02/01 Geom Satake

$G / \mathcal{O}_E$ split reductive

$G \supset B \supset U$, T

Div^d_Y moduli of deg d

Cart divisors $D_S \subset Y_S := S \dot{\times}_{\mathrm{Spa}\,\mathcal{O}_E}$

<u>BD Gr</u>

param. G-torsors on Y_S with merom triv on $Y_S \setminus D_S$

$$\begin{tikzcd} L^+G \circlearrowright \mathrm{Gr}_{G,\mathrm{Div}^d_Y} \arrow[r] & \mathrm{Div}^d_Y \\ \mathrm{Gr}_{G,S|\mathrm{Div}^d_Y} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & S \arrow[u] \end{tikzcd}$$

<u>local Hecke stack</u> $= (L^+G \backslash LG / L^+G) \times_{\mathrm{Div}^d_Y} S$

$\mathrm{Hck}_{G,S/\mathrm{Div}^d_Y} = L^+G \backslash \mathrm{Gr}_{G,S/\mathrm{Div}^d_Y}$

$D_{ét}(Hck_{G,S/Div^d_y}, \Lambda)^{bd}$ ← bounded support

has relative $/S$ perverse t-structure

$\rightsquigarrow$ $Perv(Hck_{G,S/Div^d_y}, \Lambda)$ abelian cat (functorial in S)

+ monoidal structure $\star$

= convolution prod on $D_{ét}(Hck_{G,S/Div^d_y}, \Lambda)^{bd}$

<u>Prop</u> pullback to Gr gives a fully faithful functor

$$Perv(Hck_{G,S/Div^d_y}, \Lambda) \hookrightarrow D_{ét}(Gr_{G,S/Div^d_y}, \Lambda)^{bd}$$

Reason: L^+G is connected.

<u>Prop</u> $A, B \in {}^pD^{\leq 0}(Hck)^{bd} \Rightarrow A \star B \in {}^pD^{\leq 0}(Hck)^{bd}$

If A, B perv, + A <u>flat</u> perv $\Rightarrow A \star B$ perv

(i.e $A \overset{\mathbb{L}}{\otimes}_\Lambda M$ perv $\forall$ <u>any</u> Λ-mod M)

__Sketch__ Statement can be proved on geom fibres; then Hck decomposes into prod over untilts

reduce to $d=1$, and standard obj $\begin{cases} A = j_{\mu_1,!}\Lambda \\ B = j_{\mu_2,!}\Lambda \end{cases}$

reduce to $S = \mathrm{Div}^1_y$. Then ~~everything ULA~~,

so reduce to special fiber, use [Zhu] Witt Gr;

Alternatively, use fusion prod to conclud. (no use of Witt Gr)

$\rightsquigarrow$ convolution prod on __flat__ perverse sheaves

(flat is neccessary, e.g. G is trivial gp $\circledast = \overset{L}{\otimes}$)

__Def'n__ An obj $A \in D_{\acute{e}t}(\mathrm{Hck}_{G,S/\mathrm{Div}^d_y}, \Lambda)^{bd}$ is ULA $|S$ if its pullback to $\mathrm{Gr}_{G,S/\mathrm{Div}^d_y}$ is ULA $|S$

$\wr$ $Hck = L^+G\backslash LG/L^+G$ has switching symmetry.

$g \mapsto g^{-1}$

Being ULA is invariant under this sym.

<u>Prop'</u>. if $d=1$, $A \in D_{ét}(Hck_{G,S/Div^d_Y}, \Lambda)^{bd}$

is ULA iff $\forall$ all μ

$$A|_S \xrightarrow{[\mu]} Hck_{G,S/Div^d_Y} \in D_{ét}(S,\Lambda)$$

is locally const with perfect fibres

"local system"

Q: $S = *$, ULA $\Leftrightarrow$? $d>1$?

