

01/22

Geom Satake

$G/E$  as usual,  $\text{Bun}_G$  on  $\text{Perf } \overline{\mathbb{F}}_q$

$$\text{Dét}(\text{Bun}_G, \mathbb{Z}/\ell^n \mathbb{Z}) \quad (\ell \neq p)$$

Question (Drinfeld) Can one define

$$\text{Dét}(\text{Bun}_G, \mathbb{Z}[\frac{1}{p}]) \quad \text{s.t.}$$

$$1) \quad \text{Dét}(\text{Bun}_G, \mathbb{Z}[\frac{1}{p}]) \otimes \mathbb{Z}/\ell^n \mathbb{Z} \\ = \text{Dét}(\text{Bun}_G, \mathbb{Z}/\ell^n \mathbb{Z})$$

2) stratified into pieces

$$\text{Dét}(\text{Bun}_G^b, \mathbb{Z}[\frac{1}{p}]) \cong D(G_b(E), \mathbb{Z}[\frac{1}{p}])$$

partial answer: Such categories exist with

$\mathbb{Z}_\ell$ -coeff in particular  $\mathbb{Q}_\ell$

But "independent of  $\ell$ " is not clear

Can only work canonically with  $\mathbb{Z}_l$ -coeff  
implicitly know about Tate twist

$$\mathbb{Z}_l(1) = \text{Hom}(\mathbb{Q}_l / \mathbb{Z}_l, \overline{\mathbb{F}_q}^\times)$$

free  $\mathbb{Z}_l$ -mod rk 1

would need  $\mathbb{Z}[\frac{1}{p}]$ -structure on  $\mathbb{Z}_l(1)$ 's

eg choose an isomorphism  $\mathbb{Z}_l(1) \cong \mathbb{Z}_l$   
 $\forall l \neq p$

with such choice, it seems that

$D_{\text{ét}}(\text{Bun}_G, \mathbb{Z}[\frac{1}{p}])$  exist.

Related Fact :  $\forall l \neq p$ , have canonical  
(on Langlands dual side) Artin stack  $\text{Par}_G$  over  $\mathbb{Z}_l$   
of  $L$ -parameters, ant 1-cycles

$$W_E \longrightarrow \widehat{G}(A) / \widehat{G}\text{-conj} \quad A / \mathbb{Z}_\ell$$

tame inertia is  $\prod_{\ell \neq p} \mathbb{Z}_\ell(1)$  can map non-trivially to  $\mathbb{Z}_\ell\text{-alg}$

If fix top generator  $\tau \in \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$ , can form a partially discretized version (discrete)

$$W_E^\tau \subseteq W_E \text{ of Weil gp}$$

• replacing tame inertia by  $\mathbb{Z}[\frac{1}{p}] \cdot \tau$

Then  $\{ W_E^\tau \rightarrow \widehat{G} \} / \widehat{G}$  defines an

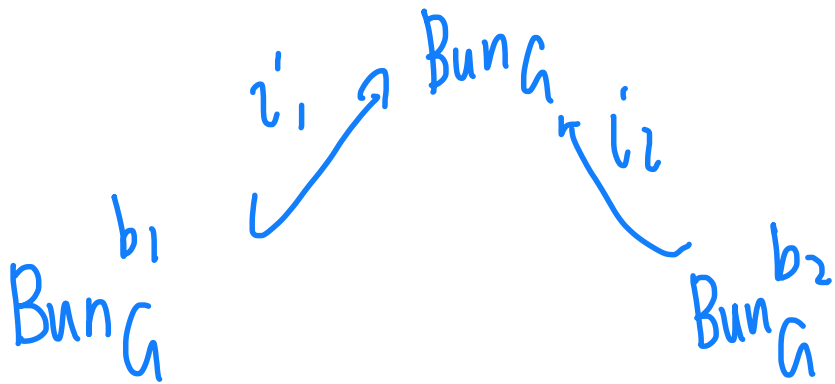
Artin stack over  $\mathbb{Z}[\frac{1}{p}]$

(shall depend on  $\tau$ )

