01/11 | Universal Local Acyclicity (ULA)

$$\begin{tikzcd}
D_{et}(\mathrm{Bun}_G) & & \text{Jacobian criterion} \arrow[ll, squiggly, "\text{chart}"'] \\
& \text{ULA sheaves} \arrow[ul, squiggly, "\text{notion of admissibility}"] \arrow[ur, "\text{for proof}"'] \arrow[d, squiggly] & \\
& \text{geometric Satake} &
\end{tikzcd}$$

Recall $f: X \longrightarrow S$ finite type, separated map of noetherian schemes, $A \in D^b_c(X_{et}, \Lambda)$

constructible

where $n\Lambda = 0$ $n \in \mathcal{O}_S^\times$ Then

A is $\underline{\underline{f\text{-LA}}}$ if

$\forall$ all geometric pt

$$\begin{tikzcd}
& \bar{X} \arrow[r] \arrow[d] & X \arrow[d] \\
\bar{t} \arrow[r, squiggly] & \bar{s} \arrow[r] & S
\end{tikzcd}$$

the map

$$A_{\bar{x}} = R\Gamma(X_{\bar{x}}, A)$$
$$\longrightarrow R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} \bar{t}, A)$$

is an isomorphism

i.e $X_{\bar{x}} \longrightarrow S_{\bar{s}}$ cohom of all geom fibres agrees.

" étale analogue of $\mathcal{F} / X$ (quasi)-coherent
asking $\mathcal{F}$ "flat/ S" "

See Lu-Zhang Duality on general bases

<u>Thm</u> (Gabber) A f-LA

$\Longrightarrow$ $\forall$ any base change

$$\begin{tikzcd} X' \arrow[r, "\tilde{g}"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

also $\tilde{g}^* A$ is f'-LA

requires noetherian base. In general, following notion is better:

Def'n A is f-ULA if $\forall$ any base change $g^* A$ is f'-LA

Example 1) f smooth then any locally const sheaf e.g Λ is ULA

Reason: $$\begin{tikzcd} X_{\bar{x}} \arrow[d] \\ S_3 \end{tikzcd}$$ looks like "a ball"

2) $f = id : X = S \to S$ then A is id-ULA

$\Longleftrightarrow$ A is locally const

Reason: specialization all iso $\Rightarrow$ loc const

3) $$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$ h proper

B/Y g-ULA

(Related to geo satake)
?

$\Rightarrow$ $A = Rh_*B$ is f-ULA

2) + 3): If f is proper then

A f-ULA $\Rightarrow$ Rf_*A locally constant

4): A f-ULA

$\Rightarrow$ A-twist version of Poincare duality:

$$\mathbb{D}_{X/S}(A) \otimes f^*B \xrightarrow{\sim} R\mathcal{H}om(A, Rf^!B)$$

where $\mathbb{D}_{X/S}(A) \simeq R\mathcal{H}om(A, Rf^!\Lambda)$
relative Verdier dual

($A = \Lambda$: get $Rf^!\Lambda \otimes f^*B \xrightarrow{\sim} Rf^!B$)
if Λ f-ULA e.g. f is smooth

5) A f-ULA $\Rightarrow$ Verdier biduality, and $\mathbb{D}_{X/S}(A)$ f-ULA

(Lu-Zhang) 2020

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$$

In fact, they characterize ULA sheaves as dualizable objs in a certain sym monoidal category : obj : (X, A) A sheaf on X

$$\begin{tikzcd} (X,A) \arrow[d] \\ S \end{tikzcd}$$

base S

mor : correspondence

6) If S geom point, all

$$A \in D^b_c(X_{ét}, \Lambda) \text{ are ULA,}$$

__want__ : Variants for diamonds

__important point__ : have good analogue $D_{ét}(X, \Lambda)$ of full unbounded derived cat

but "constructiblity" is a subtle notion

e.g $i : \mathrm{Spa}\, C \hookrightarrow \mathbb{B}_C$ $\underbrace{i_* \Lambda}$ not constructible

is overconvergent

but shall be ULA / Spa C

prop X spatial diamond of finite cohom dim (unif on $X_{\acute{e}t}$)

then $D_{\acute{e}t}(X, \Lambda)$ is compactly generated

compact objs = constructible complexes

i.e locally constant after passing to a constructible stratification

in boolean alg gen by qc open subset

example $j : \mathbb{T}_C \hookrightarrow \mathbb{B}_C$

$\{T \mid |T| = 1\} = \mathbb{T}_C$, $\quad \mathbb{B}_C = \{T \mid |T| \leq 1\}$

$j_! \Lambda$ is constructible

Def'n Let $f : X \longrightarrow S$ map of locally spatial diamonds (compatifiable, locally finite dimtrg.)

