Interface Fluctuations in Competitive Erosion

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Competitive Erosion Definition

1. Competitive Erosion is a Markov chain.
2. The state space is the set of two-colorings (red and blue) of sites of a cylindrical lattice $\mathbb{Z}/N\mathbb{Z} \times \{-N/2, -N/2+1, \ldots, N/2\}$. We will define a Markov chain on this state space.

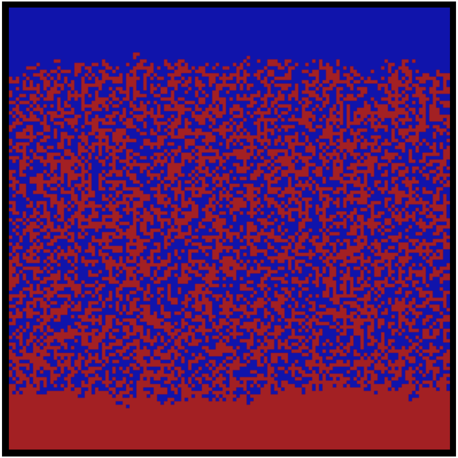
Competitive Erosion Markov Chain

1. Start with a coloring σ. One step is as follows:
    1. Choose a random point on the bottom base, then run a simple random walk until it hits a red square. This square turns blue.
    2. Choose a random point on the top base, then run a simple random walk until it hits a blue square. This square turns red.

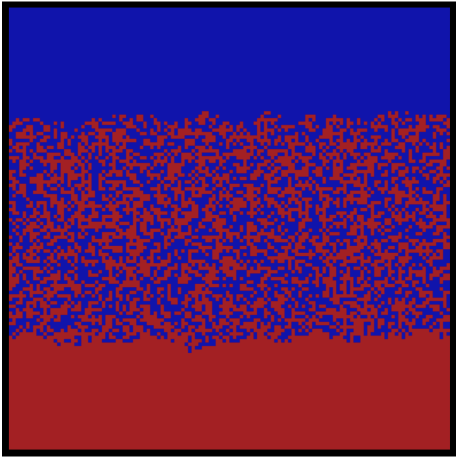
Competitive Erosion

1. We are interested in studying the stationary measure π_N of the erosion Markov chain.
2. A generic, near equilibrium configuration looks something like this:

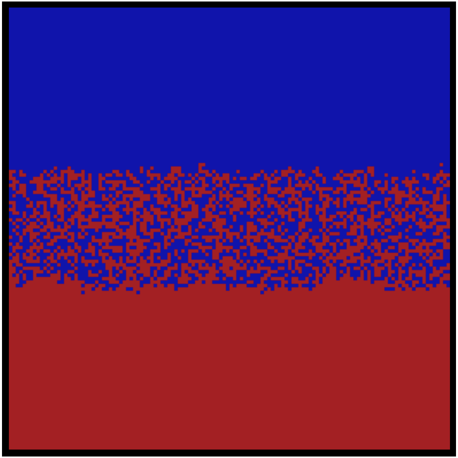
Competitive Erosion



Competitive Erosion



Competitive Erosion



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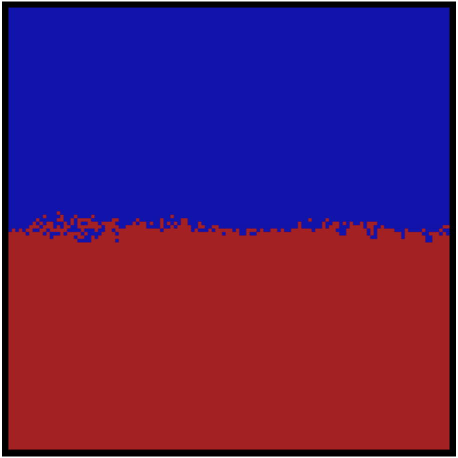

