sqrt{5}}{5}\log\alpha\,, \tag{21}$$

$$\sum_{n=1}^{\infty}\frac{2^n}{n(n+1)C_n} = \frac{\pi}{2}\,, \tag{22}$$

$$\sum_{n=1}^{\infty}\frac{3^n}{n(n+1)C_n} = \frac{2\sqrt{3}\pi}{3}\,. \tag{23}$$

The series (20) and (21) can be found in [14, p. 89] and [20], respectively, while the series (22) and (23) were also obtained in [28].

Also, from (20) and (21) we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n(2n+1)C_{2n}} = \frac{\sqrt{3}}{9} - \frac{2\sqrt{5}}{5}\log\alpha\,,$$

$$\sum_{n=1}^{\infty}\frac{1}{n(2n-1)C_{2n-1}} = \frac{\sqrt{3}}{9} + \frac{2\sqrt{5}}{5}\log\alpha\,.$$

Theorem 6. *For any integer s,*

$$\sum_{n=1}^{\infty}\frac{F_{2n+s}}{n(n+1)C_n} = \frac{\sqrt{5}\pi\alpha}{25}(\sqrt{5}F_{s+1} - F_{s-1})\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=1}^{\infty}\frac{L_{2n+s}}{n(n+1)C_n} = \frac{\sqrt{5}\pi\alpha}{25}(\sqrt{5}L_{s+1} - L_{s-1})\sqrt{\alpha\sqrt{5}}\,.$$

Proof. Evaluation of $\alpha^s X(\alpha^2) \pm \beta^s X(\beta^2)$ gives

$$\sum_{n=1}^{\infty}\frac{\alpha^{2n+s} \pm \beta^{2n+s}}{n(n+1)C_n} = \frac{3\pi}{5}\alpha^s\cot\left(\frac{\pi}{5}\right) \pm \frac{\pi}{5}\beta^s\cot\left(\frac{2\pi}{5}\right).$$

Thus, using the Binet formulas and identity (3) of Lemma 1, we have

$$\sum_{n=1}^{\infty}\frac{F_{2n+s}}{n(n+1)C_n} = \frac{\pi}{5\sqrt{5}}(3\alpha^s + \beta^{s+3})\cot\left(\frac{\pi}{5}\right),$$

$$\sum_{n=1}^{\infty}\frac{L_{2n+s}}{n(n+1)C_n} = \frac{\pi}{5}(3\alpha^s - \beta^{s+3})\cot\left(\frac{\pi}{5}\right),$$

and hence the stated identities, upon use of Lemma 5 with $r=s$. □

Example 7. We have

$$\sum_{n=1}^{\infty} \frac{F_{2n-1}}{n(n+1)C_n} = \frac{\sqrt{5}\pi\alpha}{25}\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=1}^{\infty} \frac{L_{2n-1}}{n(n+1)C_n} = \frac{(7\sqrt{5}-5)\pi}{50}\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{n(n+1)C_n} = \frac{2\sqrt{5}\pi}{25}\sqrt{\alpha\sqrt{5}}\,, \tag{24}$$

$$\sum_{n=1}^{\infty} \frac{L_{2n}}{n(n+1)C_n} = \frac{2\sqrt{5}\pi\alpha^2}{25}\sqrt{\alpha\sqrt{5}}\,, \tag{25}$$

$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{n(n+1)C_n} = \frac{\pi\alpha}{5}\sqrt{\alpha\sqrt{5}}\,,$$

$$\sum_{n=1}^{\infty} \frac{L_{2n+1}}{n(n+1)C_n} = \frac{(15-2\sqrt{5})\pi\alpha}{25}\sqrt{\alpha\sqrt{5}}\,.$$

Both series (24) and (25) appeared recently as a problem proposal [16]. Identity (24) was also found by Stewart [28, Identity (37b)].

7. Some Combinatorial Identities

Before closing we state some combinatorial identities (finite and infinite) which can be inferred from the series studied in the previous sections. Concerning the finite class we note that similar results were studied by Wituła and Słota and their collaborators [30, 31], and more recently by Alzer and Nagy [2], Batir et al. [4], Batir and Sofo [5], Bhandari [7], Chen [11], Chu [12], and Qi et al. [23].

