1/18 $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$

G/E reductive gp   residue field $\mathbb{F}_q$

char $= p > 0$

Λ coefficient ring, $n\Lambda = 0$ $(n, p) = 1$

(how about $\underbrace{\Lambda = \mathbb{Z}[\frac{1}{p}]}_{\mathbb{Z}_\ell}$)

Recall   Q. (Drinfeld)

$D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ has infinite semi-orthogonal decomposition into $D_{\acute{e}t}(\mathrm{Bun}_G^b, \Lambda)$

$\simeq D(G_b(E), \Lambda)$

§1 compact object

Recall $A \in \mathcal{C}$ is compact iff $\mathrm{Hom}_C(A, -)$ commutes with all direct sums

Fact C homotopy category of a stab ∞

category $\mathcal{C}$ with all colimits, then

$A \in \mathcal{C}$ compact iff $A \in \mathcal{C}$ s.t.

$\Downarrow$ $\qquad$ $\mathrm{Hom}_{\mathcal{C}}(A, -)$ commutes with

commutes with filtered colimit $\Leftrightarrow$ all colimits

exact

If $\mathcal{C}$ is generated under colimits by its compact objects $\mathcal{C}^{\omega} \subseteq \mathcal{C}$, then

$$\mathrm{Ind}(\mathcal{C}^{\omega}) \xrightarrow{\simeq} \mathcal{C} \quad \text{equiv of } \infty\text{-cats}$$

<u>Prop</u> $D(G_b(E), \Lambda)$ is compactly generated

compact generators are $c\text{-}\mathrm{Ind}_{K}^{G_b(E)} \Lambda$ $\quad$ $K \subseteq G_b(E)$ open pro-p (hence K cpt)

<u>proof</u>. $\mathrm{Hom}_{G_b(E)}(c\text{-}\mathrm{Ind}_{K}^{G_b(E)} \Lambda, -)$

$= \mathrm{Hom}_{K}(\Lambda, -) = (-)^{K}$

Q: for which cpt K, $(-)^{K}$ is compact.

K pro-p $\Rightarrow$ taking K-inv commutes with all direct sums
($=K$-cohomo)

(If A repr. by

$$\cdots \to A_2 \to A_1 \to A_0 \to A_{-1} \to \cdots$$

$\Rightarrow$ A^K repr. by

$$\cdots \to A_2^K \to A_1^K \to A_0^k \to A_{-1}^k \to \cdots \quad)$$

So $c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$ is compact $\forall K$

If A s.t $A^K = 0$ $\forall K$, then $A = 0$
so generate $\square$

<u>Thm</u> $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is cpt generated

$A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ cpt iff $\forall\, b \in B(G)$ $\quad i_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$

$i^{b*}A$ is compact and $=0$ $\forall$ almost all b

proof First, exhibit compact generators

Fix $b \in B(G)$, $K \subseteq G_b(E)$ open pro-p

Goal Show that $\exists\ A_K^b \in D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$

s.t. $R\mathrm{Hom}(A_K^b, B) = (i^{b*} B)^K$

Q: what is it for GL_1 tori?

$\forall\ B \in D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$

if $\exists\ A_K^b$, it's cpt and these objs generates

to find A_K^b, use

$$\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G \quad \text{cohom, smooth chart}$$

$$\uparrow G_b(E)\text{-torsor}$$

$$\widetilde{\mathcal{M}}_b$$

$\rightarrow$ $f_K : [\widetilde{\mathcal{M}}_b / \underline{K}] \longrightarrow \mathrm{Bun}_G$ cohom. smooth

Claim $A_K^b = Rf_{K!} Rf_K^! \Lambda$ works

Pf: $R\mathrm{Hom}(A_K^b, B) = R\mathrm{Hom}(Rf_{K!} Rf_K^! \Lambda, B)$

$= R\mathrm{Hom}(Rf_K^! \Lambda, Rf_K^! B)$

$\overset{f_K \text{ ULA}}{=} R\mathrm{Hom}(Rf_K^! \Lambda, f_K^* B \otimes Rf_K^! \Lambda)$

$\overset{Rf_K^! \Lambda \text{ inv}}{=} R\Gamma([\tilde{\mathcal{M}}_b / K], f_K^* B)$

"flow" $\Rightarrow$ $\tilde{\mathcal{M}}_b$ strict local

$= R\Gamma([*/K], f_K^* B|_{[*/\underline{K}]})$

$= (i^{b*} B)^K$.