Ref: Dat - Helm - Kurincouch - Mas, Zhu

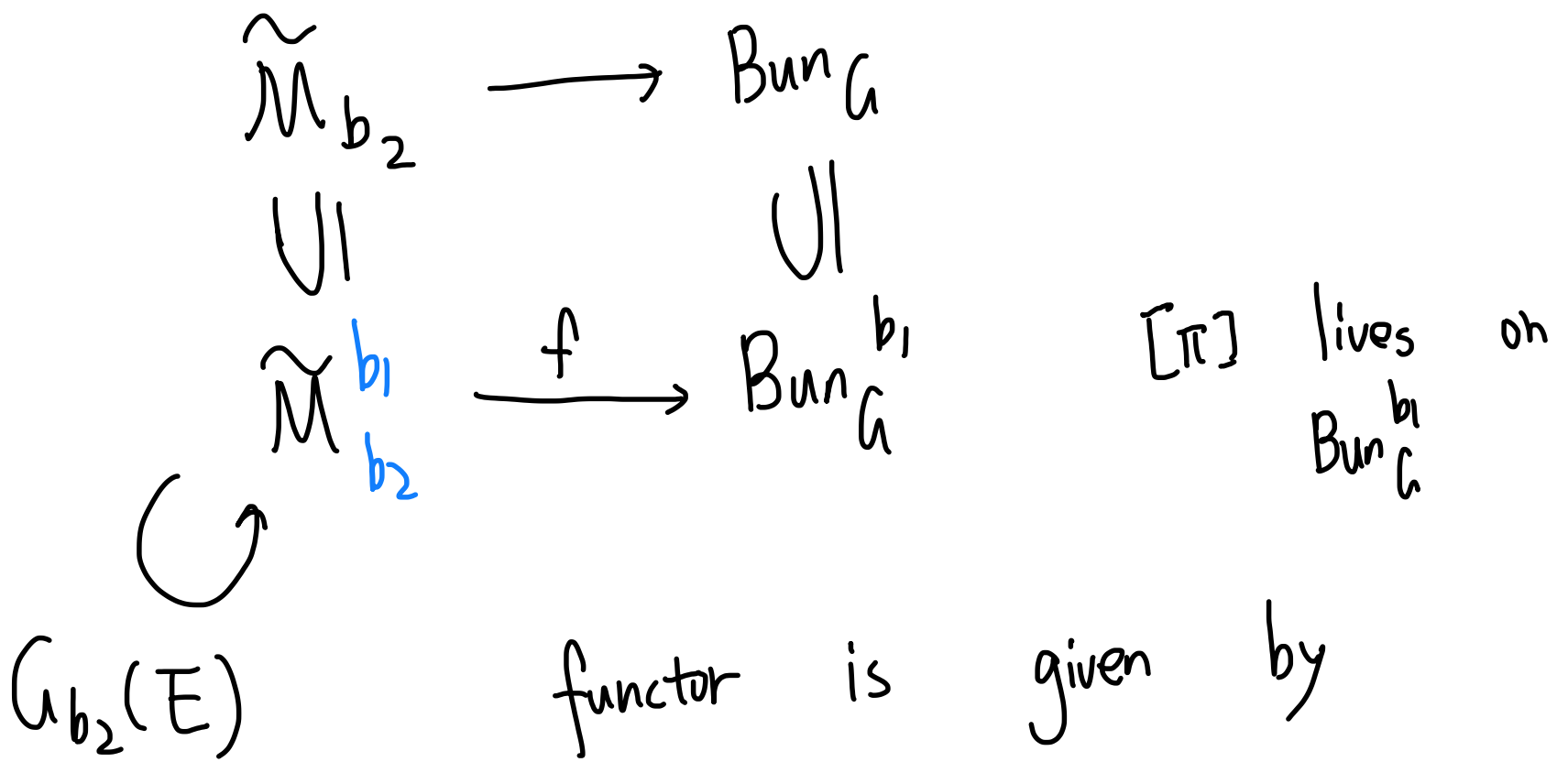
Question (Drinfeld) Can one make this explicit  
for  $G = SL_2$  ?