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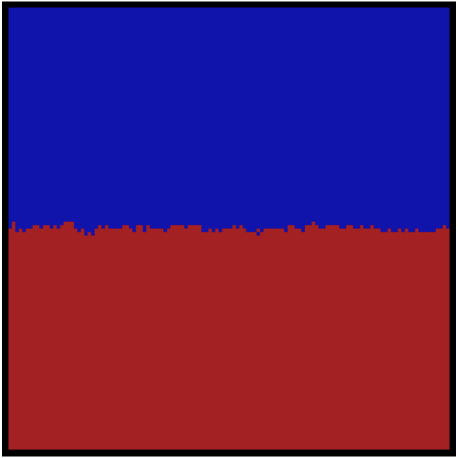

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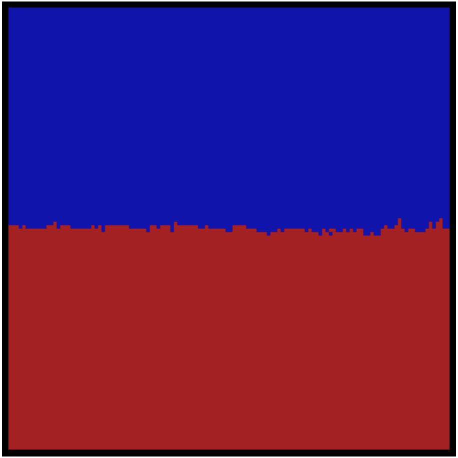
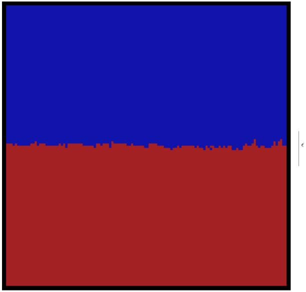


“Limit shape” Phenomenon

The probability, under the stationary distribution π_N, of seeing a blue square below height $-\epsilon N$ converges to 0 (exponentially fast) as $N \to \infty$ ([5]).



Natural next step is to study fluctuations.

Discussion of Fluctuations

1. The remainder is dedicated to a heuristic derivation of a stochastic PDE satisfied by the interface, which we present now.
2. Let $\Delta = \partial_x^2 + \partial_y^2$.
3. Take N large, and let γ be a smooth interface approximating the lattice interface I. Define the *blue Green's function* G_B on the blue region $\mathcal{B} \subset \mathbb{R}/\mathbb{Z} \times [-\frac{1}{2}, \frac{1}{2}]$ by

$$\begin{aligned} \Delta G_B &= 0 \text{ subject to} \\ G_B|_\gamma &\equiv 0 \\ \partial_y G_B|_{y=-\frac{1}{2}} &\equiv 1 \quad . \end{aligned}$$

Define the *red Green's function* G_R similarly.

Discussion of Fluctuations

Conjecture

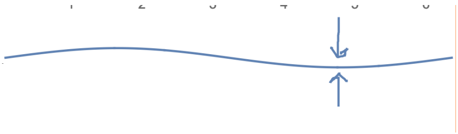
We argue that at a point (x, y) on the interface, over the course of a time step ΔT (which must be chosen at the right scale relative to N), the change $\delta\gamma$ of the interface in the normal direction ν is equal to

$$(\partial_\nu G_B(x,y) - \partial_{-\nu} G_R(x,y))\Delta T + F(\gamma)\mathcal{W}_t$$

where $\mathcal{W}$ is a Gaussian white noise.

Recovering the Limit Shape

1. The limiting interface should be the "equilibrium" point, so at the interface the drift term in the SPDE above should be 0. So $\partial_\nu G_B(x, y) - \partial_{-\nu} G_R(x, y) = 0$ for $(x, y) \in I$.



2. So the function $G = G_B - G_R$ is harmonic on the whole cylinder, satisfying the boundary conditions $\partial_\nu G|_{\text{bottom / top base}} = \pm 1$, $G|_I = 0$, and away from I, $\Delta G = 0$. In fact, G must be harmonic and smooth on the interior of the cylinder.
3. We have found that *the interface is a level set of G*. See the main theorem in [4].

