1/29

Geometric Satake

Last time

corrections:

(minor) for schemes, stratification need not exist as described. OK if X is normal

(HL OK if stratif exist)

(major) claimed

$$R\Gamma(X, A) \cong R\Gamma(X^0, L(A))$$

This is false: exa $X = \mathbb{P}^1 \xhookleftarrow{j} (\mathbb{A}^1)^- \circlearrowleft \mathbb{G}_m$

$A = j_! \Lambda$

Previous thought: stratif $X = \bigcup_{i=1}^{m} X_i^+$

$\rightsquigarrow$ filtration of $R\Gamma(X, A)$ with graded pieces

$$R\Gamma_c(X_i^+, A|_{X_i^+})$$

$$\simeq R\Gamma_c(X_i^0, \underbrace{R(p_i^+)_! A|_{X_i^+}}_{L(A)|_{X_i^0}})$$

graded piece

$$\bigoplus_{i=1}^{m} R\Gamma_c(X_i^0, L(A)) = R\Gamma_c(X^0, L(A))$$

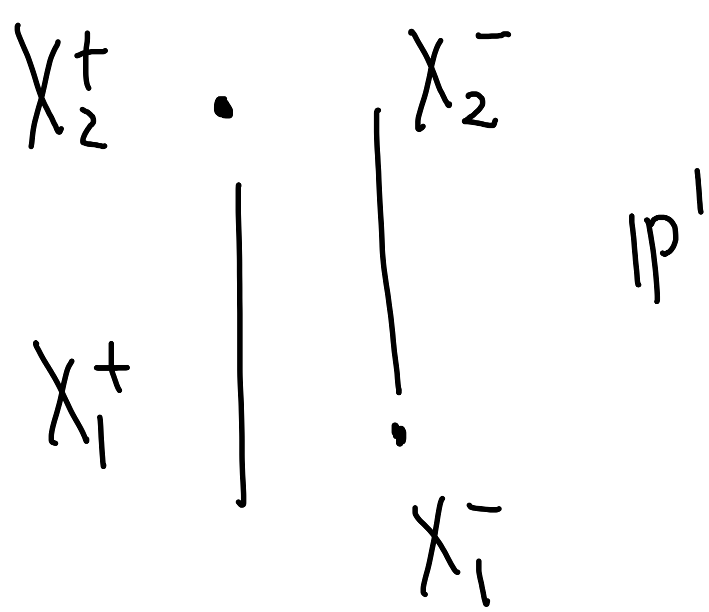
$$\begin{array}{ccc} X_i^+ & \xhookrightarrow{q_i^+} & X \\ \downarrow{\scriptstyle p_i^+} & & \\ X_i^0 & & \end{array}$$

and $X = \bigcup_{i=1}^{m} X_i^-$ will give opposite filtration

thus a splitting !

<u>However</u>, the filtration are <u>the same</u>

(the claim in [MV] may be wrong)



Beilinson – Drinfeld Grassmannians (local obj)

Assume $G / \mathcal{O}_E$ split reductive

(In general, we use localization to reduce to this case)

don't work for Bun_G

<u>Recall</u> moduli space of $\deg = d$ Cartier divisors

on integral $Y_S = S \dot{\times} \mathrm{Spa}\, \mathcal{O}_E$

(keep char p pt) → allowing deform to WITT Gr

(for $S = \mathrm{Spa}(R, R^+)$ $\varpi \in R$ pseudounif,

$$Y_S = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi] = 0\} \;)$$

$$\mathrm{Div}^d_Y = (\mathrm{Div}^1_Y)^d / \Sigma_d$$ small v-stack

$$= (\mathrm{Spa}\, \mathcal{O}_E)^{\diamond, d} / \Sigma_d$$

$\mathrm{Div}^d_Y \to *$ rep in local spatial diamond

" param. d pts on $\mathrm{Spa}\, \mathcal{O}_E$ "

Given S, section of $\mathrm{Div}^d_Y(S)$

$\rightsquigarrow$ relative Cart div $D_S \subset Y_S$ $(/S)$

If $S = \mathrm{Spa}(R, R^+)$ affinoid, let

B^+ = completion of $\mathcal{O}(Y_S)$ along $\mathcal{I}(D_S)$

$= W_{\mathcal{O}_E}(R^+)[\frac{1}{[\varpi]}]^\wedge_{\xi}$, where $D_S = V(\xi)$

here is related to d.