Key problem:



$$i_2^* R i_{1*} = ?$$

Abstract answer (using chart as last lecture)





non saturated one extend to maps

$$O \hookrightarrow O^2$$

diagonal  
↓

$$\Rightarrow \tilde{M}_{b_1, b_2}^{\approx} = BC(O(1))^2 \setminus GL_2(E) \Delta(BC(O(1)))$$

$\mathbb{P}^2$

$\mathbb{P}^2$

$\mathbb{P}^2$

$\mathbb{P}^1(E)$

many  
copies of

$\mathbb{P}^1$

glued

at 0

Need to compute

$$RT(SL_2(E), \pi \otimes RT(\tilde{M}_{b_1, b_2}^{\approx}, \Lambda))$$

by excision, get 1, st,  
(twists)

$$2) \quad h_1 \cong O(-1) \oplus O(1) \quad h_2 = O(-2) \oplus O(2)$$

$\tilde{M}_{b_1, b_2}$  param saturated injections

$$O(-2) \hookrightarrow O(-1) \oplus O(1)$$

(nonsaturated ones extend to)

$$O(-1) \hookrightarrow O(-1) \oplus O(1)$$

$$\tilde{M}_{b_1, b_2} \cong BC(O(1)) \times BC(O(3))$$

minus image of  $\underline{E} \times BC(O(2))$

$\times BC(O(1))$

$RP(\tilde{M}_{b_1, b_2}, \Lambda)$

can compute  $v$  by

$(x, y, z)$

$\mapsto (xz, yz)$

excision

□

Now, we want to extract L-parameters

$$\mathcal{Y}: W_E \rightarrow \widehat{G} \quad \text{1-cocycles}$$

need to make dual gp  $\widehat{G}$  appears

Idea "Spectral information" always arise  
as "eigenvalues" of Hecke operators

acting on  $\text{Det}(\text{Bun}_G, \Lambda)$

+ Hecke operators are enumerated by  $\widehat{G}$

Hecke operators

Warning: not related to elements of

classical Hecke alg

notions

are different.

$$\Lambda [K \setminus G(E)/K]$$



Definition

Hecke<sub>G</sub> small V-stack  
on Perf $\overline{\mathbb{F}}_q$

$$\text{Hecke}_G(S) = \{ (\mathcal{E}_1, \mathcal{E}_2, S^\#, f) \mid$$

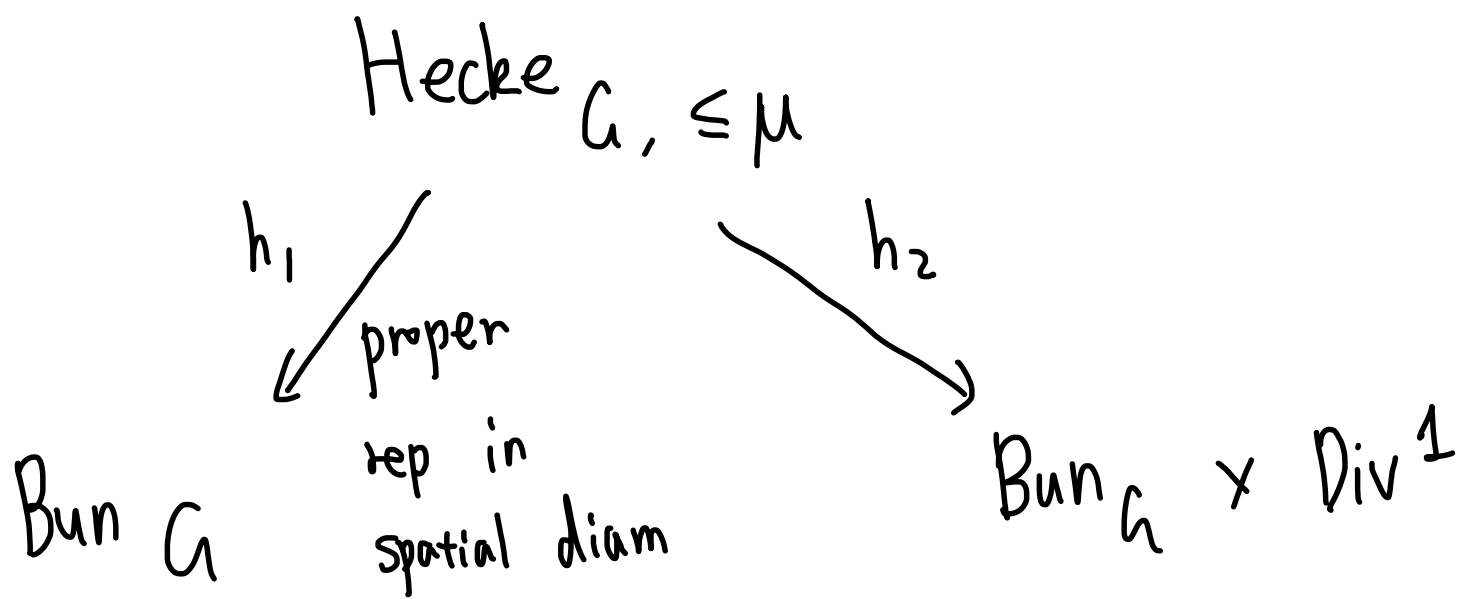
$$\mathcal{E}_1, \mathcal{E}_2 \in \text{Bun}_G(S)$$