$\rightsquigarrow Rf_!$ is defined

$A \in D_{ét}(X, \Lambda)$

1) A is f-LA if

a) $\forall$ geo pts

$$\begin{tikzcd} & \bar{x} \arrow[r] \arrow[d] & X \arrow[d] \\ \bar{t} \arrow[r, squiggly] & \bar{s} \arrow[r] & S \end{tikzcd}$$

$$A_{\bar{x}} = R\Gamma(X_{\bar{x}}, A) \simeq R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}}, A)$$

is an iso

b) $\forall$ all étale $j : U \longrightarrow X$ s.t

$f \circ j : U \longrightarrow X$ is qcqs

$R(f \circ j)_! (A|_U) \in D_{ét}(S, \Lambda)$ is constructible

i.e constructible after any pull back $S' \longrightarrow S$

spatial

2) A is f-ULA if any base change is LA

Remark

- $\forall$ schemes, b) is automatic, all information is in a)
- $\forall$ diamonds, a) is almost automatic

  b) powerful
- Analogue of Gabber's thm fails:

$S = \mathrm{Spa}\, C$, X coh smooth $/ S$

$\Rightarrow$ any constructible A is f-LA

but only locally constant A are f-ULA

Condition a) (Henselization is easy in analytic world)

$X_{\bar{x}}$ is rep by $\mathrm{Spa}(C, C^+)$

C complete alg-closed field, $C^+ \subseteq C$ valuation subring

$$\begin{tikzcd} {|X_{\bar{x}}|} \arrow[d] \\ {|S_{\bar{s}}|} \arrow[r, phantom, "\supseteq"] & {|S_{\bar{t}}|} \end{tikzcd}$$

$|X_{\bar{x}}|$: totally ordered chain of points

$$\cdot \to \cdot \to \cdot \to \cdots\cdots \to \bullet$$

$|S_{\bar{s}}|$: similar

$|S_{\bar{t}}|$ subset

$\Rightarrow \quad X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}} = X_{\bar{y}}$ for some

$$\begin{tikzcd} \bar{y} \arrow[r, rightsquigarrow] \arrow[d] & \bar{x} \arrow[d] \\ \bar{t} \arrow[r, rightsquigarrow] & \bar{s} \end{tikzcd}$$

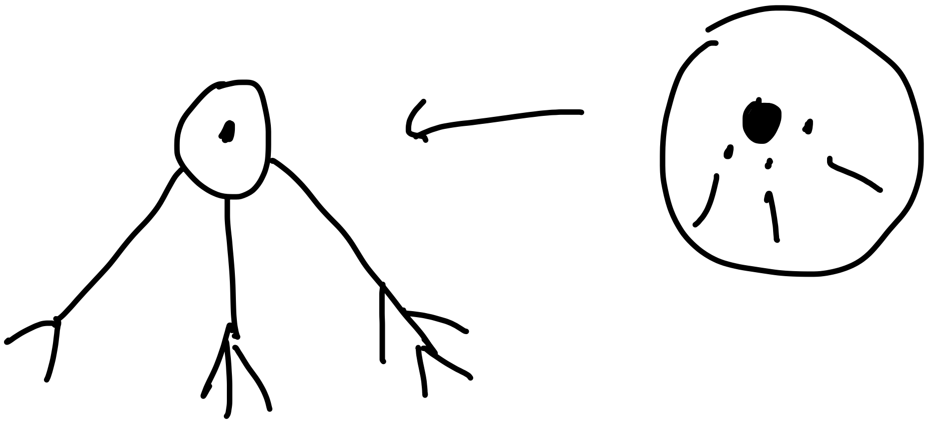
Condition is just

$$A_{\bar{x}} \xrightarrow{\sim} A_{\bar{y}} = R\Gamma(X_{\bar{y}}, A)$$
$$= R\Gamma(X_{\bar{s}} \times_{S_{\bar{s}}} S_{\bar{t}}, A)$$

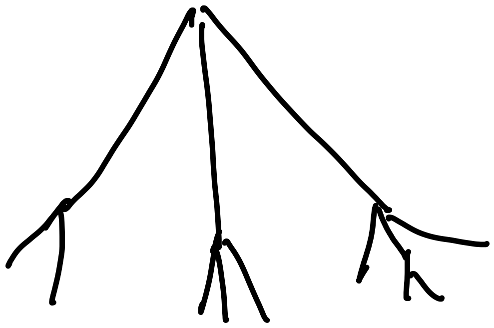
"a) universally" $\Longleftrightarrow$ A is overconvergent i.e.