<u>Cor</u> ULA sheaves is preserved under

6 operations

that one can build

from the strata

$(Rj_*, Ri^!, \ldots, R\mathcal{H}om(-,-))$

Cor

$$\begin{tikzcd} (\mathrm{Spa}\, C)^{\diamond} \arrow[r, hook] \arrow[dr] & (\mathrm{Spa}\, O_C)^{\diamond} \arrow[d] & (\mathrm{Spa}\, k)^{\diamond} \arrow[l, hook'] \arrow[dl] \\ & \mathrm{Div}^1_y = (\mathrm{Spa}\, O_E)^{\diamond} & \end{tikzcd}$$

induces equiv (t-exact)

$$\begin{tikzcd} D^{ULA}_{\acute{e}t}(\mathrm{Hck}_{G,\mathrm{Spa}\, C}, \Lambda)^{bd} & D^{ULA}_{et}(\mathrm{Hck}_{G,\mathrm{Spa}\, O_C}, \Lambda)^{bd} \arrow[l, "\sim"'] \arrow[d, "\wr"] \\ & D^{ULA}_{\acute{e}t}(\mathrm{Hck}_{G,\mathrm{Spa}\, k}, \Lambda)^{bd} \end{tikzcd}$$

Sketch of pf

$(S = \mathrm{Div}^1_y)$

Key pt: $j_{\mu!}\Lambda$ is ULA

$$j_\mu : \mathrm{Gr}_{G,\mu} \hookrightarrow \mathrm{Gr}_{G,\leq\mu} \;/S$$

check this after pullback to affine flag var

$$I_W \curvearrowright \mathcal{F}\ell_G = LG/I_W \xrightarrow[(G/B)^{\diamond}\text{-bundle}]{} \mathrm{Gr}_G = LG/L^+G$$

$$\begin{tikzcd} I_w \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & L^+G \arrow[d, "ev"] \\ B \arrow[r, hookrightarrow] & G \end{tikzcd}$$

$${}^\circ\mathcal{F}l_G = \bigcup_{w \in \widetilde{W}} {}^\circ\mathcal{F}l_{G,w}$$

$\uparrow$ IW-orbits

$$\widetilde{W} = N(T)(B_{dR}) / T(B_{dR}^+)$$ extended Weyl gp

$$1 \to X_*(T) \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 1$$

$$X_*(T) = T(B_{dR}) / T(B_{dR}^+)$$

$$1 \to W_{aff} \to \widetilde{W} \to \Omega \to 1$$

can reduce instead to

$j_{w!}\Lambda$ $\qquad j_w : \mathcal{F}l_{G,w} \hookrightarrow \mathcal{F}l_{G,\le w}$

Q: how? $\mu \to$ which w?

Key pt

$\exists$ Demazure–Bott–Samelson resolution

$$\begin{tikzcd} {}^\circ\mathcal{F}l_{G,w} \arrow[rr, hookrightarrow, "j_w"] \arrow[dr, "\tilde{j}_w"'] & \circlearrowright & {}^\circ\mathcal{F}l_{G,\le w} \\ & \widetilde{\mathcal{F}l}_{G,w} \arrow[ur, "\text{proper}"'] & \end{tikzcd}$$

sheave we interested are $j_{w!}$

make through proper

$\boxed{\text{proper } * \text{ preserves ULA}}$

for $w = s_{i_1} \cdots s_{i_l} \cdot w$ $\qquad w \in \Omega \subseteq \tilde{W}$

s_i simple affine reflection $\qquad \Omega \cong$ closed orbits

$l = l(w)$

$Fl_{G,w}$ = pts

$$\widetilde{Fl}_{G,\dot{w}} = P_{s_{i_1}} \overset{I_w}{\times} P_{s_{i_2}} \overset{I_w}{\times} \cdots \overset{I_w}{\times} P_{s_{i_l}} / I_w$$

iterated $(\mathbb{P}^1)^\circ$-bundle

$$P_{s_{i_j}} / I_w = (\mathbb{P}^1)^\circ$$

($P_s \subset L^+G$ parahoric subgp corresponding to simple affine reflection s)

enough:

$\tilde{\mathcal{J}}_w$: Λ ULA on $\underbrace{\widetilde{Fl}_{G,\dot{w}}}$ coh smooth, boundary behaves like normal crossing div

remove factors from $\dot{w}$

(if s usual reflection

$$\begin{tikzcd}
P_s \arrow[r, phantom, "\subseteq"] \arrow[d] \arrow[dr, phantom, "\square"] & L^+G \arrow[d] \\
P_s \arrow[r, hookrightarrow] & G
\end{tikzcd}$$

minimal parabolic)

All strata smooth $\Rightarrow$ $\tilde{J}_{w,!}\Lambda$ is ULA.

