01/22

Geom Satake

G/E as usual, Bun_G on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$D_{ét}(\mathrm{Bun}_G, \mathbb{Z}/l^n\mathbb{Z})$ $(l \neq p)$

Question (Drinfeld) Can one define

$D_{ét}(\mathrm{Bun}_G, \mathbb{Z}[\frac{1}{p}])$ s.t

1) $D_{ét}(\mathrm{Bun}_G, \mathbb{Z}[\frac{1}{p}]) \otimes \mathbb{Z}/l^n\mathbb{Z}$

$= D_{ét}(\mathrm{Bun}_G, \mathbb{Z}/l^n)$

2) stratified into pieces

$D_{ét}(\mathrm{Bun}_G^b, \mathbb{Z}[\frac{1}{p}]) \simeq D(G_b(E), \mathbb{Z}[\frac{1}{p}])$

partial answer : Such categories exist with

$\mathbb{Z}_l$ - coeff in particular $\mathbb{Q}_l$

But " independent of l " is not clear

Can only work canonically with $\mathbb{Z}_\ell$-coeff

implicitly know about Tate twist

$$\mathbb{Z}_\ell(1) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \overline{\mathbb{F}_q}^{\times})$$

free $\mathbb{Z}_\ell$-mod rk 1

would need $\mathbb{Z}[\frac{1}{p}]$-structure on $\mathbb{Z}_\ell(1)$'s

eg choose an isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$ $\forall \ell \neq p$

with such choice, it seems that $D_{ét}(\mathrm{Bun}_G, \mathbb{Z}[\frac{1}{p}])$ exist.

Related Fact: $\forall \ell \neq p$, have canonical Artin stack Par_G over $\mathbb{Z}_\ell$ of L-parameters, and 1-cocycles

(on Langlands dual side)

$$\begin{tikzcd} W_E \arrow[r] & \hat{G}(A) \big/ \hat{G}\text{-conj} & A \,|\, \mathbb{Z}_\ell \\ \text{tame inertia} \arrow[u] & & \end{tikzcd}$$

tame inertia is $\prod_{\ell \neq p} \mathbb{Z}_\ell(1)$ can map non-trivially to $\mathbb{Z}_\ell$-alg

If fix top generator $\tau \in \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$, can form a partially discretized version

(discrete)

$$W_E^\tau \subseteq W_E \text{ of Weil gp}$$

- replacing tame inertia by $\mathbb{Z}[\frac{1}{p}] \cdot \tau$

Then $\{W_E^\tau \to \hat{G}\} / \hat{G}$ defines an Artin stack over $\mathbb{Z}[\frac{1}{p}]$

(shall depend on τ)

Ref: Dat - Helm - Kurincouch - Mao, Zhu

Question (Drinfeld) Can one make this explicit for $G = SL_2$?

Key problem:

$$\begin{tikzcd}
 & \mathrm{Bun}_G & \\
\mathrm{Bun}_G^{b_1} \arrow[ur, "i_1"] & & \mathrm{Bun}_G^{b_2} \arrow[ul, "i_2"']
\end{tikzcd}$$

$$i_2^* R i_{1*} = ?$$

Abstract answer (using chart as last lecture)

$$\begin{tikzcd}
\tilde{M}_{b_2} \arrow[r] & \mathrm{Bun}_G \\
\cup & \cup \\
\tilde{M}^{b_1}_{b_2} \arrow[r, "f"] \arrow[loop left, "G_{b_2}(E)"] & \mathrm{Bun}_G^{b_1}
\end{tikzcd}$$

$[\pi]$ lives on $\mathrm{Bun}_G^{b_1}$

functor is given by

$$\begin{tikzcd} \pi \arrow[r, mapsto] & R\Gamma(\widetilde{\mathcal{M}}^{b_1}_{b_2}, f^*[\pi]) \\ \circlearrowright & \circlearrowright \\ G_{b_1}(E) & G_{b_2}(E) \end{tikzcd}$$

(up to some shift)