$S^\# \in \text{Div}_X^1(S)$  unlift of  $S$  over  $E$   
up to Frob  
 $\downarrow$   
 $X_S$

$$f: \mathcal{E}_1|_{X_S \setminus S^\#} \cong \mathcal{E}_2|_{X_S \setminus S^\#}$$

morphism at  $S^\#$

it's infinite dim, need to bound  $f$   
to get finiteness



$\rightsquigarrow$  operators like

$$R h_{2*} h_1^* : \mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda)$$

Q: Function-sheaf dictionary?

$$\downarrow$$

$$\mathcal{D}_{\text{ét}}(\text{Bun}_G \times \text{Div}^1, \Lambda)$$

$(\text{Spa} \widehat{E})^\diamond / \underline{W_E}$

$$\cong \mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda)^{W_E}$$

by invariance of  $\mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda)$  along base change  $(\text{Spa} \widehat{E})^\diamond \rightarrow (\text{Spa} \overline{F}_q)^\diamond$

Slightly better if we insert kernels on Hecke  $G$

Thm  $\exists$  canonical exact functor  $Q$ :  
(1st incarnation of geom. Satake) no assump on  $G$  e.g. quasi-split

$$\text{Rep}_\Lambda(\hat{G}) \longrightarrow \text{Dét}(\text{Hecke}_G, \Lambda)$$
$$V \longmapsto \mathcal{S}_V$$

$\rightsquigarrow$  get  $T_V := R h_{2*} (h_1^* \otimes \mathcal{S}_V)$

$\text{Dét}(\text{Bun}_G, \Lambda) \longrightarrow \text{Dét}(\text{Bun}_G, \Lambda)^{W_E}$

and it's monoidal i.e.  $T_W \circ T_V \cong T_{V \otimes W}$

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Statement of geometric Satake  
(Mirkovic-Vilonen, Lusztig, ... Ginzburg)

Usual Set up:

$$G/\mathbb{C}$$

reductive gp

$$A$$

$$\downarrow$$

$$L^+G$$

positive

loop

gp

$$G(A[[t]])$$

$$\cap$$

$$LG$$

loop

gp

$$G(A((t)))$$

Def'n

Affine

Grassmanian

$$Gr_G = LG/L^+G$$

$A \mapsto \{ \mathcal{E} \text{ } G\text{-torsor on } A[[t]] \}$   
 trivialized over  $A((t))$

is an ind-scheme

+ transition map closed imm  
 + each scheme at finite step is projective/ $\mathbb{C}$

Def'n

$$\text{Sat}_G = \text{Perv}_{L^+G}(\text{Gr}_G, \mathbb{Z})$$

Satake category of  $L^+G$ -equiv

perverse sheaves

Note  $L^+G(\mathbb{C}) \setminus LG(\mathbb{C}) / L^+G(\mathbb{C}) \cong X_{\mu}^+$

$\mu(t)$

$\longleftarrow \mu$

dominant char

closure of  $L^+G$ -orbit of  $\mu(t)$

is a proj scheme

$$\text{Gr}_{G, \leq \mu} \hookrightarrow \text{Gr}_G$$

(affine) Schubert variety  $\text{Gr}_{G, \mu} = L^+G \cdot \mu$

In particular,  $\forall$  each  $\mu$ , get

$$IC_{\mu} = IC_{Gr_{G, \leq \mu}} \in \text{Sat}_G$$

$$\simeq \text{im} \left( P_{j_{\mu}!} \mathbb{Z}[d_{\mu}] \rightarrow P_{j_{\mu*}} \mathbb{Z}[d_{\mu}] \right)$$

