01/25 # Geometric Satake

Thm (Roughly) $(\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \mathbb{Z}_\ell), \star)$
$\cong (\mathrm{Rep}\,\hat{G}, \otimes)$

Two preparations
1) perverse sheaves
2) hyperbolic localization (important)

Usual setting: X separated scheme of finite type over an alg closed field k
why?

Λ ring s.t $n\Lambda = 0$ for some $n \in k^\times$
(or $\Lambda = \overline{\mathbb{Q}}_\ell$)

$D_{\acute{e}t}(X, \Lambda) = D(X_{\acute{e}t}, \Lambda)$ compactly generated

cpt objects $=: D^b_{c, f\,\mathrm{tor}}(X_{\acute{e}t}, \Lambda)$: bounded complex constructible cohomology finite tor dim $/\Lambda$

$$\cap\!\shortmid$$

$$D^b_c(X_{\acute{e}t}, \Lambda)$$

<u>Def'n</u> 1) ${}^pD^{\le 0}_{ét}(X,\Lambda) \subseteq D_{ét}(X,\Lambda)$ full subcat of all $A \in D_{ét}(X,\Lambda)$ s.t $\forall$ geom pts $\bar{x} \longrightarrow X$, $A_{\bar{x}} \in D^{\le -d(\bar{x})}(\Lambda)$

(not just closed pts) where $d(\bar{x}) = \dim \overline{\{x\}} = \text{tr deg } k(\bar{x})/k$ ☆

2) ${}^pD^{\le n} := {}^pD^{\le 0}[-n]$

3) ${}^pD^{\ge 0}$ right orth of ${}^pD^{\le -1}$ i.e

$$B \in {}^pD^{\le 0} \iff \forall \text{ all } A \in {}^pD^{\le -1} \quad \operatorname{Hom}(A,B)=0$$

4) ${}^pD^{\ge n} := {}^pD^{\ge 0}[-n]$

<u>Thm</u> ① $({}^pD^{\ge 0}, {}^pD^{\le 0})$ defines a t-structure on $D_{ét}(X,\Lambda)$ (usually on D^b)

$\rightsquigarrow$ $\exists$ functors ${}^p\tau^{\ge n}, {}^p\tau^{\le n}$: $D(X_{et},\Lambda) \downarrow$ left resp. right adj to inclusion ${}^pD^{\ge n}$, ${}^pD^{\le n}$

and ${}^p\tau^{\le 0}A \longrightarrow A \longrightarrow {}^p\tau^{\ge 1}A$ dist. triangle

2) $A \in D_{ét}(X,\Lambda)$ lies in ${}^pD^{\geq 0}_{ét}(X,\Lambda)$

iff $\forall\ \bar{x} \xrightarrow{i_{\bar{x}}} X$ geom pts

$$R i^!_{\bar{x}} A \in D^{\geq -d(\bar{x})}(\Lambda)$$

here $\bar{x} \overset{j_{\bar{x}}}{\hookrightarrow} \overline{\{x\}} \overset{i}{\hookrightarrow} X$ $\quad R i^!_{\bar{x}} = j^*_{\bar{x}} R i^! A$

3) It induces a t-structure on

$D^b_c(X,\Lambda)$ (equiv. truncation ${}^p\tau^{\leq 0}, {}^p\tau^{\geq 0}$ preserves the subcat)

They don't preserve $D^b_{c,ftor}$ e.g $X = \operatorname{Spec} k$

truncations of perfect Λ-complex may not be perfect

OK if Λ is regular e.g Λ is a field

<u>Def'n</u> $\operatorname{Perv}(X,\Lambda) := {}^pD^{\geq 0} \cap {}^pD^{\leq 0}$

heart of t-structure is an abelian cat

Exa 1) $i: \operatorname{Spec} k \hookrightarrow X$ $\quad i_* \Lambda$ perverse

2) X smooth equidim $= d$, then $\Lambda[d]$ perverse

pf: ${}^p D^{\leq 0}$: easy ${}^p D^{\geq 0}$: use dualizing complex invariant under $Ri^!$