For characterization of cpt objs, argue by induction on quasi-cpt open substacks

$U \subset \mathrm{Bun}_G$

Pick some $b \in B(G)$ with

$i^b : \mathrm{Bun}_G^b \hookrightarrow U$ closed

on Bun_G no Bun_G^b is closed, but on qc open it's ok

$j : V = \mathcal{U} \setminus \mathrm{Bun}_G^b \hookrightarrow \mathcal{U}$

we know the result for $D_{ét}(V, \Lambda)$

<u>enough</u>: j^* preserves compact objects.

Indeed, then if A cpt then $j^* A$ cpt so $i^* A$ cpt

$j^* A$ cpt $\Rightarrow i^{b' *} A$ cpt $\forall b' \neq b$ by induction ✓

$+ \, i^* A$ compact. So it's enough to show

<u>Claim</u>. $j^* A_K^b \in D_{ét}(V, \Lambda)$ is cpt

<u>Proof</u>. $f_K : [\widetilde{\mathcal{M}}_b / K] \longrightarrow \mathcal{U} \subseteq \mathrm{Bun}_G$

$$\begin{array}{ccc} [\widetilde{\mathcal{M}}_b / K] & \xrightarrow{f_K} & \mathcal{U} \\ \cup & \Box & \cup \\ [\widetilde{\mathcal{M}}_b^{\circ} / K] & \xrightarrow{f_K^{\circ}} & V \end{array}$$

if remove more pts, will not be qc;

$\widetilde{\mathcal{M}}_b^{\circ} = \widetilde{\mathcal{M}}_b \setminus *$

~~of $\widetilde{\mathcal{M}}_b$~~ spatial diamond of finite dim'l

$$j^* A^b_K = Rf^\circ_{K!} Rf^{\circ !}_K \Lambda$$ by formula for A^b_K

By similar computation,

$$R\mathcal{H}om(j^* A^b_K, B) \cong R\Gamma([\tilde{\mathcal{M}}^\circ_b / \underline{K}], f^{\circ *}_K B)$$

$$= R\Gamma(\tilde{\mathcal{M}}^\circ_b, \text{pullback of } B)^K$$

But $R\Gamma(\tilde{\mathcal{M}}^\circ_b, -)$ commutes with all direct sums

as $\tilde{\mathcal{M}}^\circ_b$ is a spatial (qcqs!) diamond (of finite dim.trg)

(need finite dim to make sure some convergence)

tricky

$\square$

§2. Bernstein-Zelevinsky duality

A duality on cpt objs

<u>Prop'n</u> $\forall$ $A \in D(G_b(E), \Lambda)^\omega$,

$\exists!$ $A' \in D(G_b(E), \Lambda)^\omega$

s.t. $RHom(A', B) = (A \otimes B)_{\underline{G_b(E)}}$

derived homology

left adjoint of $D(\Lambda) \to D(G_b(E), \Lambda)$

For $A = c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$

get $A' = c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$

In general $A' = RHom_{G_b(E)}(A, H(G_b(E)))$

Hecke alg of cpt supported locally constant functions on $G_b(E)$

$(A')' \longrightarrow A$ is an isomorphism

<u>proof</u> Yoneda: A' unique if it exists

existence: enough to take $A = c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$

$$(A \otimes B)_{G_b(E)} = B_K \underset{\text{averaging}}{\cong} B^K = R\mathrm{Hom}(c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda, B)$$

$\square$

<u>Thm</u> $\forall$ any $A \in D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)^{\omega}$

$\exists !$ $A' = \mathbb{D}_{BZ}(A) \in D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ s.t

$$R\mathrm{Hom}(\mathbb{D}_{BZ}(A), B) = \pi_{\natural}(A \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} B)$$

here $\pi_{\natural} : D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda) \longrightarrow D(\Lambda)$ is the left adj to π^*

$\pi : \mathrm{Bun}_G \to *$

$R\pi_!(- \otimes R\pi^! \Lambda)$

+ biduality map $\mathbb{D}_{BZ}(\mathbb{D}_{BZ}(A)) \to A$ is an isomorphism

Q: $\pi_{L}(A \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} -)$ for general A not representable ?

For $\mathcal{U} = \mathrm{Bun}_G^b$, b basic, reduces to usual BZ-duality on $\mathrm{D\acute{e}t}(\mathrm{Bun}_G^b, \Lambda)^\omega$

$\simeq D(G_b(E), \Lambda)^\omega$

Q: How does $\mathbb{D}_{BZ}$ change the support.