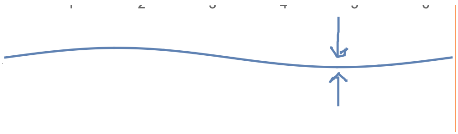
Fluctuations

1. Now we assume the interface I is near equilibrium, and we expand the drift term

$$\partial_\nu G_B - \partial_{-\nu} G_R$$

around the equilibrium point.

2. Suppose we have an interface I given by some height function h at time t. Then we know by the above that $h = h_0 + \delta h$, where h_0 is some limiting height function and δh is small.



Fluctuations

1. A calculation shows that to first order in δh, the net drift is $-2R[\delta h]$, where R is a Dirichlet to Neumann operator:

$$R : \delta h \mapsto \partial_\nu G_B$$

2. R is the operator which takes in a potential at the boundary $y = 0$ and gives the electric field at the boundary.

A Stochastic PDE

Revisiting our equation for the interface increment we get near equilibrium

$$\Delta h = -2R[\delta h]\Delta T + \mathcal{W}$$

Conjecture

The interface fluctuations converge to the solution of

$$dh = -2R[\delta h]dt + \mathcal{W}$$

where $\mathcal{W}(x, t)$ is a spacetime white noise.

Ornstein Uhlenbeck Process

1. We have
$$dh = -2R[\delta h]dt + \mathcal{W}$$
which is an infinite dimensional analog of the $1d$ mean reverting process $dx_t = -\alpha x_t dt + dB_t$.
2. This is a well known SPDE. The random height function h that is stationary under this is the Gaussian process with covariance kernel R^{-1}.

Ornstein Uhlenbeck Process

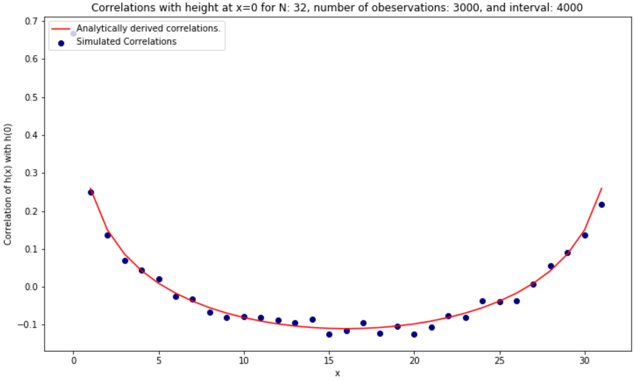
1. So roughly speaking, the stationary h should be a Gaussian process on $\mathbb{R}/\mathbb{Z}$ with $E[h(x)h(x')] = R^{-1}(x - x')$.
2. On the cylinder R^{-1} is easy to compute via Fourier transform (Fourier modes $e^{2\pi ikx}$ are eigenfunctions).
3. We ultimately get (up to some constant)

$$\log(2 - 2\cos(x - x'))$$

for the kernel R^{-1}.