Prop. $S \to \mathrm{Div}^d_Y$ arbitrary.

$$\begin{tikzcd} & \mathrm{Gr}_B \arrow[dl, "q"'] \arrow[dr, "p"] & & \\ \mathrm{Gr}_G & & \mathrm{Gr}_T \arrow[r, "\deg"] & \mathbb{Z} \end{tikzcd}$$

$$CT_B = Rp_! q^* [\deg]$$

Then $A \in D_{et}(\mathrm{Hck}_{G,S/\mathrm{Div}^d_Y}, \Lambda)^{bd}$

is ULA

iff $CT_B(A) \in$ ULA

iff $R\pi_{T*} CT_B(A) \in D_{et}(S,\Lambda)$

is locally const with perf values

where $\pi_T : \mathrm{Gr}_{T,S/\mathrm{Div}^d_Y} \to S$

(general prop: hyperbolic loc preserves ULA)

Back to preservation of perversity :

wanted $j_{\mu_1 !} \Lambda \star j_{\mu_2 !} \Lambda$ still in $P_D^{\leq 0}$

let $\widetilde{Gr}_{G,(Div^1_Y)^2} \xrightarrow{\pi_2} Gr_{G,(Div^1_Y)^2}$

$\|$

$\{(\mathcal{E}_1, \mathcal{E}_2$, two untilts $S_1^\#, S_2^\#) \mid \mathcal{E}_1$ triv away from $S_1^\#$

$\mathcal{E}_1 \simeq \mathcal{E}_2$ away from $S_2^\# \}$

isom away from diagonal.

Over diagonal, get

$\widetilde{Gr}_{G, Div^1_Y} \xrightarrow{\pi} Gr_{G, Div^1_Y}$

conv aff Gr

want : $R\pi_*(j_{\mu_1 !}\Lambda \,\tilde{\boxtimes}\, j_{\mu_2 !}\Lambda) \in {}^pD^{\le 0}$

globalize to

$$\underbrace{R\pi_{2*}(j_{\mu_1 !}\Lambda \,\tilde{\boxtimes}\, j_{\mu_2 !}\Lambda)}_{\text{ULA on } \widetilde{Gr}_{G,(Div^1_Y)^2}} \quad \text{ULA on } Gr_{G,(Div^1_Y)^2}$$

want: $\in {}^pD^{\le 0}$

can be checked after applying

$$R\pi_{T*}\, CT_B[\deg] \, ;$$

result is in $D_{ét}((Div^1_Y)^2, \Lambda)$, loc const

Away from diagonal, it's just the tensor prod by Kunneth formula

$$D^{\le 0} \ni \left(R\pi_{T*}CT_B(j_{\mu_1 !}\Lambda) \otimes R\pi_{T*}CT_B(j_{\mu_2 !}\Lambda) \right)$$

complement of diagonal $\Rightarrow$ same over diagonal

(degenerate tensor prod $/_{\text{comp of diag}} \overset{\text{to}}{\Longrightarrow}$ convolution $/_{\text{diag}}$)

p

From now on, work again over $(\mathrm{Div}_X^1)^d$

(now X is FF curve not integral)

G/E, any reductive gp

Q: ?

use previous results via implicit étale localization to G split

I finite set

Λ any ring killed by some n st $(n,p)=1$

<u>Def'n</u> (Satake cat)

$$\mathrm{Hck}_G^1 \supseteq$$

$$\mathrm{Sat}_G^I(\Lambda) := \mathrm{Perv}^{ULA}_{flat}(\mathrm{Hck}_{G,(\mathrm{Div}_X^1)^I}, \Lambda)$$

flat perverse sheaves A on Hck_G^I that are ULA with fibre functor (exact + conservative)

locally free, finite proj $/\Lambda$

$$F : \mathrm{Sat}^{I}_{G}(\Lambda) \longrightarrow \mathrm{LocSys}\left(\left(\mathrm{Div}^1_X\right)^I, \Lambda\right)$$

$\cong$ Thm (Λ finite ✓ by étale

but Λ infinite shall use W_E rather than Gal_E)