Thm if $\Lambda = \overline{\mathbb{F}}_\ell$, $\underline{\operatorname{Perv}(X, \Lambda) \cap D^b_c(X_{et}, \Lambda)}$

artinian cat, each obj has finite length

irr objs $\overset{\text{bijection}}{\longleftrightarrow}$ closed irr subset $Z \subseteq X$

+ irr rep of the absolute Galois gp of $k(Z)$ on $\overline{\mathbb{F}}_\ell$-vs.

Sketch. Given $i: Z \hookrightarrow X$ such irr rep

get dense open $j: U \hookrightarrow Z$

+ irr $\overline{\mathbb{F}}_\ell$-local system $\mathbb{L}$ on $\mathcal{U}$,

$\mathcal{U}$ smooth

$$j_! \mathbb{L}[d_Z] \in {}^pD^{\le 0}(Z, \overline{\mathbb{F}}_\ell)$$

(easy to check as $j_!$ is just ext by zero)

$$Rj_* \mathbb{L}[d_Z] \in {}^pD^{\ge 0}(Z, \overline{\mathbb{F}}_\ell)$$

(duality $\because$)

$i^! Rj_*$

$${}^pj_! \mathbb{L}[d_Z] = {}^p\tau^{\ge 0}(j_! \mathbb{L}[d_Z])$$

$$\downarrow$$

$${}^pRj_* \mathbb{L}[d_Z] = {}^p\tau^{\le 0}(Rj_* \mathbb{L}[d_Z])$$

image in $\mathrm{Perv}(X, \overline{\mathbb{F}}_\ell)$, defined as $IC(Z, \mathbb{L})$ "intersection complex"

Then $i_* IC(Z, \mathbb{L}) \in \mathrm{Perv}(X, \overline{\mathbb{F}}_\ell)$ are all irred objs.

Relative perversity

Setting $f : X \to S$ separated of finite type

S arbitrary scheme

Goal : Define notion of "perversity $/S$"

Def'n 1) ${}^{p/S}D^{\leq 0}(X,\Lambda) \subseteq D_{et}(X,\Lambda)$

full subcat of all $A \in D_{et}(X,\Lambda)$

s.t for all geom pts $\bar{s} \to S$

$$A|_{X_{\bar{s}}} \in {}^{p}D^{\leq 0}(X_{\bar{s}},\Lambda)$$

i.e $\forall$ all geom pts

$$\begin{tikzcd} \bar{x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d] \\ \bar{s} \arrow[r] & S \end{tikzcd}$$

$$A_{\bar{x}} \in D^{\leq -d(\bar{x}|\bar{s})}(\Lambda)$$

2) ${}^{p/S}D^{\geq 0}(X,\Lambda)$ = right ortho of ${}^{p/S}D^{\leq -1}$

Thm (Hansen–Scholze)

1) This defines a t-structure on $D_{ét}(X, \Lambda)$

2) $A \in D_{ét}(X, \Lambda)$ lies in ${}^{p/S}D^{\geq 0}$

iff $\forall$ all $\bar{s} \longrightarrow S$ geom pts

(∗-restriction! not !) $A|_{X_{\bar{s}}} \in {}^{p}D^{\geq 0}(X_{\bar{s}}, \Lambda)$

3) It induces a t-structure on $D^b_c(X, \Lambda)$

Cor of 2) Pull back along

$$\begin{tikzcd} S' \arrow[r] & S \\ X' \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & X \arrow[u] \end{tikzcd}$$

induces

t-exact functor

$${}^{p/S}D^{\leq 0}(X, \Lambda) \longrightarrow {}^{p/S'}D^{\leq 0}(X', \Lambda)$$

$$\geq 0 \qquad\qquad \geq 0$$

Cor There is a notion of "family of perverse sheaves" $\mathrm{Perv}(X/S, \Lambda) := {}^{p/S}D^{\geq 0} \cap {}^{p/S}D^{\leq 0}$