<u>Example</u> $G = SL_2$

1) $b_1 \simeq \mathcal{O}^2$ $\qquad$ $b_2 \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$

$$\begin{tikzcd} \widetilde{\mathcal{M}}^{b_1}_{b_2} & \text{param inj } \mathcal{O}(-1) \hookrightarrow \mathcal{O}^2 \text{ with cokernel } \mathcal{O}(1) \\ \widetilde{\widetilde{\mathcal{M}}}^{b_1}_{b_2} \arrow[u] & \text{trivialize also the second bundle} \end{tikzcd}$$

all saturated injection $\mathcal{O}(-1) \hookrightarrow \mathcal{O}^2$.

non saturated one extend to maps

$$\mathcal{O} \hookrightarrow \mathcal{O}^2$$

$$\Rightarrow \tilde{\mathcal{M}}^{b_1}_{b_2} = \underset{\underset{\mathbb{D}^2}{\cong}}{BC(\mathcal{O}(1))^2} \backslash \underset{\cong}{GL_2(E)\, \overset{\text{diagonal}}{\Delta}(BC(\mathcal{O}(1)))}$$

$\mathbb{P}^1(E)$ many copies of $\mathbb{D}^1$ glued at 0

Need to compute

$$R\Gamma(SL_2(E), \pi \otimes \underbrace{R\Gamma(\tilde{\mathcal{M}}^{b}_{b_2}, \Lambda)}_{\text{by excision, get } 1,\ \text{st},\ (\text{twists})})$$

2) $b_1 \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$ $\quad b_2 = \mathcal{O}(-2) \oplus \mathcal{O}(2)$

$\widetilde{\mathcal{M}}^{b_1}_{b_2}$ param saturated injections

$$\mathcal{O}(-2) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)$$

(nonsaturated ones extend to

$$\mathcal{O}(-1) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)\text{)}$$

$$\widetilde{\mathcal{M}}^{b_1}_{b_2} = BC(\mathcal{O}(1)) \times BC(\mathcal{O}(3))$$

minus image of $\underline{E} \times BC(\mathcal{O}(2)) \times BC(\mathcal{O}(1))$

$RP(\widetilde{\mathcal{M}}^{b_1}_{b_2}, \Lambda)$

can compute $\vee$ by $\quad (x, y, z) \mapsto (xz, yz)$

$\underline{\text{excision}}$

□

Now, we want to extract L-parameters

$\varphi : W_E \longrightarrow \hat{G}$ 1-cocycles

need to make dual gp $\hat{G}$ appears

Idea "Spetral information" always arise as "eigenvalues" of Hecke operators acting on $D_{et}(Bun_G, \Lambda)$

+ Hecke operators are enumerated by $\hat{G}$

Hecke operators

Warning: not related to elements of classical Hecke alg

notions are different.

$\Lambda[K \backslash G(E)/K]$

<u>Definition</u> Hecke_G small v-stack on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$$\mathrm{Hecke}_G(S) = \{ (\mathcal{E}_1, \mathcal{E}_2, S^\#, f) \mid$$

$\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}_G(S)$

$S^\# \in \mathrm{Div}_X^1(S)$ untilt of S over E up to Frob

$$\begin{tikzcd} S^\# \arrow[d, hook] \\ X_S \end{tikzcd}$$

$$f : \mathcal{E}_1 |_{X_S \setminus S^\#} \cong \mathcal{E}_2 |_{X_S \setminus S^\#}$$

meromorphic at $S^\#$

it's infinite dim, need to bound f to get finiteness

$$\begin{tikzcd}
 & \mathrm{Hecke}_{G, \leq \mu} \arrow[dl, "h_1"', "\substack{\text{proper} \\ \text{rep in} \\ \text{spatial diam}}"] \arrow[dr, "h_2"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Div}^1
\end{tikzcd}$$

$\rightsquigarrow$ operators like

$$Rh_{2*}h_1^* : \mathcal{D}_{ét}(\mathrm{Bun}_G, \Lambda)$$

$$\big\downarrow$$

$(\mathrm{Spa}\,\hat{E})^\diamond / \underline{W_E}$

Q: Function-sheaf dictionary ?