<u>proof</u>. Check existence for a class of generators. Take $i^b_! [c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda]$ $\quad i^b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$

Claim: $\mathbb{D}_{BZ}(i^b_![c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda]) = A^b_K$

$R\mathrm{Hom}(A^b_K, B) = (i^{b*}B)^K$

$$\underset{??}{=} \pi_{\natural}(\underbrace{i^b_![c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda]}\otimes B)$$

up to shift

Need to check biduality i.e

$$\mathbb{D}_{BZ}(A^b_K) \overset{?}{\simeq} i^b_![c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda]$$

Ok on Bun_G^b, need to check that

after pull back to complement, LHS $=0$ ✓

$j: U \hookrightarrow \mathrm{Bun}_G$ open subset

proper

(= generalization of b)

to see: $\forall B \in D_{\acute{e}t}(U, \Lambda)$

$$R\mathcal{H}om(\mathbb{D}_{BZ}(A_K^b), Rj_*B) \stackrel{!}{=} 0$$

$$\Big\| \text{ def'n}$$

$$\pi_{\natural}(A_K^b \otimes Rj_*B)$$

$$\wr\wr$$

$$R\Gamma_c([\widetilde{\mathcal{M}}_b / \underline{K}], Rj_*B) = $$ pull back of

$\|$ cohom of $[\widetilde{\mathcal{M}}_b^\circ / \underline{K}]$ with cpt supp towards boundary of $[\widetilde{\mathcal{M}}_b / K]$, no supp condition near $[* / \underline{K}] \subseteq [\widetilde{\mathcal{M}}_b / \underline{K}]$

So it's zero by "partial cpt supp vanishing"

§3. Verdier duality $\pi : \mathrm{Bun}_G \to *$

$$A \mapsto R\mathrm{Hom}(A, \underline{\underline{R\pi^!\Lambda}})$$ is

contravariant endofunctor on $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$

Verdier duality

On $D_{\acute{e}t}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ is

just <u>smooth duality</u> (up to shift

Dualzing complex $\cong$ Haar measures

(for basic strata)

$R\pi^!\Lambda[*/H']$

$\uparrow$

$\cong \Lambda$-valued Haar measures on H'

$[*/H] \xrightarrow{\pi} *$ H local gp

Thm $\forall$ any open imm $j : U \hookrightarrow V$

of open substacks of Bun_G

$A \in D_{\acute{e}t}(U, \Lambda)$, we have

$$Rj_* RHom(A, \mathbb{D}_U) \cong RHom(j_! A, \mathbb{D}_V) \quad (1)$$

(easy)

and $$j_! RHom(A, \mathbb{D}_U) = RHom(Rj_* A, \mathbb{D}_V) \quad (2)$$

doesn't matter, just twist

Cor $A \in D_{ét}(Bun_G, \Lambda)$ is reflexive i.e

$$A \cong \mathbb{D}(\mathbb{D}(A)) \quad \text{iff}$$

$\forall\ b \in B(G)$, $i^{b*} A \in D_{ét}(Bun_G, \Lambda)$

is reflexive, i.e $(i^{b*}A)^K \in D(\Lambda)$

reflexive $\forall$ all $K \subseteq G_b(E)$ open pro-p

(here use smooth duality)

Thm $\Rightarrow$ i^{b*} commutes with $\mathbb{D}(\mathbb{D}(-))$

proof of Thm. Can assume by induction $\mathcal{U} = V \setminus Bun_G^b$

(1) clear (6-functors)

(2) clear after j^*, so enough to show it's an isom after $R\mathcal{H}om(A^b_K, -)$

As $R\mathcal{H}om(A^b_K, B) = (i^{!*}B)^K$ then LHS $= 0$

right side of (2)

just twist

$$= R\mathcal{H}om(A^b_K, R\mathcal{H}om(Rj_* A, \underline{\Lambda}))$$

$$= R\mathcal{H}om(A^b_K \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} Rj_* A, \pi^* \Lambda)$$

$$= R\mathcal{H}om(\pi_\natural(A^b_K \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} Rj_* A), \Lambda)$$

enough: $\boxed{\pi_\natural(A^b_K \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} Rj_* A) = 0}$

use the power of BZ duality

use $\mathbb{D}_{BZ}(A) = i^b_! [c\text{-Ind}^{G_b(E)}_K \Lambda]$

$$R\mathcal{H}om(\underline{i^b_!}[c\text{-Ind}^{G_b(E)}_K \Lambda], \underline{Rj_*} A) = 0$$

§4 ULA sheaves

(being ULA is cohom. smooth local on the source)