Our first example is an identity derived by Wituła and Słota [31] using a completely different method.

Theorem 7. *For each $n \geq 1$,*

$$\sum_{k=1}^{n} \frac{2^{2k}}{\binom{2k}{k}} = \frac{1}{3}\Big(\frac{2^{2n+1}}{C_n} - 2\Big). \tag{26}$$

Proof. We work with the function $f(z)$. From (2) we get

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{C_n} = \frac{2(8+z^2)}{(4-z^2)^2} + \frac{24z\arcsin(\frac{z}{2})}{(4-z^2)^{5/2}}. \tag{27}$$

Now, for all $|z| < 1$,

$$\begin{aligned}\frac{2z}{(1-z^2)^{5/2}}\arcsin(z) &= \frac{1}{(1-z^2)^2}\frac{2z}{\sqrt{1-z^2}}\arcsin(z)\\ &= \sum_{n=0}^{\infty}(n+1)z^2\sum_{n=1}^{\infty}\frac{(2z)^{2n}}{n\binom{2n}{n}}\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}(n+1-k)\frac{2^{2k}}{k\binom{2k}{k}}z^{2n}\end{aligned}$$

and by replacing z by $\frac{z}{2}$

$$\frac{16z}{(4-z^2)^{5/2}}\arcsin\left(\frac{z}{2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{n}(n+1-k)\frac{2^{2k-2n}}{k\binom{2k}{k}}\,z^{2n}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{z^{2n}}{C_n} = \frac{2(8+z^2)}{(4-z^2)^2} + \frac{3}{4}\sum_{n=1}^{\infty}\sum_{k=1}^{n}(n+1-k)\frac{2^{-2(n-k)}}{k\binom{2k}{k}}z^{2n}.$$

Next, from the partial fraction decomposition

$$\frac{16+2z^2}{(4-z^2)^2} = \frac{1}{4(2+z)} + \frac{1}{4(2-z)} + \frac{3}{2(2+z)^2} + \frac{3}{2(2-z)^2}$$

it follows that

$$\frac{16+2z^2}{(4-z^2)^2} = \sum_{n=0}^{\infty}\frac{2+3n}{2^{2n+1}}z^{2n}, \quad |z| < 2.$$

Comparing the coefficients of z^{2n} and rearranging yields for all $n \geq 1$

$$\sum_{k=1}^{n}(n+1-k)\frac{2^{2k}}{k\binom{2k}{k}} = \frac{1}{3}\left(\frac{2^{2n+2}}{C_n} - 4\right) - 2n.$$

The identity (26) follows from

$$\sum_{k=1}^{n}\frac{2^{2k}}{2k\binom{2k}{k}} = \frac{2^{2n}}{\binom{2n}{n}} - 1,$$

which is known as Parker's formula [32]. □

Theorem 8. *For each $n \geq 0$,*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}\binom{2(n-k)}{n-k}}{(2k+1)\big(2(n-k)+1\big)} = \frac{16^n}{(n+1)(2n+1)\binom{2n}{n}}. \tag{28}$$

Proof. This identity can be derived straightforwardly by working with the function $Y(z)$ in conjunction with the power series expansion

$$A(z) = \arcsin(z) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n+1}}{4^n(2n+1)}, \qquad |z| < 1.$$

□

It is interesting to compare identity (28) with the following identity derived by Wituła et al. [30]:

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}\binom{2(n-k)}{n-k}}{(2k+1)} = \frac{16^n}{(2n+1)\binom{2n}{n}}. \tag{29}$$

The identity (29) was rediscovered by Qi et al. [23] and also by Batir et al. [4] applying the Wilf-Zeilberger method.