$$\mathcal{D}_{ét}(\mathrm{Bun}_G \times \boxed{\mathrm{Div}^1}, \Lambda)$$

$$\cong \mathcal{D}_{ét}(\mathrm{Bun}_G, \Lambda)^{W_E}$$

by invariance of $\mathcal{D}_{ét}(\mathrm{Bun}_G, \Lambda)$ along base change $(\mathrm{Spa}\,\hat{E})^\diamond \to (\mathrm{Spa}\,\bar{\mathbb{F}}_q)^\diamond$

Slightly better if we insert kernels on Hecke_G

Thm ∃ canonical exact functor

Q: no assump on G e.g quasi-split

(1st incarnation of geom. Satake)

$$\mathrm{Rep}_\Lambda(\hat{G}) \longrightarrow D_{ét}(\mathrm{Hecke}_G, \Lambda)$$

$$V \longmapsto \mathcal{S}_V$$

⇝ get $T_V := Rh_{2*}(h_1^* \otimes \mathcal{S}_V)$

$$D_{ét}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{ét}(\mathrm{Bun}_G, \Lambda)^{W_E}$$

and it's monoidal i.e $T_W \circ T_V \cong T_{V \otimes W}$

Statement of geometric Satake

(Mirkovic–Viloen, Lusztig, Ginzburg, ..)

Usual Set up:

$G/\mathbb{C}$ reductive gp

$$\begin{tikzcd} & & A \arrow[d, mapsto] \\ L^+G \arrow[d, phantom, sloped, "\subset"] & \text{positive loop gp} & G(A[[t]]) \\ LG & \text{loop gp} & G(A((t))) \end{tikzcd}$$

<u>Def'n</u> Affine Grassmanian

$$Gr_G = LG/L^+G$$

$$A \longmapsto \{ \mathcal{E} \ \ G\text{-torsor on } A[[t]] \text{ trivalized over } A((t)) \}$$

is an ind-scheme

+ transition map closed imm

+ each scheme at finite step is projective$/\mathbb{C}$

Def'n $\mathrm{Sat}_G = \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \mathbb{Z})$

Satake category of L^+G - equiv perverse sheaves

Note $L^+G(\mathbb{C}) \backslash LG(\mathbb{C}) / L^+G(\mathbb{C}) \cong X_*^+$

$$\mu(t) \longleftarrow\!\shortmid \mu$$

dominant cochar

closure of L^+G - orbit of $\mu(t)$ is a proj scheme

$$\mathrm{Gr}_{G,\le\mu} \hookrightarrow \mathrm{Gr}_G$$

(affine) Schurbert variety $\mathrm{Gr}_{G,\mu} = L^+G.\mu$

In particular, $\forall$ each μ, get

$$IC_\mu = IC_{Gr_{G,\le\mu}} \in Sat_G$$

$$\begin{tikzcd}
\simeq \mathrm{im}\big({}^p j_{\mu!}\mathbb{Z}[d_\mu] \arrow[r] \arrow[d, equal] & {}^p j_{\mu*}\mathbb{Z}[d_\mu] \big) \arrow[d, equal] \\
\Delta_\mu & \nabla_\mu
\end{tikzcd}$$

$j_\mu : Gr_\mu \hookrightarrow Gr_{G,\le\mu}$

(standard and costandard objs)

$$d_\mu = \dim Gr_\mu = \langle 2\rho, \mu \rangle$$

With $\mathbb{Q}$-coefficients, we have

$$\begin{tikzcd}
{}^p j_{\mu!}\mathbb{Q}[d_\mu] \arrow[rr, phantom, "\simeq"] \arrow[dr, "\sim"] & & {}^p j_{\mu*}\mathbb{Q}[d_\mu] \\
& IC_{\mu,\mathbb{Q}} \arrow[ur, "\sim"] &
\end{tikzcd}$$

but not with $\mathbb{Z}$-coeff

$\rightsquigarrow$ Sat_G a structure of "highest weight category"