$B = B^+[\frac{1}{\xi}]$

<u>Def'n</u> L^+G, LG / Div^d_Y :

$$S = \mathrm{Spa}(R, R^+) / \mathrm{Div}^d_Y \mapsto G(B^+) \text{ resp. } G(B)$$

BD Gr is $\mathrm{Gr}_{G, \mathrm{Div}^d_Y} = LG / L^+G$

local Hecke stack $\mathrm{Hck}_{G, \mathrm{Div}^d_Y} = L^+G \backslash \mathrm{Gr}_{G, \mathrm{Div}^d_Y}$

small v-stack

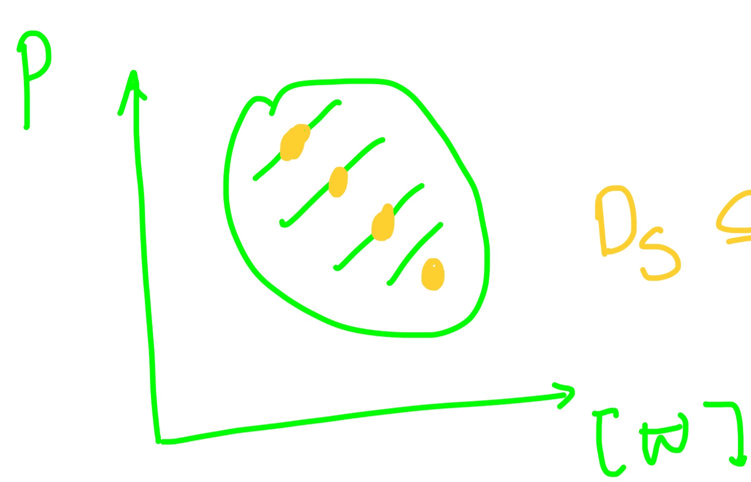
Prop'n (as classical)

- Gr_{G,Div^d_Y} param G-torsors $\mathcal{E}$ over B^+ + triv. over B;

equiv (by Beauville-Laszlo glueing) param G-torsors $\mathcal{E}$ over Y_S + merom. triv. over $Y_S \setminus D_S$

- Hck_{G,Div^d_Y} param G-torsors $\mathcal{E}_1, \mathcal{E}_2$ over B^+ + isom over B

equiv (by ...) ....



$D_S \subseteq Y_S$

$B \simeq$ "$\mathcal{O}(Y_S|^{\wedge}_{D_S} \setminus D_S)$"

For $S \longrightarrow \mathrm{Div}^d_Y$ any small v-stack let $\mathrm{Gr}_{G,S/\mathrm{Div}_Y^d} := \mathrm{Gr}_{G,\mathrm{Div}_Y^d} \times_{\mathrm{Div}_Y^d} S$

Schubert var Assume $S = \mathrm{Spa}\, C$ geom pt

$S \to \mathrm{Div}_Y^d$ corr to a collection of d untilts $C_1^\#, \dots, C_d^\#$ of C

If some agree, can remove them and $\mathrm{Gr}_{G,S/\mathrm{Div}_Y^d}$ doesn't change. so can assume distinct

chose $\xi_1, \dots, \xi_d \in W_{O_E}(O_C)$

s.t $O_{C_i^\#} = W_{O_E}(O_C) / (\xi_i)$

$\xi = \xi_1 \cdots \xi_d$ chose $T \subset B \subset G$

(prod in $W_{O_E}(O_C)$) (we assume G split)

Prop'n $|\mathrm{Hck}_{G,S/\mathrm{Div}_Y^d}| \xleftarrow{\sim} X_*^+(T)^d$ ← domin

orbit of $\mu_1(\xi_1) \cdots \mu_d(\xi_d)$ $\in LG(S) = G(B)$ $\longleftarrow\!\shortmid (\mu_1, \dots, \mu_d)$

$$\mathrm{Hck}_{G,S/\mathrm{Div}_Y^d} = \prod_{i=1}^{d} {}^{/S}\, \mathrm{Hck}_{G,S/\mathrm{Div}_Y^1}$$

implicit map given by s_i^*

⇝ can define L^+G-orbits

$$\mathrm{Gr}_{G,S/\mathrm{Div}_Y^d,(\mu_1,\dots,\mu_d)} \subseteq \mathrm{Gr}_{G,S/\mathrm{Div}_Y^d},$$

$$\mathrm{Gr}_{G,S/\mathrm{Div}_Y^d,\le(\mu_1,\dots,\mu_d)} := \overline{\mathrm{Gr}_{G,S/\mathrm{Div}_Y^d,(\mu_1,\dots,\mu_d)}}$$

$$= \bigcup_{(\mu_1',\dots,\mu_d') \le (\mu_1,\dots,\mu_d)} \mathrm{Gr}_{G,S/\mathrm{Div}_Y^d,(\mu_1',\dots,\mu_d')}$$

in dominant order

quotients $\mathrm{Hck}_{G,S/\mathrm{Div}^d_Y,(\mu_1,\dots,\mu_d)}$, $\mathrm{Hck}_{G,S/\mathrm{Div}^d_Y,\le(\mu_1,\dots,\mu_n)}$

can also define this in families:

