01/15 # The Jacobian criterion

$E \supseteq O_E$, π, $\mathbb{F}_q$ as usual

<u>Thm</u> S perfectoid space $/ \mathbb{F}_q$

$$Z \longrightarrow X_S = X_{S,E} \quad \text{smooth map}$$

of (sousperfectoid) adic spaces

s.t. Z quasi-projective

($\exists$ Zariski closed embedding $Z \hookrightarrow U \overset{\text{open}}{\subseteq} \mathbb{P}^n_{X_S}$)

letting $\mathcal{M}_Z^{\text{smooth}} \subseteq \mathcal{M}_Z = \left\{ \text{sections of } \begin{array}{c} Z \\ \downarrow \\ X_S \end{array} \right\}$

$$T/S \longmapsto \left\{ \begin{tikzcd} & Z \arrow[d] \\ X_T \arrow[ur, dashed, "s"] \arrow[r] & X_S \end{tikzcd} \right\}$$

be the open subspace

where $s^* T_{Z/X_S}$ has everywhere > 0 HN slope then the map $\mathcal{M}_Z^{\text{smooth}} \longrightarrow S$ is

ℓ-cohom. smooth $\forall\, \ell \neq p$

(Recall: $\mathcal{M}_Z \longrightarrow S$ is repr. in loc spat. diamond, locally finite dim.trg

$\Rightarrow$ same for $\mathcal{M}_Z^{\text{smooth}}$

Strategy: 1) formal smoothness of
$$\begin{array}{c}\mathcal{M}_Z^{\text{smooth}} \\ \downarrow \\ S\end{array}$$

2) formal smooth + "geo finite dim"

$\Rightarrow \mathbb{F}_\ell$ f-ULA
3) $Rf^! \mathbb{F}_\ell$ invertible $\Bigg\} \Leftrightarrow f$ cohom. smooth

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1) "formal smoothness" $T_0 \subset T$ Zar closed im of aff'd perf'd spaces

$$\begin{array}{ccc} T_0 & \longrightarrow & \mathcal{M}_Z^{sm} \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

(important, can't assume just local spatial diamonds)

$\Rightarrow \exists\ T' \to T$ étale containing T_0 in image

s.t.

$$\begin{tikzcd} T_0 \times_T T' \arrow[r] \arrow[d, hook] & \mathcal{M}_Z^{\text{smooth}} \arrow[d] \\ T' \arrow[r] \arrow[ur, dashed, "\exists"] & S \end{tikzcd}$$

pf: tricky, explicite BC spaces argument

Q: which space shall satisfy this prop?

2) Prop $\exists$ a cohom. smooth + formally smooth

locally finite dim ty

compatible

surjective map $T_0 \to \mathcal{M}_Z$ repr in loc spatial diamond

s.t T_0 is a perf'd space s.t T_0

locally admit a Zariski closed embedding

into finite dim perf'd ball $|_S$

proof: $\mathcal{M}_Z \subset \mathcal{M}_{\mathbb{P}^n_S}$ locally Zar closed

(in suitable sense)

(pullback of surj is surj)

only need do it for $\mathcal{M}_{\mathbb{P}^n_S}$

$$\mathcal{M}_{\mathbb{P}^n_S} \underset{\text{open}}{\subset} \bigcup_{d \geq 0} \left(BC(\mathcal{O}(d))^{n+1}\right)_S \setminus \{0\} \,/\, \underline{E}^\times$$

$\rightsquigarrow$ enough for $BC(\mathcal{O}(d)^{n+1})_S \,/\, \underline{E}^\times$

this can be done explicitly ?