Ex if $S = $ DVR then well-behaved thm

$\Leftrightarrow$ nearby cycle / van cycle

preserves the perversity

Q: what if $X \to S$ finite? A: usual t-start

Perverse Sheaves in p-adic geometry

Warning: currently no good def'n of dim

e.g. the "current" dim of a pt of $\mathbb{B}^2_{\mathbb{C}_p}$

Exam 1) $|\mathbb{B}^{ad}_{\mathbb{C}_p}|$

classical pts: dim 0

All other rank 1 pts should be dim 1

What about rk 2 pts? either 0 or 1

depending on your perspective

(two choices are changed under $\mathbb{D}$ duality) !!

2) $|\mathbb{B}^2_{\mathbb{C}_p}|$ no classification of rk 1 pts,

and "top trans degree" has weird behaviors:

$\exists$ tower $\underbrace{\mathbb{C}_p \underset{\substack{\text{top tr deg} \\ =1}}{\subseteq} K_1 \underset{\substack{\text{top tr deg} \\ =1}}{\subseteq} K_2}_{\text{top tr deg} = 1}$

cf. Temkin "Topological Transcendence Degree"

$\rightsquigarrow$ no hope for complete general theory of perverse sheaves

But we only need relative one for

$\underbrace{\mathrm{Heck}_G \longrightarrow \mathrm{Div}^1}$

$\rightsquigarrow$ only need to define dimensions of

points of $\mathrm{Hck}_G \times_{\mathrm{Div}^1} \mathrm{Spd}\, C$

But we have Cartan decomposition

$$\mathrm{Hck}_G = L^+G \backslash \mathrm{Gr}_G$$

$$\mathrm{Gr}_G = \bigcup \mathrm{Gr}_{G,\mu}$$

decomp into L^+G-orbits

$$\dim \mathrm{Gr}_{G,\mu} = \langle 2\rho, \mu \rangle$$

(for any possible dim)

Hyperbolic Localization

Usual Set up: k alg closed field

$\mathbb{G}_m \circlearrowright X/k$ proper scheme

$\rightsquigarrow$ fixed pts $X^0 = X^{\mathbb{G}_m} \subseteq X$ closed

+ two stratifications

$$X = \bigcup_{i=1}^{m} X_i^+ \qquad X^0 = \coprod_{i=1}^{m} X_i^0$$

loc closed $\qquad$ open + closed

$$X = \bigcup_{i=1}^{m} X_i^-$$

all $\mathbb{G}_m$-stable

$X^+ := \bigcup X_i^+ \qquad X^- := \bigcup X_i^-$

s.t $\mathbb{G}_m$-action extends to

$$\begin{array}{ccc} (\mathbb{A}^1)^+ \times X_i^+ & \longrightarrow & X_i^+ \\ \cup\!\! \mid & & \cup\!\! \mid \\ 0 \times X_i^+ & \longrightarrow & X_i^0 \end{array}$$

contracting

$$\begin{array}{ccc} (\mathbb{A}^1)^- \times X_i^- & \longrightarrow & X_i^- \\ \cup\!\! \mid & & \cup\!\! \mid \\ 0 \times X_i^- & \longrightarrow & X_i^0 \end{array}$$

$X_i^+ =$ locus where $\lim_{t \to 0} t \cdot x$ exists and in X_i^0

$X_i^- =$ " " " $\lim_{t \to \infty}$ " " " " "

Exam $\mathbb{G}_m \circlearrowright X = \mathbb{P}^1$

$$X^0 = \{0, \infty\} = \{0\} \sqcup \{\infty\}$$

$$X^+ = \mathbb{A}^1 \sqcup \{\infty\} \qquad X_1^+ \sqcup X_2^+$$

$$X^- = \{0\} \sqcup \mathbb{A}^1 \qquad X_1^- \sqcup X_2^-$$

Example (hyperbolic action)

$$\mathbb{G}_m \circlearrowright \mathbb{P}^1 \times \mathbb{P}^1$$

$$t \cdot (a_1, a_2) = (t^{-1}a_1, ta_2)$$

Goal of hyperbolic localization:

Describe the cohomology of $\mathbb{G}_m$-equiv. sheaves on X in terms of local information at $X^0 \subseteq X$

Q: function-sheaf dictionary?