$\forall\ \underset{\text{specialization}}{\bar{y} \rightsquigarrow \bar{x}}$, $A_{\bar{x}} \simeq A_{\bar{y}}$ is iso

$$\begin{tikzcd}
{|\mathbb{B}_C|} \arrow[d] & \\
{|\mathbb{B}_C|^{\text{hausdorff}}} \arrow[r, phantom, "\simeq"] & \text{Berkovich space}
\end{tikzcd}$$



compact Hausdorff



overconvergence sheaves on $|\mathbb{B}_C|$

$\cong$ sheaves on $|\mathbb{B}_C|^{\text{hausdorff}}$

(similar statement for étale sheaves)

over conv. étale sheaves $\cong$ étale sheaves on Berkovich space

when this makes sense

prop 1) f cohom. smooth

A locally const $\Rightarrow$ A f-ULA

loc const $\Rightarrow$ overconvergence ✓

need to prove: f qcqs + cohom. smooth

$\rightsquigarrow$ $Rf_!$ preserves constructible complexes

Know: constr complex = compact objs

i.e $R\mathrm{Hom}(A, -)$ commutes with all direct sums

Lem $F : \mathcal{C} \leftrightarrows D : G$

adj functors of compactly generated triang cats

Then F preserves compact objs

$\Longleftrightarrow$ G commutes with all direct sums pf: use $\mathrm{Hom}(F(-), (-)) \simeq \mathrm{Hom}((-), G(-))$

formal. $\square$

Now $Rf_! \rightleftarrows Rf^!$

but f is coh smooth

$\Rightarrow$ $Rf^! \cong f^* \otimes Rf^! \Lambda$

commutes with direct sums

2) If $f = \mathrm{id} : X = S \longrightarrow S$ then

A f-ULA $\Leftrightarrow$ A is locally const

'th <u>perfect</u> fibres

perfect complex of

finite proj Λ - mods

pf: b) $\Rightarrow$ A constructible
a) $\Rightarrow$ A overconvergent
$\Bigg\}$ $\Rightarrow$ locally const

3). f proper , A f-ULA $\Rightarrow$ Rf_*A locally const
(by proper base change)

4) twisted version of Poincare duality

A f-ULA

$$\Rightarrow \mathbb{D}_{X|S} \overset{\mathbb{L}}{\otimes} f^*B \simeq R\mathcal{H}om_\Lambda(A, Rf^!B)$$

$$\forall \; B \in D_{et}(S, \Lambda)$$

Note this implies b):

$\Rightarrow R\mathcal{H}om_\Lambda(A, Rf^! -) : D_{et}^c(S,\Lambda) \to D_{et}^c(X,\Lambda)$

commutes with all direct sums

$\Rightarrow$ its left adjoint

$Rf_!(A \overset{\mathbb{L}}{\otimes} -)$ preserves constructiblity

Apply it to $j_!\Lambda$.

5) Verdier biduality : If A f-ULA

then $\mathbb{D}_{X/S}(A)$ is f-ULA

and $A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$

6) $S = \mathrm{Spa}\, C$, $X = X_0^\diamond$ for some alg

variety $X_0/\mathbb{C}$ Then $\forall$ $A_0 \in D^b_c(X_{0,\acute{e}t}, \Lambda)$

its analytification $A \in D_{\acute{e}t}(X, \Lambda)$

is ULA (but may not constructible)

e.g. $i : \mathrm{Spa}\, C \hookrightarrow \mathbb{A}^1_C$

$i_* \Lambda$ is ULA

($\Rightarrow$ same for $\mathrm{Spa}\, C \hookrightarrow \mathbb{B}_C$)

About 5) : Biduality

Two proofs : Both use "dualizability" in

2-cats

Lu-Zhang's approach : Fix base S

Consider sym mon 2-category LZ_S :