Height Correlation

1. Numerics:



Height Correlation

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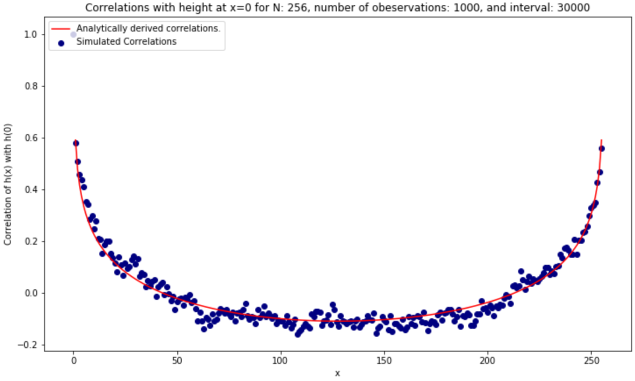
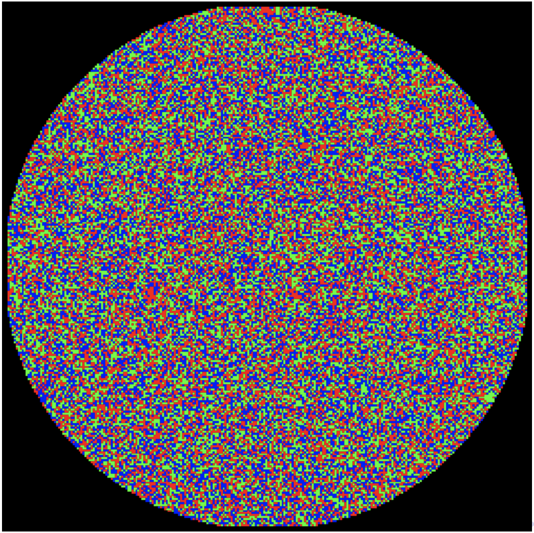


Further Questions...

1. More pictures:



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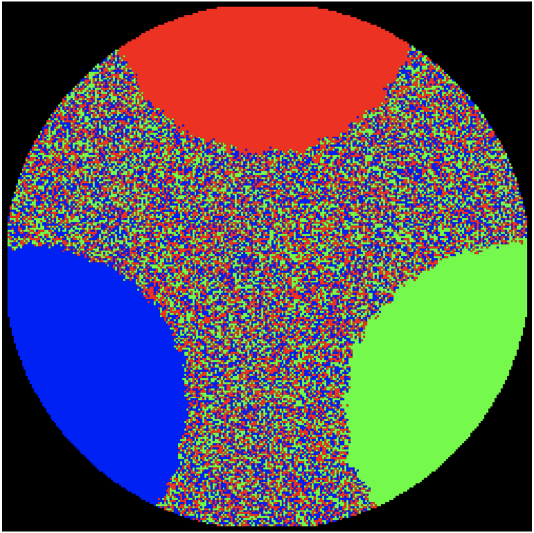

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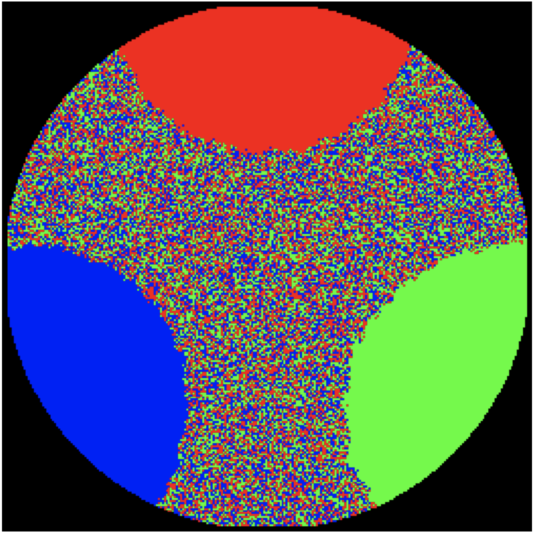

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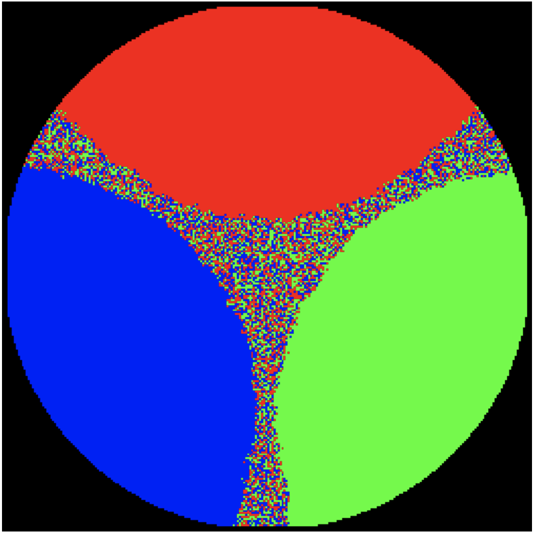