Thm (Braden), $\exists$ functor

$$L : D_{ét}(X/\mathbb{G}_m, \Lambda) \longrightarrow D_{ét}(X^0, \Lambda)$$

s.t $\quad R\Gamma(X, A) \cong R\Gamma(X^0, L(A))$

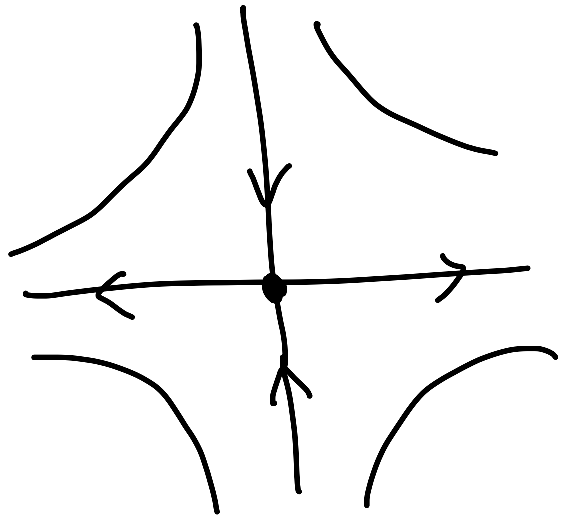
In fact, L has 4 descriptions:

$$\begin{tikzcd}
X_i^0 \arrow[r, bend right, "i^-"'] \arrow[d, shift left, "i^+"] & X_i^- \arrow[l, bend right, "p^-"'] \arrow[d, "q^-"] \\
X_i^+ \arrow[u, shift left, bend left, "p^+"] \arrow[r, "q^+"] & X
\end{tikzcd}$$

$$R(p^-)_!(q^-)^* \xleftarrow{\simeq} R(i^-)^!(q^-)^* \xleftarrow{\simeq} (i^+)^* R(q^+)^!$$
$$\uparrow \wr$$
$$R(p^+)_* R(q^+)^!$$

Exam $\quad \mathbb{G}_m$-equiv A on

(analyze locally at 1 fixed pt) $\quad \mathbb{G}_m \circlearrowright \mathbb{A}^2 \quad t(a_1, a_2) = (t^{-1}a_1, ta_2)$



$$\begin{array}{ccc} 0 & \xrightarrow{i^-} & \mathbb{A}^1 \times \{0\} \\ {\scriptstyle i^+}\downarrow & & \downarrow{\scriptstyle q^-} \\ 0 \times \mathbb{A}^1 & \xrightarrow[q^+]{} & \mathbb{A}^2 \end{array}$$

$$\Longrightarrow \quad (i^+)^* R(q^+)^! A \xrightarrow{\sim} R(i^-)^! (q^-)^* A$$

Exam $\quad X = \mathbb{P}^1 \circlearrowleft \mathbb{G}_m \qquad A = \Lambda$

$$R\Gamma(\mathbb{P}^1, \Lambda) = \Lambda[0] \oplus \Lambda[-2]$$

$$L(A)_{\{0\}} = R\Gamma_c(\mathbb{A}^1, \Lambda) = \Lambda[-2]$$

$$= R\Gamma_{\{0\}}(\mathbb{A}^1, \Lambda) = \Lambda[-2]$$

$$L(A)_{\{\infty\}} = R\Gamma_c\{\infty, \Lambda] = \Lambda[0]$$

$$R\Gamma(X^0, L(A)) = \Lambda[0] \oplus \Lambda[-2] \quad \checkmark$$

<u>Ex</u> $X = $ flag var $\circlearrowleft G \supseteq \mathbb{G}_m$
$\simeq G/P$ dominant

$$X^{\mathbb{G}_m} = X^T = W/W_P$$

$$\rightsquigarrow R\Gamma(X, \Lambda) = \bigoplus_{\substack{w \in W/W_P \\ \cong \\ W^P}} \Lambda[-2l(w)]$$