with weights given by X_*^+

simple objs $= \{ IC_\mu \}$

With $\mathbb{Q}$-coeff, semi-simple

Def'n Convolution monoidal structure on Sat_G

$$A * B := Rm_* \pi^* (A \boxtimes B)$$

$$\begin{tikzcd}
L^+G \backslash LG / L^+G \times L^+G \backslash LG / L^+G \\
L^+G \backslash LG \overset{L^+G}{\times} LG / L^+G \arrow[u, "\pi"] \arrow[d, "m"] \\
L^+G \backslash LG / L^+G
\end{tikzcd}$$

Thm (Mirkovic - Viloen)

$$(\mathrm{Sat}, \star) \xrightarrow{\bigoplus H^i(\mathrm{Gr}_G, -)} (\mathrm{Vect}, \otimes)$$

is a fibre functor,

$(\mathrm{Sat}, \star)$ can be upgraded to a symm mon. structure making $\bigoplus H^i(\mathrm{Gr}_G, -)$ a sym monoidal functor

corresponding Tannakian group is $\hat{G}$,

so

$$\begin{tikzcd}
(\mathrm{Sat}_G, \star) \arrow[rr, phantom, "\cong"] \arrow[dr, "\bigoplus H^i(\mathrm{Gr}_G, -)"'] & & (\mathrm{Rep}\,\hat{G}, \otimes) \arrow[dl, "\text{forget}"] \\
& (\mathrm{Vect}, \otimes) &
\end{tikzcd}$$

with $\mathbb{Q}$-coefficient, $IC_\mu \cong V_\mu$ highest weight rep of $\hat{G}$ of weight $\mu \in X_*^+(G) = X^*_+(\hat{G})$

with $\mathbb{F}_p$ - coeff $\quad IC_\mu \cong L_\mu$ irr rep

${}^p j_{\mu!} \mathbb{F}_p[d_\mu] \cong \Delta_\mu$ standard rep

${}^p j_{\mu *} \mathbb{F}_p[d_\mu] \cong \nabla_\mu$ costandard rep

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want a version for B^+_{dR} - affine Grass-manian

Def'n G/E as usual, Div^1

$$\mathrm{Gr}_G \longrightarrow \mathrm{Div}^1$$

param. $S^\# \in \mathrm{Div}^1(S)$

+ G - torsor $\mathcal{E}$ at completion of X_S at $S^\#$

+ trivialization on $\underline{(X_S)^\wedge_{S^\#}} \setminus S^\#$

what does this mean ?

Cheating ...

only define it for $S = \mathrm{Spa}(R, R^+)$ affinoid

Q: why? A: in general, choose an affinoid neighborhood

then $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ also affinoid

$$\theta : W_{O_E}(R^+)\left[\frac{1}{[\varpi]}\right] \twoheadrightarrow R^\#$$

$$\underset{\overset{\wr\wr}{O((X_S)^\wedge_{S^\#})}}{B^+_{dR}(R^\#)} := (\underbrace{\ker\theta}_{\text{principal } = (\xi)})\text{-adic completion of } W_{O_E}(R^+)\left[\frac{1}{[\varpi]}\right]$$

(isom to completing A at $R^\#$ for $S^\# \subseteq \mathrm{Spa}(A, A^+) \subseteq X_S$)

$$B_{dR}(R^\#) = B^+_{dR}(R^\#)\left[\frac{1}{\xi}\right]$$

$$\wr\wr$$

$$O\left((X_S)^\wedge_{S^\#} \setminus S^\#\right)$$

$\mathrm{Gr}_G = LG / L^+G$,

$S \mapsto \{S^\#, G\text{-torsor on } B^+_{dR}(R^\#), \text{ trivialized over } B_{dR}(R^\#)\}$

$$\begin{tikzcd} LG : & R^\# \arrow[r, mapsto] & G(B_{dR}(R^\#)) \\ L^+G : \arrow[u, phantom, "\cup" rotate=90] & R^\# \arrow[r, mapsto] & G(B^+_{dR}(R^\#)) \end{tikzcd}$$