- objs: (X, A) $X \to S$

  $A \in D_{et}(X, \Lambda)$

- morphisms $(X, A) \longrightarrow (Y, B)$ cohom corr

$$\begin{tikzcd} & Z \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

+ map $c_1^* A \longrightarrow Rc_2^! B$

Thm TFAE

1) A is $(X \to S)$-ULA

2) (X,A) is dualizable on LZ_S

3) $(X,A) \otimes (X,A)^\vee \simeq R\mathcal{H}om((X,A),(X,A))$

i.e $p_1^* D_{X/S}(A) \overset{L}{\otimes}_\Lambda p_2^* A$

$\simeq R\mathcal{H}om_\Lambda(p_1^* A, Rp_2^! A)$

for

$$\begin{tikzcd} X \times_S X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

(an instance of A-twisted Poincare duality)

In that case, dual $(X,A)^\vee = (X, \mathbb{D}_{X/S}(A))$

thus $\mathbb{D}_{X/S}(A)$ is $(X \to S)$-ULA and

$$A \simeq \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$$

Cor 1) Λ is f-ULA iff

$$\boxed{p_1^* \mathbb{D}_{X/S} \xrightarrow{\sim} \mathbb{D}_{X \times_S X / X} = Rp_2^! \Lambda}$$

2) f is cohom smooth (wrt. Λ)

$\iff$ $\mathbb{D}_{X/S}$ is invertible, and $p_1^* D_{X/S} \simeq Rp_2^! \Lambda$

(can be used to give an easy proof that ball is coh smooth)

Second proof 2-cat $\mathcal{C}_S$

- objects: $X \to S$ as above
- morphism: $\mathrm{Fun}_{\mathcal{C}_S}(X, Y) = D_{\text{ét}}(X \times_S Y, \Lambda)$

— composition = convolution

$X, Y, Z \to S$

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl] \arrow[d] \arrow[dr] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd}$$

$A \in \mathrm{Fun}_{\mathcal{E}_S}(X, Y) \quad B \in \mathrm{Fun}_{\mathcal{E}_S}(Y, Z)$

$\Rightarrow \quad A \star B := R\pi_{13!}\left(\pi_{12}^{*} A \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} \pi_{23}^{*} B\right)$

proper base change $\Rightarrow$ associativity

$\mathrm{id}_X = \Delta_{X/S,!}\,\Lambda$

maps to 2-cat / obj: $X \to S$ as above

\ mor: functors $D^c_{et}(X, \Lambda) \to D^c_{et}(Y, \Lambda)$

by using sheaves as kernels:

$$D \mapsto R\pi_{2!}(A \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} \pi_2^* D)$$

Recall: In any 2-category, have notion of adjointness: $f: X \to Y$ left adj of $g: Y \to X$

if there are $\alpha: id_X \to gf$

$\beta: fg \to id_Y$

s.t. $$f \xrightarrow{f\alpha} fgf \xrightarrow{\beta f} f = id$$

$$g \xrightarrow{\alpha g} gfg \xrightarrow{g\beta} g$$

Thm TFAE:

1) $A \in D_{et}(X,\Lambda)$ is ULA

2) $A \in \mathrm{Fun}_{\mathcal{C}_S}(X,S)$ is a left adjoint

In that case, the right adj is

$$\mathbb{D}_{X/S}(A) \in \mathrm{Fun}_{\mathcal{C}_S}(S,X)$$

3) $p_1^* \mathbb{D}_{X/S}(A) \overset{L}{\underset{\Lambda}{\otimes}} p_2^* A \xrightarrow{\sim} R\mathcal{H}om_{\Lambda}(p_1^* A, Rp_2^! A)$

Conclusion: ULA $\Longleftrightarrow$ concrete duality thm

geometric Satake $\leftarrow$ second approach

(use sheaf as kernel)

Q: ULA base change ✓

Q: 3) $\overset{?}{\Rightarrow}$ what in geom Satake ?

$$\begin{tikzcd}
\widetilde{Gr}_G^I \arrow[r, "h"] \arrow[dr] & Gr_G^I \arrow[d] \\
 & (Div_X^1)^I
\end{tikzcd}$$

Convolution BD Gr

Beilinson–Drinfeld Gr

h is proper

fusion product

$\Rightarrow$ ($(A_i)_{i\in I}$ ULA on $\begin{matrix} Gr_G \\ \downarrow \\ Div^1 \end{matrix}$

$\Longrightarrow$ $\widetilde{\boxtimes}_{i\in I} A_i$ ULA $\Rightarrow$ $Rh_*(\widetilde{\boxtimes}_{i\in I} A_i)$ ULA)