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Cor $\mathbb{F}_\ell$ is ULA for $\mathcal{M}_Z^{sm} \to S$

proof: enough for $T_0 \to S$ (check coh, smooth locally)

Lem $T_0 \to S$ map of aff'd perf'd space

s.t. 1) $T_0 \to S$ formally smooth

2) $T_0 \hookrightarrow \mathbb{B}^n_S$ Zariski closed in a finite dim perf'd ball

Then $\mathbb{F}_\ell$ is ULA for $T_0 \to S$

proof.

$$\begin{tikzcd} T_0 \times_T T' \arrow[r] \arrow[d] & T_0 \arrow[r, equal] \arrow[d] & T_0 \arrow[d] \\ T' \arrow[r, "\text{étale}"] & T = \mathbb{B}^n_S \arrow[r] & S \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} T_0 \times_T T' \arrow[r, "\text{étale}"] \arrow[d, bend right] & T_0 \\ T' \arrow[u, dotted, bend right] \arrow[ur] & \end{tikzcd}$$

Shrinking T', can even find a retraction $T' \to T_0 \times_T T'$

$$\Rightarrow \quad \begin{tikzcd} & T_0 \times_T T' \arrow[dl, "\text{étale}"'] \arrow[d, "\text{ULA}"] \\ T_0 \arrow[r, "\text{ULA}"'] & S \end{tikzcd}$$

retract of

$$\Leftarrow \quad \begin{tikzcd} T' \arrow[d, "\text{étale}"] \arrow[dd, bend right, "\text{ULA}"'] \\ \mathbb{B}^n_S \arrow[d] \\ S \end{tikzcd}$$

ULA is étale local (even sm. smooth)

being ULA is stable under retracts: follows either by def'n or from categorical characterization

∎

3) $Rf^! \mathbb{F}_\ell$ is invertible is locally isomorphic to $\mathbb{F}_\ell[n]$

Fact: If A f-ULA for $f: X \to S$ then $\mathbb{D}_{X/S}(A)$ is again f-ULA and its formalism commutes with any $S' \to S$ base change

Being invertible can be checked v-locally, so can be checked after pullback along v-cover.

$\Rightarrow$ Passing to universal section of

$$\mathcal{M}_Z^{\text{smooth}} \to S$$

enough to prove that for a section

$$s: S \longrightarrow \mathcal{M}_Z^{\text{smooth}}$$

$$\cong (s: X_S \to Z)$$

the pullback $s^* Rf^! \mathbb{F}_\ell \in D_{ét}(S, \mathbb{F}_\ell)$

is invertible.

Now we use deformation to the normal cone:

$$\begin{tikzcd} Z \arrow[d] \\ X_S \arrow[u, bend left, "s"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \tilde{Z} \arrow[d, "\text{smooth}"'] \\ X_S \times \mathbb{A}^1 \arrow[u, bend left, "\tilde{s}"] \end{tikzcd} \circlearrowleft \mathbb{G}_m$$

s.t. • $\tilde{Z} \times_{\mathbb{A}^1} \{1\} = Z$

• $\tilde{Z} \times_{\mathbb{A}^1} \{0\}$ = normal cone of s in Z

= geometric vector bundle

corr to $s^* T_{Z|X_S}$

$$\begin{tikzcd} \underline{E}^\times \arrow[d, bend right] & & \mathbb{G}_m \arrow[d, bend right] \\ X_{(S \times \underline{E})} \arrow[rr, hookleftarrow, "= X_S \times \underline{E}"'] & & X_{S \times \mathbb{A}^1} \\ \tilde{Z}' \arrow[u] \arrow[rr, hook] & & \tilde{Z} \arrow[u] \end{tikzcd}$$

Get $\tilde{Z}^1 \xrightarrow{\text{smooth}} X_{S\times \underline{E}}$ fibre over $S \times \{1\}$ is Z

$\circlearrowright$ $\underline{E}^\times$ over $S \times \{0\}$ is $s^* T_{Z|X_S}$

(0 is still "in closure of" $\underline{E}$ using action of $\underline{E}^\times$)

$\tilde{Z}^1$ still quasi-proj, all previous result apply

$$\tilde{f} : M_{\tilde{Z}^1} \xrightarrow{\text{smooth}} S \times \underline{E}$$

and $R\tilde{f}^! \mathbb{F}_\ell$ is $\tilde{f}$-ULA

$$R\tilde{f}^! \mathbb{F}_\ell |_{S\times\{1\}} = Rf^! \mathbb{F}_\ell$$

$+$ $R\tilde{f}^! \mathbb{F}_\ell |_{S\times\{0\}} =$ dualizing complex for $BC(s^* T_{Z|X_S})$ is invertible?