We will use this for

$$X = Gr_{G, \leq \mu} \subseteq Gr_G$$

$$\circlearrowleft$$

$$\mathbb{G}_m \subseteq L^+G$$

dominant

to understand cohomology of L^+G-equiv perv. sheaves on Gr_G

relative version: hyperbolic localization commutes with any base change

Hyperbolic Localization for diamonds

<u>Set up</u>: $f : X \longrightarrow S$ proper

small v-stack rep. in spatial diamonds

$\dim \mathrm{trg}.f < +\infty$

+ action of $\mathbb{G}_m$ on X/S

(trivial on S)

Thm $\forall\ A \in D_{\acute{e}t}(X/\mathbb{G}_m, \Lambda)$ the maps

$$R(p^-)_!(q^-)^*A \xleftarrow{\ \simeq\ } R(i^-)^!(q^-)^*A \xleftarrow{\ \simeq\ } (i^+)^* R(q^+)^!$$

$$\uparrow \wr$$

$$R(p^+)_* R(q^+)^! A$$

are still isomorphisms, defining

"hyperbolic local functor"

$$L_{X/S} : D_{ét}(X/\mathbb{G}_m, \Lambda) \longrightarrow D_{ét}(S, \Lambda)$$

$L_{X/S}$
- comm w/ all (co)limits (in ∞-cat)
- comm w/ all base change $S' \to S$
- preserve the perverse sheaves

+ $f : X \longrightarrow S$, $X^0 \subset X$, $f^0 : X^0 \to S$

$\Rightarrow$ $Rf_* \simeq Rf^0_* L_{X/S}$

Sketch of proof.

Claim just use the geometric principle

If $Y \circlearrowleft \mathbb{G}_m$, $[Y/\mathbb{G}_m]$ qcqs $/S$

local spatial diamond, partially proper $/S$ say Spd C

$\Rightarrow$ Y has two ends, and

for all $A \in D_{et}([Y/\mathbb{G}_m], \Lambda)$

$$\boxed{R\Gamma_{\partial-c}(Y, A) = 0}$$

comp supp at one end

no supp at other end

Pf: reduce to $\mathbb{G}_m \curvearrowright \mathbb{A}^1$

diff between $R\Gamma_c(\mathbb{A}^1, A)$ & $R\Gamma_{\{0\}}(\mathbb{A}^1, A)$

is $R\Gamma_{\partial-c}(\mathbb{G}_m, A)$

Q: (Travkin) $X = \mathbb{P}^1 \overset{j}{\supseteq} U = \mathbb{A}^1$,

$A = j_! \Lambda$

$$R\Gamma(\mathbb{P}^1, A) = R\Gamma_c(\mathbb{A}^1, \Lambda) = \Lambda[-2]$$

$$L(A)_{\{0\}} = \Lambda[-2]$$

$$L(A)_{\{\infty\}} = 0.$$

Q: (Zhiyu) Lef trace formula computes less, hyperbolic localization more powerful categorical

Q: (Zhiyu) $\mathbb{Z}_\ell$-coeff

no, construction fails, two perverse t-struct duality fails, so for geom Satake $\mathbb{Z}_\ell$ need to choose carefully

Q: (Zhiyu) $S =$ DVR

thm on relative perversity

$\Rightarrow$ nearby cycle preserves the perversity.

Q: (Feng): no duality, $X \to S$ quasi-finite rel perversity is usual t-structure

Q: (Dat) "partially comp supp" Cohom

do you need two ends be disjoint?