S/Div_Y^d $\quad (\mathrm{Div}_Y^1)^d \longrightarrow \mathrm{Div}_Y^d$,

$\mu_1,\dots,\mu_d \in X_*^+$ can define

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_Y,(\leq)(\mu_1,\dots,\mu_d)} \subseteq \mathrm{Gr}_{G,S/\mathrm{Div}^d_Y}$$

by applying previous def'n fiberwise

When pts collide, need to add corresponding μ_i

Thm $\mathrm{Gr}_{G,S/\mathrm{Div}^d_Y,\leq(\mu_1,\dots,\mu_d)} \underset{\text{closed}}{\subseteq} \mathrm{Gr}_{G,S/\mathrm{Div}^d_Y}$

Q: assume G split red?

proper + repr in spatial diamonds over S

(finite dim.trg)

Main thm of

Berkeley course '2014

Remark No explicit pro-étale charts are known!

OK if base change to $(\mathrm{Spa}\, E)^{\diamond,d} = \mathrm{Div}^d_Y$

Instead, proved by some

Artin criterions for stacks

(simpler pf here by using Shtukas for GL_n and reduction

Master thesis of Bence Heuesi

<u>prop</u> On open Schubert cells, away from diagonals, the L^+G-action is transitive

<u>Cor</u> The strata of $Hck_{G,S/Div^d_Y}$ are, away from diagonals, of form

$$[S / \underline{\text{some large gp}}]$$

ext of finite dim cohom'ly smooth gp (like $G^\diamond$)
+ inf-dim'l "unipotent" gp (like $\ker L^+G \to G^\diamond$))

$\rightsquigarrow$ on level of D_{et}, all strata behave <u>like Artin v-stacks</u>.

Similarly, the L^+G-action on each $Gr_{G,S/Div^d_Y, \leq(\mu_1,\cdots,\mu_d)}$ factors over a quotient $\underline{(L^+G)_{\le(\mu_1,\cdots,\mu_d)}} \twoheadleftarrow L^+G$

fin dim'l cohom. smooth + Kernel is "unipotent"

$\Rightarrow D_{ét}(Hck_{G,S/Div^d_Y,\le\mu},\Lambda)$

$\cong D_{ét}((L^+G)_{\le\mu}\backslash Gr_{G,S/Div^d_Y},\Lambda)$

<u>Def'n</u> $D_{ét}(Hck_{G,S/Div^d_Y},\Lambda)^{\underline{bd}}$ ← "bounded"

$$:= \bigcup_{\mu} D_{ét}(Hck_{G,S/Div^d_Y,\le\mu},\Lambda)$$

inside $D_{ét}(Hck_{G,S/Div^d_Y},\Lambda)$

This is a <u>monoidal category</u>:

<u>Def'n</u> (Convolution): For $A,B\in D_{ét}(Hck,\Lambda)^{bd}$

$$\begin{tikzcd} \mathcal{H}ck \times \mathcal{H}ck & L^+G \backslash LG \times_{L^+G} LG/L^+G \arrow[l, "\pi"'] \arrow[d, "m"] \\ & L^+G \backslash LG / L^+G = \mathcal{H}ck \end{tikzcd}$$

$$A \star B := Rm_* \pi^* (A \boxtimes B)$$

Note: m is ind-proper, as fibres are aff Gr, so proper base change thm

$\Longrightarrow$ $\star$ is associative

Perverse sheaves on $\mathcal{H}ck$

Def'n Fix $S \longrightarrow \mathrm{Div}^d_Y$
$\in \mathrm{Perf}_{\mathbb{F}_q}$

Let ${}^pD^{\leq 0}_{\acute{e}t}(\mathcal{H}ck_{G,S/\mathrm{Div}^d_Y}, \Lambda)^{bd} \subseteq D(\mathcal{H}ck)^{bd}$

be the full subcat of all $A \in D(\mathcal{H}ck)^{bd}$

s.t for all geom pts $\mathrm{Spa}(C,C^+) \to S$

all $\mu_1, \cdots, \mu_m \in X_*^+$ (m = # distinct untilts of $\mathrm{Spa}(C,C^+)$ corr to $\mathrm{Spa}(C,C^+) \to S \to \mathrm{Div}_Y^d$)

we have

$$A\Big|_{\mathrm{Hck}_G,\, \mathrm{Spa}(C,C^+)/\mathrm{Div}_Y^d,\, (\mu_1,\cdots,\mu_m)} \in D^{\le -d(\mu)}(\Lambda)$$

where $d(\mu) := \sum_{i=1}^{m} \langle 2\rho, \mu_i \rangle \geq 0$

This is the direct analogue of "relative perversity over S"