Q: explicit

$s^* Rf'_! \mathbb{F}_\ell$ is invertible

deformation to the normal cone

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(Similar argument by Clausen to show dualizing sphere of p-adic Lie gp agrees and . . . . . of its Lie alg

Application to $D_{et}(\mathrm{Bun}_G, \Lambda)$

Recall charts for Bun_G:

Def'n: Let $\mathcal{M}$ be the moduli space of $\mathbb{Q}$-filtered G-bundles, i.e exact $\otimes$-functor $\mathrm{Rep}_E G \xrightarrow{\rho} \mathbb{Q}\text{-Fil}\,\mathrm{Bun}_{X_S}$

(increasing filtration)

s.t. $\forall$ all $V \in \mathrm{Rep}_E G$

$$\rho(V)^{\lambda} := \rho(V)^{\le \lambda} \Big/ \bigcup_{\lambda' < \lambda} \rho(V)^{\le \lambda'}$$

semi-stable of slope λ

"opposite HN filtrations"

Then
$$\begin{tikzcd} \bigsqcup_{b \in B(G)} \mathcal{M}_b = \mathcal{M} \arrow[r] \arrow[d] & \mathrm{Bun}_G \\ \bigsqcup_{b \in B(G)} [* \mid \underline{G_b(E)}] & \end{tikzcd}$$

forgetting filtration

pass to associated graded bundle

<u>Thm</u> $\mathcal{M} \longrightarrow \mathrm{Bun}_G$ is cohom. smooth

<u>Example</u>: GL_2 $\quad b = \mathcal{O} \oplus \mathcal{O}(1)$

Then $\mathcal{M}_b$ classify extensions

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$$

$$\deg \mathcal{L} = 0 \qquad \deg \mathcal{L}' = 1$$

$G = GL_n$: similar successive extensions)

Thm is a consequence of Jacobian criterion

(Take any $S \to \mathrm{Bun}_G) \cong \mathcal{E}/X_S$

Z = moduli space of $\mathbb{Q}$-fil on $\mathcal{E}$

Then $\mathcal{M} \subseteq \mathcal{M}_Z$ actually lie in

?. $\mathcal{M}_Z^{\text{smooth}}$ by condition on slopes.

Now fix $b \in B(G)$, consider

$$\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G$$

$$\mathcal{M}_b \subseteq \mathcal{M}$$

"chart for Bun_G near Bun_G^b"

<u>Structure of $\mathcal{M}_b$:</u>

- $$\mathcal{M}_b = [\widetilde{\mathcal{M}}_b / \underline{G_b(E)}]$$

$$\downarrow$$

$$[* / \underline{G_b(E)}]$$

In $\widetilde{\mathcal{M}}_b$, graded bundle is trivialized

e.g. $\{0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0\}$

"base point" $* \in \widetilde{\mathcal{M}}_b$ corresponding to split extension

$\rightsquigarrow$

$$\begin{tikzcd}
{[* / \underline{G_b(E)}]} \arrow[r, phantom, "\subseteq"] \arrow[dr, "\sim"] \arrow[dd, bend right] & \mathcal{M}_b \arrow[d] \arrow[dd, bend left=60] \\
 & {[* / \underline{G_b(E)}]} \\
\mathrm{Bun}_G^b \arrow[r, phantom, "\subseteq"] \arrow[ur, phantom, "\square"] & \mathrm{Bun}_G
\end{tikzcd}$$

(i.e if $\mathcal{E} \in \mathrm{Bun}_G^b$ then HN fil of $\mathcal{E}$ gives splitting of given $\mathbb{Q}$-fil)