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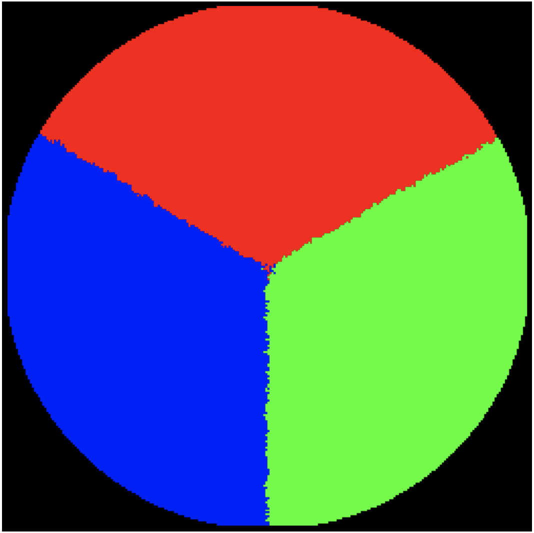
1. 

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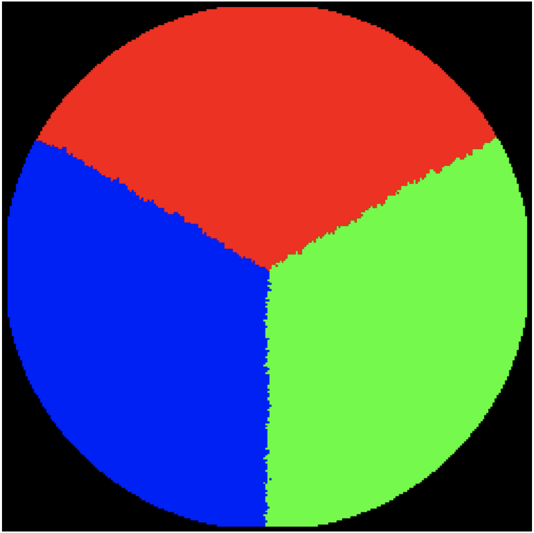

Recovering the Limit Shape: Simply connected domain

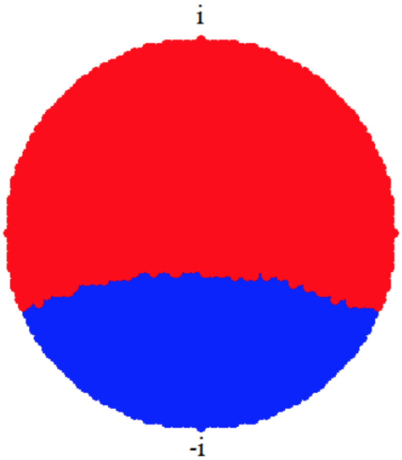

Figure: Picture from [4]

Recovering the Limit Shape

1. On the cylinder, the relevant Green's function is $G(x, y) = y$. So we know from the heuristic that the interface is at $y = 0$ (if, say, we started with half of the squares blue).
2. But this argument works in any domain. As harmonic functions are conformally invariant, the interface is conformally invariant.

End!

[1] G. Lawler, M. Bramson, D. Griffeath,
"Internal Diffusion Limited Aggregation"

[2] S. Sheffield, L. Levine, D. Jerison,
"Logarithmic Fluctuations for Internal DLA"
Preprint (2018), arXiv:1010.2483.

[3] S. Sheffield, L. Levine, D. Jerison,
"Internal DLA for Cylinders"
Preprint (2018), arXiv:1310.5063.

[4] S. Ganguly
"Competitive Erosion is Conformally Invariant"
Preprint.

[5] S. Ganguly
"Formation of an interface by competitive erosion"
Preprint.