$$\parallel$$

$$\Delta_{\mu}$$

$$\parallel$$

$$\nabla_{\mu}$$

$$j_{\mu}: Gr_{\mu} \hookrightarrow Gr_{G, \leq \mu}$$

(standard and costandard objs)

$$d_{\mu} = \dim Gr_{\mu} = \langle 2\rho, \mu \rangle$$

With  $\mathbb{Q}$ -coefficients, we have

$$P_{j_{\mu}!} \mathbb{Q}[d_{\mu}] \simeq P_{j_{\mu*}} \mathbb{Q}[d_{\mu}]$$

$$\searrow \quad \nearrow$$

$$IC_{\mu, \mathbb{Q}}$$

but not with  $\mathbb{Z}$ -coeff

$\leadsto \text{Sat}_G$  a structure of "highest weight category"

with weights given by  $X_{\star}^{+}$

simple objs =  $\{ I\mu \}$

With  $\mathbb{Q}$ -coeff, semi-simple

Def'n Convolution monoidal structure on  $\text{Sat}_G$   
 $A \star B := Rm_{\star} \pi^*(A \boxtimes B)$

$$L^+G \setminus L_G / L^+G \times L^+G \setminus L_G / L^+G$$

$\uparrow \pi$

$$L^+G \setminus L_G \overset{L^+G}{\times} L_G / L^+G$$

$\downarrow m$

$$L^+G \setminus L_G / L^+G$$

Thm (Mirkovic - Vilonen)

$$(Sat, \star) \xrightarrow{\oplus H^i(Gr_G, -)} (Vect, \otimes)$$

is a fibre functor,

$(Sat, \star)$  can be upgraded to a

Symm mon. structure making  $\oplus H^i(Gr_G, -)$  a sym monoidal functor

corresponding Tannakian group is  $\hat{G}$ ,

so  $(Sat_G, \star) \cong (Rep \hat{G}, \otimes)$

$$\begin{array}{ccc} \oplus H^i(Gr_G, -) & & \\ \searrow & & \swarrow \text{forget} \\ & (Vect, \otimes) & \end{array}$$

with  $\mathbb{Q}$ -coefficient,  $IC_\mu \cong V_\mu$  highest weight rep of  $\hat{G}$  of weight  $\mu \in X_*^+(\hat{G}) = X_+^*(\hat{G})$



with  $\mathbb{F}_p$ -coeff  $IC_\mu \cong L_\mu$  in rep

$P_{j,\mu}! \mathbb{F}_p[d_\mu] \cong \Delta_\mu$  standard rep

$P_{j,\mu} \mathbb{F}_p[d_\mu] \cong \nabla_\mu$  costandard rep

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want a version for  $B_{dR}^+$ -affine Grassmanian

Def'n  $G/E$  as usual,  $Div^1$

$Gr_G \longrightarrow Div^1$

param,  $S^\# \in Div^1(S)$

+  $G$ -torsor  $\mathcal{E}$  at completion of  $X_S$  at  $S^\#$

+ trivialization on  $\underline{(X_S)_{S^\#}^\wedge} \setminus S^\#$

what does this mean?

Cheating ...

only define it for  $S = \text{Spa}(R, R^+)$  affinoid

Q: why? A: in general, choose an affinoid neighborhood

then  $S^\# = \text{Spa}(R^\#, R^{\#\dagger})$  also affinoid

$$\theta: W_{0,E}(R^+) \left[ \frac{1}{[t_0]} \right] \twoheadrightarrow R^\#$$

$B_{\text{dR}}^+(R^\#) := (\underbrace{\text{Ker } \theta}_{\text{principal} = (s)})$  - adic completion of

$$O((X_S)_{S^\#}^\wedge)$$

principal = (s)

$$W_{0,E}(R^+) \left[ \frac{1}{[t_0]} \right]$$

(isom to completing  $A$  at  $R^\#$  for

$$S^\# \subseteq \text{Spa}(A, A^\dagger) \subseteq X_S)$$

$$B_{\text{dR}}(R^\#) = B_{\text{dR}}^+(R^\#) \left[ \frac{1}{s} \right]$$

$$O((X_S)_{S^\#}^\wedge \setminus S^\#)$$

$$\text{Gr}_G = LG / L^+G,$$

$$LG : \mathbb{R}^\# \mapsto G(B_{\text{dR}}(\mathbb{R}^\#))$$

$$\cup \\ L^+G : \mathbb{R}^\# \mapsto G(B_{\text{dR}}^+(\mathbb{R}^\#))$$

$S \mapsto \{S^\#, G\text{-torsor on } B_{\text{dR}}^+(\mathbb{R}^\#), \text{ trivialized over } B_{\text{dR}}(\mathbb{R}^\#)\}$