- $\widetilde{\mathcal{M}}_b \longrightarrow *$ rep in loc. spat diamonds, cohom. smooth

successive ext of negative BC spaces of $\dim = \langle 2\rho, \nu_b \rangle$

e.g. $\{0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0\}$

$$\simeq BC(\mathcal{O}(-1)[1])$$

- $\widetilde{\mathcal{M}}_b \setminus * = \widetilde{\mathcal{M}}_b^{\circ}$ is a spatial diamond

($\Rightarrow$ qcqs ! , but not qcqs $| *$)

In GL_2-example, it's

$$\left(\operatorname{Spa} \overline{\mathbb{F}_q}((t^{\frac{1}{p^\infty}}))\right) / \underline{SL_1(D)}$$

$D | E$ quaternion alg

aff'd perfectoid — profinite

(On $\widetilde{\mathcal{M}}_b^{\circ}$, $\mathcal{E} \cong \mathcal{O}(\frac{1}{2})$. Picking such iso

$\mathcal{O} \hookrightarrow \mathcal{O}(\frac{1}{2})$ gives section of

$BC(\mathcal{O}(\frac{1}{2})) \setminus \{0\} = \operatorname{Spa} \overline{\mathbb{F}_q}((t^{\frac{1}{p^\infty}}))$.)

this is t, so t can't go to zero

Warning: In GL_2-example

$$\widetilde{\mathcal{M}}_b^{\circ} \subseteq \widetilde{\mathcal{M}}_b = \widetilde{\mathcal{M}}_b^{\circ} \cup *$$

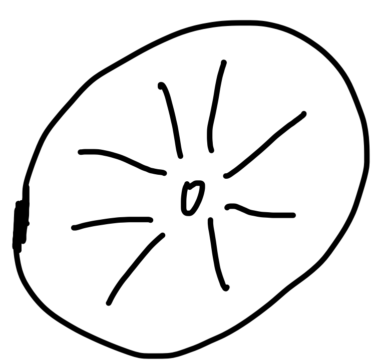
modulo some gp action $\circlearrowright$ $\mathrm{Spa}\, \overline{\mathbb{F}}_q((t^{\frac{1}{p^\infty}}))$

formal punctured open disc

this point $*$ sits near $|t|=1$

not near origin $|t|=0$!

after base change to $\mathbb{C}$:



- $\widetilde{\mathcal{M}}_b$ "strictly local": shall think it as strict localization at b

Thm

$\forall$ any $A \in D_{ét}(\tilde{M}_b, \Lambda)$, the

restriction $R\Gamma(\tilde{M}_b, A) \longrightarrow R\Gamma(*, A)$

is an isomorphism

<u>Sketch</u> the cone of this map is

$$R\Gamma_{\partial-c}(\tilde{M}_b^{\circ}, A)$$

compact supported towards $*$

no supported condition towards boundary of $\tilde{M}_b$

special case of:

Let $X = (\tilde{M}_b^{\circ})$ e.g spatial diamonds dim trg $< \infty$

$X \longrightarrow *$ partially proper "one-pt" compactification

i.e $X(R, R^+) = X(R, R^{\circ})$

Then for any $C / \mathbb{F}_q$

X_C has "two ends" ? precise

examples: $X = \mathrm{Spa}(R, R^+)$ aff'd perf'd

$C = \widehat{\overline{\mathbb{F}_q((t))}}$

$$X_C \xrightarrow[\text{profinite}]{} \underbrace{X \times_{\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q((t))}$$

$=$ punctured open unit $\underbrace{\text{disc} / X}$

one boundary = orig'n

another boundary = "boundary of unit disc"

$\Longrightarrow$ can define "partial compact supported cohomology"

$$R\Gamma_{\partial\text{-}c}(X_c, A)$$

Thm $R\Gamma_{\partial\text{-}c}(X_c, A) = 0$

Sketch: reduce to $X = \mathrm{Spa}\, \overline{\mathbb{F}}_q((t^{\frac{1}{p^\infty}}))$