Def'n local Hecke stack

$$\begin{tikzcd} \mathrm{Hck}_G = L^+G \backslash \mathrm{Gr}_G \arrow[d] \\ \mathrm{Div}^1 \end{tikzcd}$$

$$= L^+G \backslash LG / L^+G$$

we will define

$$\mathrm{Sat}_G(\Lambda) = \mathrm{Perv}(\mathrm{Hck}_G, \Lambda)$$

with convolution monoidal cat

it's a $\mathrm{Rep}_{W_E}(\Lambda)$ -linear cat

via tensoring with pullback of local system on Div^1

i.e an element in $\mathrm{Rep}_{W_E}(\Lambda)$

Thm $(\mathrm{Sat}(\Lambda), \star)$ upgrades naturally to a sym. monoidal category with

$$\bigoplus H^i(\mathrm{Gr}_{G,\bar{\eta}}, -):$$

$$\underset{G}{\mathrm{Sat}(\Lambda)} \longrightarrow \mathrm{Rep}_{W_E}(\Lambda)$$

(because if G is non-split we need to concern about W_E action so not just Vect.$\otimes$

Tanakian dual is given by $\hat{G}$ with W_E-action that can be made explicit

(a cyclotomic twist of usual action)

$\Rightarrow$

$$\begin{tikzcd}
(\mathrm{Sat}_G(\Lambda), \star) \arrow[rr, phantom, "\simeq"] \arrow[dr, "\text{cohom}"'] & \arrow[d, phantom, "\circlearrowleft"] & (\mathrm{Rep}\,\hat{G}, \otimes) \arrow[dl, "\text{forget}"] \\
 & (\mathrm{Rep}_{W_E}(\Lambda), \otimes) &
\end{tikzcd}$$

internally in $\mathrm{Rep}_{W_E}(\Lambda)$

Fix G/O_E reductive

Cor (different proof) Zhu's geometric Satake for Witt vector affine Gr. (Zhu: $\bar{\mathbb{Q}}_\ell$)
Yu: $\bar{\mathbb{Z}}_\ell$

Use: <u>degeneration</u>

$$\begin{tikzcd}
\left(\mathrm{Gr}_G^{\mathrm{Witt}}\right)^{\diamond} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Gr}_G^{\mathrm{BD}} \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Gr}_G \times_{\mathrm{Div}^1} \mathrm{Spa}\,\breve{E} \arrow[l] \arrow[d] \\
\mathrm{Spa}\,\mathbb{F}_q \arrow[r] & \mathrm{Spa}\,\mathcal{O}_{\breve{E}} & \mathrm{Spa}\,\breve{E} \arrow[l]
\end{tikzcd}$$

+ use nearby cycles / formalism of ULA sheaves

to "specialize" perverse sheaves

"Symmetric monoidal structure comes from fusion": Let <u>two points</u> on curve $1 \rightarrow \bullet \leftarrow 2$

requires a space like

$\mathrm{Spec}\, \mathbb{Q}_p \times \mathrm{Spec}\, \mathbb{Q}_p$ (which doesn't make sense)

$E = \mathbb{Q}_p$

But this exists in the world of diamonds,

$\mathrm{Div}^1 \times \mathrm{Div}^1$ is a surface

Q: (Tony) line bundle on Gr_G

doesn't make sense

Q: (Kestutis) construction of endoscopic groups from Satake Category?

Q: (Zhiyu) no assumption on G e.g. unramified ?

Q: (zhiyu)

$$\begin{tikzcd} \overline{\mathbb{Q}}_\ell \arrow[r] & \overline{\mathbb{Z}}_\ell \\ \text{semisimple} \arrow[u] & \end{tikzcd}$$

and decomp then here

semisimple ← no decomposition thm for perfectoid

we use degeneration to Witt Cr

Q: (zhiyu) vanishing result for cohomology ?

JL

Q: (zhiyu) function-sheaf dictionary

no in perf world

just rep theory