Bun_G Artin v-stack $\Rightarrow$ notion of ULA sheaves for $\pi : \mathrm{Bun}_G \to *$

↳ cohom. smooth

prop'n (consequence of "dualizability" characterisation of being ULA)

$A \in D_{et}(\mathrm{Bun}_G, \Lambda)$ is ULA

iff $p_1^* R\mathcal{H}om(A, \Lambda) \overset{\mathbb{L}}{\otimes} p_2^* A \simeq R\mathcal{H}om(p_1^* A, p_2^* A)$

$p_1, p_2 : \mathrm{Bun}_G \times \mathrm{Bun}_G \to \mathrm{Bun}_G$

Thm $A \in D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ is ULA

iff $\forall\, b \in B(G)$, $K \subseteq G_b(E)$ open pro-p

$(i^{b*} A)^K \in D(\Lambda)$ perfect complex proof of Thm

Lem Exterior $\otimes$-prod

$$- \boxtimes - \quad D_{ét}(\mathrm{Bun}_G, \Lambda) \underset{D(\Lambda)}{\otimes} D_{ét}(\mathrm{Bun}_G, \Lambda)$$

$$\longrightarrow D_{ét}(\mathrm{Bun}_G \times \mathrm{Bun}_G, \Lambda)$$

is an equiv of ∞-cats

i.e $\forall$ $A_1, A_2 \in D_{ét}(\mathrm{Bun}_G, \Lambda)^{\omega}$

also $A_1 \boxtimes A_2 \in D_{ét}(\mathrm{Bun}_G \times \mathrm{Bun}_G, \Lambda)$ is compact

and $\forall$ $B_1, B_2 \in D_{ét}(\mathrm{Bun}_G, \Lambda)$

$$R\mathrm{Hom}(A_1 \boxtimes A_2, B_1 \boxtimes B_2)$$

$$\simeq R\mathrm{Hom}(A_1, B_1) \overset{L}{\underset{\Lambda}{\otimes}} R\mathrm{Hom}(A_2, B_2)$$

proof: use cpt generators A_K^b Ω

proof of the theorem

need to check

$$p_1^* R\mathrm{Hom}(A, \Lambda) \otimes p_2^* A \simeq R\mathrm{Hom}(p_1^* A, p_2^* A)$$

apply $R\mathrm{Hom}(A_1 \boxtimes A_2, -)$ A_i cpt

get:

$$R\mathrm{Hom}\left(\boxed{\pi_{G}(A_1 \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} A)}, \Lambda\right) \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} R\mathrm{Hom}(A_2, A)$$

$$\Big\downarrow{?}\wr$$

$$R\mathrm{Hom}\left(\boxed{\pi_{G}(A_1 \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} A)}, R\mathrm{Hom}(A_2, A)\right)$$

satisfied if $\pi_{G}(A_1 \overset{\mathbb{L}}{\otimes} A) \in D(\Lambda)^{\text{perfect}}$

$\Longleftarrow$: also true, use test A_1, A_2

$A_1 = i^b_! [c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda]$ to see this perfect translates to $(i^{b*}A)^K \in D(\Lambda)$

Q. (Drinfeld) examples for SL_2 / PGL_2

$G = SL_2$:

$$< \cdot < \cdot < \cdot \quad \mathcal{O}^2$$

$\mathcal{O}(-2) \oplus \mathcal{O}(2)$ $\quad$ $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ $\quad$ (Dat) glueing of $\mathrm{Rep}(SL_2)$ and $\mathrm{Rep}(E^{\times})$

$E^{\times}$ $\quad$ $E^{\times}$ $\quad$ $Sl_2(E)$

Q: Zhu make $c\text{-}\mathrm{Ind}_K^G$ - cpt for all cpt open K

Q: Jacquet - Langlands

$$\begin{tikzcd} D_{et}(\mathrm{Bun}_G, \Lambda) \arrow[r, "T_V"] & D_{et}(\mathrm{Bun}_G, \Lambda) \arrow[d] \\ D(G(E), \Lambda) \arrow[u, hook] \arrow[r, dashed] & D_{et}(G_b(E), \Lambda) \end{tikzcd}$$

b basic

ess Jacquet-Langlands

Q: G tori explicitly.

Q: (Drinfeld) Use theory to compute Rep?

Q: T_V Hecke operator change the support.

(supercuspidal rep) only in ss locus.

Q: $\underline{\underline{\mathbb{D}_{BZ}}}$ for non-compact

whatever A is

$\underline{\pi_G(A \otimes B) \text{ is cpt}}$