(use proper base change + "correspondence")

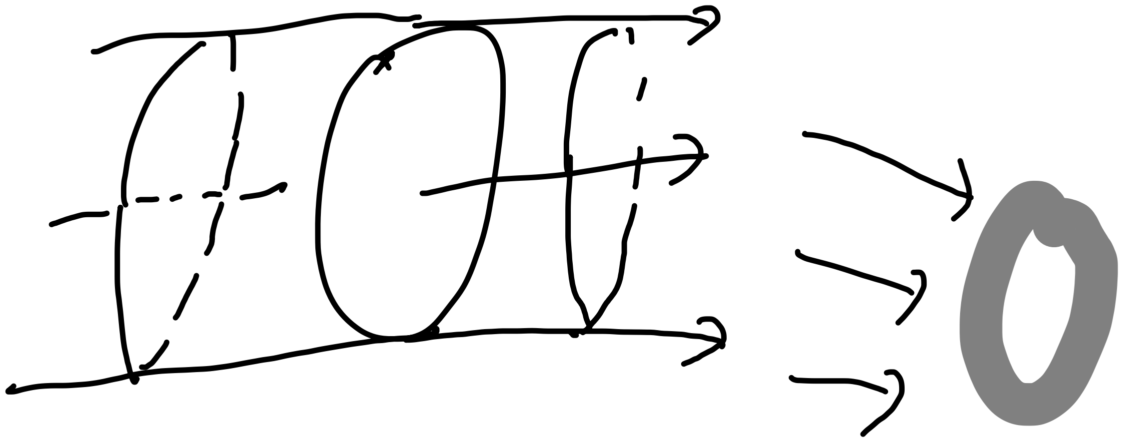
+ $A = \Lambda$ + compute.

(not many A)

Picture: Say M topological manifold

+ free action $\mathbb{R} \curvearrowright M$ "flow"

s.t. $\overline{M} = M/\mathbb{R}$ is compact



collapse to boundary disc

two boundaries

"source of flow"

"end of flow"

+ for all $A \in D(M/\mathbb{R}, \mathbb{Z})$,

$$R\Gamma_{\partial\text{-}c}(M, A) = 0$$

Pf: Flow contracts everything

"later for hyperbolic localization"

How is this analogous?

<u>Roughly</u>: $C = \overline{\mathbb{F}_q}((t^{\mathbb{R}})) \underset{\text{rescaling}}{\circlearrowleft} \mathbb{R}_{>0} \overset{\exp}{\simeq} \mathbb{R}$

$$X_{\mathbb{C}} \circlearrowleft \mathbb{R} \qquad\qquad X_{\mathbb{C}}/\mathbb{R} \quad \text{qcqs}$$

$$\begin{array}{c} X_{\mathbb{C}} \\ \cup\!\mid \\ M \end{array} \qquad\qquad \begin{array}{c} X_{\mathbb{C}}/\mathbb{R} \\ \downarrow \\ X \end{array}$$

Q : "ends" real S^1

If X ahor. smooth $/ *$

Cor : $X_{\mathbb{C}}$ as above satisfies "odd-dim'l Poincaré duality"

$$R\Gamma_c(X_{\mathbb{C}}) \longrightarrow \underbrace{\underbrace{R\Gamma_{\partial-c_1}(X_{\mathbb{C}})}_{0} \oplus R\Gamma_{\partial-c_2}(X_{\mathbb{C}})}_{\downarrow \; R\Gamma(X_{\mathbb{C}})}$$

$\Rightarrow R\Gamma_c(X_{\mathbb{C}}) \simeq R\Gamma(X_{\mathbb{C}})[-1]$

Q: $T_0 \longrightarrow M_Z$?

shall all work in pratice

If X Zariski closed in finite-dim perfectoid ball $/C$

is $\dim X = \dim_{\text{tg}} X$?