The results stated in Theorems 9–12 follow from identities $G_t(z)$, $G_{1t}(z)$, $G_{2t}(z)$, and $Y(z)$, in view of the identity

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \frac{1}{2n+1} \frac{4^n}{\binom{2n}{n}}, \qquad n \geq 0. \tag{30}$$

Theorem 9. *If r is a positive integer, then*

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+2r+1)} \frac{\binom{2n}{n}}{\binom{2n+2r}{n+r}} = \frac{1}{2(2r-1)\binom{2r-2}{r-1}} - \frac{1}{2^{2r} r}.$$

Theorem 10. *If r is a non-negative integer, then*

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+2r+1)4^n} \frac{\binom{4n}{2n}}{\binom{2n+2r}{n+r}} = \frac{1}{2^{2r-1}} \int_0^{\frac{\pi}{2}} \sin^{2r} x \sin\left(\frac{x}{2}\right) dx - \frac{1}{(2r+1)\binom{2r}{r}}.$$

In particular,

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2 4^n} \frac{\binom{4n}{2n}}{\binom{2n}{n}} = 3 - 2\sqrt{2}.$$

Theorem 11. *If r is a positive integer, then*

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+2r-1)4^n} \frac{\binom{4n-2}{2n-1}}{\binom{2n+2r-2}{n+r-1}} = \frac{1}{(2r-1)\binom{2r-2}{r-1}} - \frac{1}{2^{2r-2}} \int_0^{\frac{\pi}{2}} \sin^{2r-1} x \cos\left(\frac{x}{2}\right) dx.$$

In particular,

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)4^n} \frac{\binom{4n-2}{2n-1}}{\binom{2n}{n}} = \frac{\sqrt{2}-1}{3}.$$

Theorem 12. *If r is an integer and $r \geq -1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n+2r+1)} \frac{4^n}{\binom{2n}{n}} \frac{4^n}{\binom{2n+2r}{n+r}} = \frac{1}{2^{2r-1}} \int_0^{\frac{\pi}{2}} x^2 \sin^{2r+1} x \, dx.$$

In particular,

$$\sum_{n=1}^{\infty} \frac{16^n}{n^2(2n+1)\binom{2n}{n}^2} = 2\pi - 4,$$

$$\sum_{n=1}^{\infty} \frac{16^n}{n^2(2n-1)\binom{2n}{n}\binom{2n-2}{n-1}} = -\frac{7}{2}\zeta(3) + 2\pi G,$$

where $\zeta(s) = \sum\limits_{k=1}^{\infty} \frac{1}{k^s}$ is the Riemann zeta function and $G = \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ is Catalan's constant.

The trigonometric form of $A(z)$ is

$$A_t(z) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\sin^{2n+1} z}{4^n(2n+1)} = z. \tag{31}$$

Lemma 7 (Lewin [21, Identity A.3.3.13])**.** *For real or complex y,*

$$\int_0^y \frac{x}{\sin x} dx = \mathrm{Cl}_2(y) + \mathrm{Cl}_2(\pi - y) + y \log\left(\tan\left(\frac{y}{2}\right)\right),$$

where $\mathrm{Cl}_2(z)$ is the Clausen function defined by

$$\mathrm{Cl}_2(z) = -\int_0^z \log|2\sin(x/2)| \, dx.$$

The following values are known, among others, see Lewin [21, p. 291, Sect. A.2.4]:

$$\mathrm{Cl}_2\left(\frac{\pi}{2}\right) = G = -\mathrm{Cl}_2\left(\frac{3\pi}{2}\right).$$

Our next result is a straightforward consequence of $A_t(z)$, given in (31), upon application of Lemma 7 and the well-known result:

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{\pi}{2^{2n+1}} \binom{2n}{n}, \quad n \geq 0. \tag{32}$$

Theorem 13. *If r is an integer and $r \geq 0$, then*

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}\binom{2n+2r}{n+r}}{2^{4n+2r+1}(2n+1)} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} x \sin^{2r-1} x \, dx.$$

In particular,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n(2n+1)} = \frac{4G}{\pi}, \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}\binom{2n+2}{n+1}}{16^n(2n+1)} = \frac{8}{\pi}.$$

From the function $W_t(z)$ and identities (30) and (32) come the next result.

Theorem 14. *If r is an integer and $r \geq -1$, then*

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}\binom{2n+2r+2}{n+r+1}}{(n+1)(2n+1)2^{4n+2r+4}} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} z \sin^{2r+1} z dz - \frac{1}{2^{2r+1}}\binom{2r}{r} + \frac{1}{\pi(2r+1)}.$$

In particular,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n(n+1)(2n+1)} = \frac{8G-4}{\pi},$$
$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}\binom{2n+2}{n+1}}{16^n(n+1)(2n+1)} = \frac{32}{\pi} - 8.$$

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