Def'n

local Hecke stack

$$\text{Heck}_G = L^+G \setminus \text{Gr}_G$$

$$= L^+G \setminus LG / L^+G$$

↓  
Div 1

we will define

$$\text{Sat}_G(\Lambda) = \text{Perv}(\text{Heck}_G, \Lambda)$$

with convolution monoidal cat

it's a  $\text{Rep}_{WE}(\Lambda)$  - linear cat

via tensoring with pull back of local system on  $\text{Div}^1$

i.e. an element in  $\text{Rep}_{WE}(\Lambda)$

Thm  $(\text{Sat}(\Lambda), \star)$  upgrades naturally to a sym. monoidal category with

$$\bigoplus H^i(\text{Gr}_{a, \bar{y}}, -):$$

$$\text{Sat}(\Lambda)_{\mathfrak{h}} \longrightarrow \underline{\text{Rep}_{WE}(\Lambda)}$$

(because if  $\mathfrak{g}$  is non-split

we need to concern about  $WE$  action

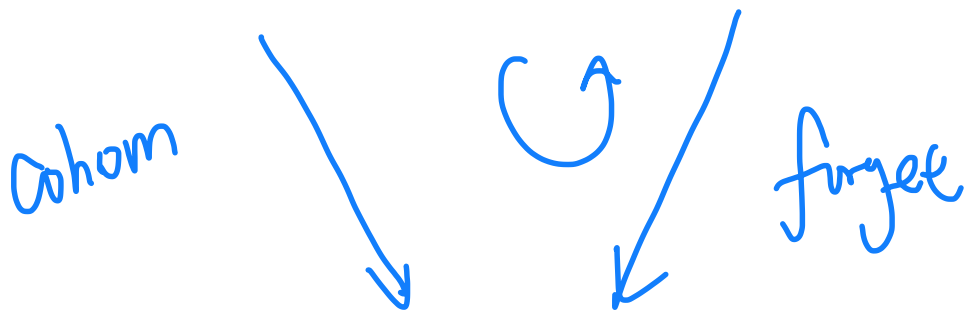
so not just  $\text{Vect}$ .  $\otimes$

Tamkian dual is given by  $\hat{G}$   
 with  $W_E$ -action that can be made explicit