${}^pD^{\geq 0}$ = right orth to ${}^pD^{\leq 0}[1]$

<u>Thm</u> This defines a t-structure

$A \in {}^pD^{\geq 0}$ iff $\forall$ all geom pts

$\mathrm{Spa}(C, C^+) \longrightarrow S$ as above

$A|_{\mathrm{Hck}_{G, \mathrm{Spa}(C,C^+)/\mathrm{Div}^d_y}} \in {}^pD^{\geq 0}$

$\Longleftrightarrow$! - restriction on all Schubert

cells lies in $D^{\geq -d(\mu)}$

In particular, pullback under $S' \to S$ is t-exact

Remark For schemes in [HS],

analogous results use perversity of nearby cycles

this fails in p-adic geometry!

+ Artin vanishing

Exa $\mathbb{A}^1_k \overset{i}{\subset} \widehat{\mathbb{A}^1_{O_C}} \overset{j}{\supset} \mathbb{B}_C$

Artin vanishing would suggest

$$R\Gamma(\mathbb{B}_C, A) \in D^{\leq 1}$$

But this fails for $A = j^!_! \Lambda$!

$j^!: \mathbb{B}^1_C \hookrightarrow \mathbb{B}_C$

$\{T, |T| \leq 1/2\}$

Then

$$R\Gamma(\mathbb{B}_C, A) = R\Gamma_c(\mathbb{B}^1_C, \Lambda) = \Lambda[-2]$$

Similarly, in this ex,

$R\psi A = i^* Rj_* A =$ skycraper sheaf $\Lambda[-2]$ at origin

not perverse !

<u>proof Sketch</u> Use hyperbolic localization

This is easy when $G = T$ torus

Then $Gr_T,\ S/\mathrm{Div}_y^d,\ \leq (\mu_1, \cdots, \mu_d)$

$\longrightarrow S$ finite

t-structure = usual t-structure

<u>Key Lem</u>

$$\begin{tikzcd} & Gr_B \arrow[dl, "q"'] \arrow[dr, "p"] & & \\ Gr_G & & Gr_T \arrow[r, "\deg"] & \mathbb{Z} \end{tikzcd}$$

☆ $\mathbb{G}_m \circlearrowright Gr_G$

$$\Rightarrow \begin{tikzcd} & X^+ \arrow[dl] \arrow[dr] & \\ X & & X^0 \end{tikzcd}$$

let $CT_B := Rp_!\, q^*[\deg]$

$$D_{\acute{e}t}(Hck_G, \Lambda)^{bd} \longrightarrow D_{\acute{e}t}(Hck_T, \Lambda)^{bd}$$

Then CT_B is <u>t-exact</u> + <u>conservative</u>

(this allows to reduce G to T)

pf of key Lem: reduce to the case of geom pts. Then $D_{\acute{e}t}(\mathrm{Hck}, \Lambda)^{bd}$ has stratif in terms of $D_{\acute{e}t}(\mathrm{Hck}_{(\mu_1,\dots,\mu_d)}, \Lambda)^{bd}$

$[*/\sim]$ generated by $D(\Lambda)$

so suffices to check

$$CT_B({}^pD^{\le 0}) \subseteq {}^pD^{\le 0}$$

$$ {}^pD^{\ge 0} \subseteq {}^pD^{\ge 0}$$

on standard objs

$$j_{\mu!}\Lambda[d_\mu] \qquad Rj_{\mu,*}\Lambda[d_\mu]$$

These sheaves are ULA on $\mathrm{Gr}_{G,S}/S$

+ hyperbolic localization perverse ULA sheaves

$\Rightarrow$ cohom local const, can reduce to geom pts in char p

But for $S = (\mathrm{Spa}\,\mathbb{F}_q)^{\lozenge} \to \mathrm{Div}^d_y$

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_y} = \left(\mathrm{Gr}_G^{\mathrm{Witt}}\right)^{\lozenge}$$

Witt Vector Gr

+ 6 operations are compatible for schemes vs. asso v-sheaves

$\Rightarrow$ reduce to the same statement for $\mathrm{Gr}_G^{\mathrm{Witt}}$ [Zhu]

Q: (Tony) Sign

Q: (Kestutis) • relative per truncation

and ULA

• middle $j_{!*}$

Q: (Zhiyu) • repn for smooth affine G

- CT_β t-exact and, conservative
- duality and rel perversity
- t-exact can't be detect by function-sheaf dictionary

Q (Le Bras) Artin-vanishing for ULA sheaves ?

un-known

- deform to char p trick

1. t-exact today
2. semi-simple for Sat Q_ℓ