$\mathrm{Rep}_{W_E^I}(\Lambda)$

$$A \longmapsto \bigoplus_{i \in \mathbb{Z}} R^i \pi_{G*} A$$

$$\pi_G : \mathrm{Gr}^I_G \longrightarrow \left(\mathrm{Div}^1_X\right)^I$$

$\parallel$

$\mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I}$

$=$ if G quasi split

$\boxed{R\pi_{T*} CT_B(A)}$

using hyperbolic loc

spectral seq deg

CT_B preserves perv.

and $ULA \to ULA$

Fusion product:

(put some signs in symmetry constraints)

$$\begin{tikzcd}
\ast : \mathrm{Sat}_G^{I_1}(\Lambda) \times \dots \times \mathrm{Sat}_G^{I_m}(\Lambda) \arrow[d] \arrow[dr, "\substack{\text{exterior} \\ \text{tensor} \\ \text{prod}}"] & \\
\mathrm{Sat}_G^{I_1 \sqcup \dots \sqcup I_m}(\Lambda) \arrow[r, hook] & \mathrm{Sat}_G^{I_1, \dots, I_m}(\Lambda)
\end{tikzcd}$$

defined as $\mathrm{Sat}^{I_1 \sqcup \dots \sqcup I_m}$ but over $(\mathrm{Div}_X^1)^I \setminus$ locus where $x_i = x_j$ for i, j in diff I_k's

use convolution affine Gr to show factorization

$\Rightarrow$ $\boxed{\text{fusion} = \text{convolution}}$

here, $\mathrm{Gr}_G^I|_{\dots}$

$\simeq \mathrm{Gr}_G^{I_1} \times \dots \times \mathrm{Gr}_G^{I_m}|_{\dots}$

$$\begin{tikzcd}
\ast : \mathrm{Sat}_G^1(\Lambda) \times \dots \times \mathrm{Sat}_G^1(\Lambda) \arrow[r] & \mathrm{Sat}_G^{I \sqcup \dots \sqcup I}(\Lambda) \arrow[d, "\text{diagonal rest}"] \\
& \mathrm{Sat}_G^1(\Lambda)
\end{tikzcd}$$

turn each $\mathrm{Sat}_G^1(\Lambda)$ a sym monoid cat compatibly with its monoidal structure.

$\Rightarrow$ fusion = convolution for cat reasons

Prop $F = \bigoplus R^i \pi_{a*}$ is a symmetric monoidal functor

$$\mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{Rep}_{W_E^I}(\Lambda)$$

Thm $F : \mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{Rep}_{W_E^I}(\Lambda)$

satisfies all Tannakian reconstruction axioms

$\Rightarrow$ $\exists$ Hopf alg $H \in \mathrm{Ind}\,\mathrm{Rep}_{W_E^I}(\Lambda)$

s.t. $\mathrm{Sat}^I_G(\Lambda) \cong \mathrm{CoMod}_H(\mathrm{Rep}_{W_E^I}(\Lambda))$

clear in char 0 (Λ)

integrally, have to work a little bit

key $\mathrm{Sat}^I_G(\Lambda)$ is a highest weight cat 12

$\mathrm{Sat}^I_G(\mathbb{Q}_\ell)$ semi-simple

Prop'n $H^I = \bigotimes_{i \in I} H^{\{i\}}$

$H^{\{i\}} \cong$ affine gp scheme $\check{G}/\Lambda$

so can reduce to G split with continous W_E-action ?

Thm $\exists$ can iso $\check{G} \cong \hat{G}$ the dual gp

W_E-equiv if the pinning of $\hat{G}$

induces cyclotomic $\mathrm{Lie}\,\hat{\mathcal{U}}_a \cong \Lambda(1)$

(can be trivialized if $\sqrt{q} \in \Lambda$)

<u>Cor</u>

$$\mathrm{Sat}_G(\Lambda) \simeq \mathrm{Rep}(\hat{G})$$

$$\mathrm{Sat}^1_h(\Lambda) \simeq \mathrm{Rep}(\hat{h}^I)$$

Q. (Zhiyu) flat perv sheave

$IC_\mu \mapsto$?

Q. (Konrad)

Q. (Travkin) $U_a \simeq G_{(a,\Lambda)}(1)$

Q. (Tony) Decomp thm by degeneration

Q. (Zhiyu) Λ general

Q. (Zhiyu) fiberwise criterion (not-regular) ✓

wk for $\mathfrak{d}$ (away from diag)

Q. (Partofard) Geometric Satake

for Y ? or Y ?