(a cyclotomic twist)  
 of usual action

$\Rightarrow$

$$(\text{Sat}_G(\Lambda), \star) \cong (\text{Rep } \hat{G}, \otimes)$$



$$(\text{Rep}_{W_E}(\Lambda), \otimes)$$

internally  
 in  $\text{Rep}_{W_E}(\Lambda)$

Fix  $G/O_E$  reductive

Cor

Zhu's geometric

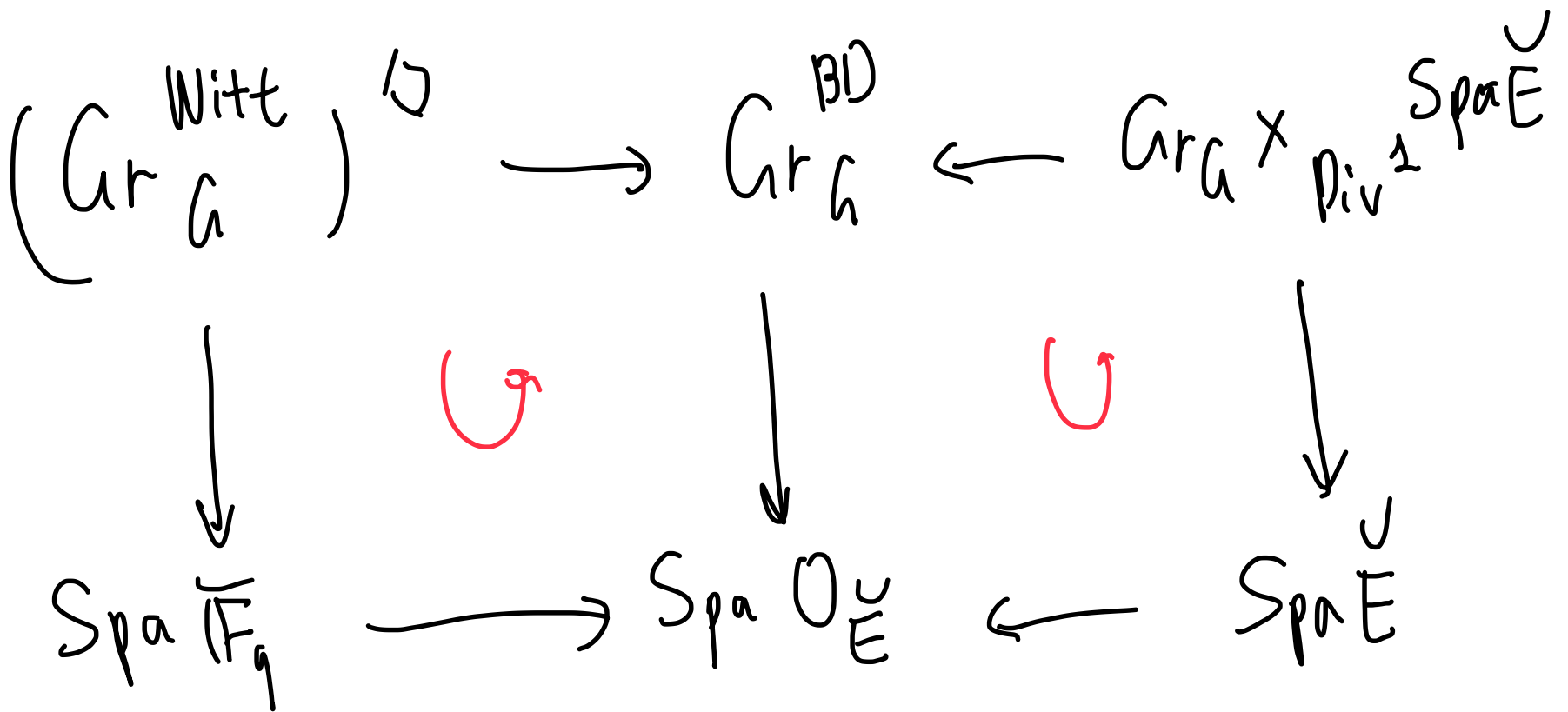
Satake for

(different  
 proof)

Witt vector affine

Gr, (Zhu:  $\overline{\mathbb{Q}}$ )  
 Yu:  $\overline{\mathbb{Z}_\ell}$

Use: degeneration



+ use nearby cycles / formalism of ULA sheaves

to "specialize" perverse sheaves

"Symmetric monoidal structure comes from fusion": Let two points on

cube  $i \rightarrow \bullet \leftarrow i$

requires a space like

$$\text{Spec } \mathbb{Q}_p \times \text{Spec } \mathbb{Q}_p$$

(which doesn't make sense)

$$E = \mathbb{Q}_p$$

But this exists in the world of diamonds

$\mathbb{P}^1 \times \mathbb{P}^1$  is a surface

$Q_i$  (Tony) line bundle on  $\text{Gr}_G$

doesn't make sense

$Q_i$  (Kestatis) construction of endoscopic groups

from Satake Category?

$Q_i$  (Zhiyu) no assumption on  $G$  e.g. unramified?

Q:  $\overline{\mathbb{Q}_L} \rightarrow \overline{\Sigma}_L$  and decomp them here

↑  
semi simple

← no decomposition thm  
for pappaloid

we use degeneration to Witt-Cr

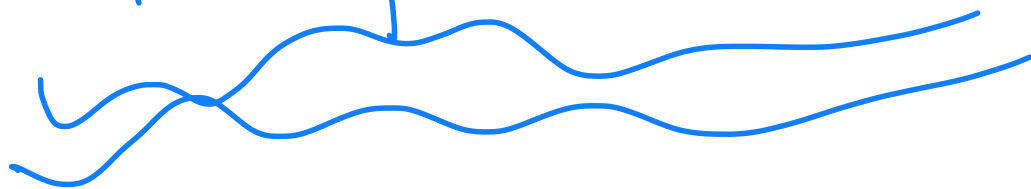
Q: (zhijun)

vanishing result for cohomology?

function-sheaf dictionary

Q: (zhijun)

no in perf world



just rep theory