02/05 next week is the last week

Geometric Satake

E nonarch local field, res field $\mathbb{F}_q \subset \overline{\mathbb{F}_q}$

$G/E \overset{\text{conn}}{\vee}$ reductive gp, $\mathrm{Div}^1_X = (\mathrm{Spa}\,\check{E})^\diamond / \varphi^{\mathbb{Z}}$

$\rightsquigarrow$ BD Gr v-sheaf on $\mathrm{Perf}\,\mathbb{F}_q$

$$\mathrm{Gr}^I_G \longrightarrow (\mathrm{Div}^1_X)^I \quad \text{any finite set } I$$

$$\begin{tikzcd} \mathrm{Gr}^I_G \\ (L^+G)^I \arrow[u, bend left] \end{tikzcd}$$

$$\mathrm{Hck}^I_G = (L^+G)^I \backslash \mathrm{Gr}^I_G \to (\mathrm{Div}^1_X)^I$$

notion of perverse + ULA sheaves

Def'n (Satake cat) Λ ring $n\Lambda = 0$, $(n,p)=1$

$$\mathrm{Sat}^I_G(\Lambda) = \mathrm{Perv}^{\mathrm{ULA}}_{\mathrm{flat}}(\mathrm{Hck}^I_G, \Lambda)$$

$$F^I = \bigoplus_i R^i\pi_{G*} \searrow$$

$$\mathrm{LocSys}((\mathrm{Div}^1_X)^I, \Lambda) \simeq \mathrm{Rep}\, W^I_E(\Lambda) \leftarrow \text{finite proj } \Lambda\text{-mod}$$

"Better" description of F^I:

Use constant term functor

$$CT_B := Rp_! q^* [\deg] : Sat_G^I(\Lambda) \to Sat_T^I(\Lambda)$$

$B \subset G$ Borel

$$\begin{tikzcd} & Gr_B^I \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Gr_G^1 & & Gr_T^I \end{tikzcd}$$

Varying B, get functor to

$$\mathrm{LocSys}((\mathrm{Div}_X^1)^I \times \underline{\mathcal{F}\ell^{\diamond}}, \Lambda)$$

flag var is simply connected

$$\cong \mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda)$$

$\Rightarrow$ independent of B

and $\pi_{T*} \cdot CT_B \cong \bigoplus R^i \pi_{G*} = F^I$

(hyperbolic localization)

Fusion product $\rightsquigarrow$ Sat^I_G symm monoidal

$$F^I : \mathrm{Sat}^I_G \longrightarrow \mathrm{Rep}_{W_E^I}(\Lambda)$$

a sym monoidal fibre functor

$+\xi \rightsquigarrow \exists!$ Hopf alg

$$H^I \in \mathrm{Ind}\,\mathrm{Rep}_{W_E^I}(\Lambda)$$

s.t. $\mathrm{Sat}^I_G \cong \mathrm{CoMod}_{H^I}(\mathrm{Rep}_{W_E^I}(\Lambda))$

"Künneth formula" : $H^I \cong \bigotimes_{i \in I} H^{\{i\}}$

$H^{\{i\}} \cong$ affine flat group scheme $\check{G}_\Lambda / \Lambda$

$+$ cont. W_E - action

formulation commutes with base change in Λ

so WLOG assume $\Lambda = \mathbb{Z}/\ell^n$

Thm ∃ canonical isom W_E-equiv

$$\check{G}_{\mathbb{Z}/\ell^n} \cong \hat{G}_{\mathbb{Z}/\ell^n}$$

if the W_E-action on $\hat{G}$ has a cyclotomic twist

$\check{G}$: geom constructed gp, from $\mathrm{Perv}(G)$

$\hat{G}$: abstractly constructed dual gp

Cor.

$$\begin{tikzcd}
\mathrm{Sat}^I_G(\Lambda) \arrow[rr, "\cong"] \arrow[dr] & \circlearrowright & \mathrm{Rep}(\hat{G}^I_\Lambda) \arrow[dl] \\
& \mathrm{Rep}_{W_E^I}(\Lambda) &
\end{tikzcd}$$

internally in $\mathrm{Rep}_{W_E^I}(\Lambda)$

⇝ appropriate defined,

$$\mathrm{Sat}^I_G(\Lambda) \cong \mathrm{Rep}((\hat{G}_\Lambda \rtimes W_E)^I)$$

Dual gp

G/E E any field

$\rightsquigarrow$ "universal Cartan" T of G

have flag var $\mathcal{F}l/E$ parametrizing Borel $B \subseteq G$; each has its torus quotient $T | \mathcal{F}l$

$\Rightarrow$ $T \simeq \mathbb{Z}$-local system $X^*(T) / \mathcal{F}l$

$\uparrow$ geom simply connected

$\rightsquigarrow$ arise uniquely from $\mathbb{Z}$-local system $X^*(T) / (\mathrm{Spec}\, E)_{\text{ét}}$

$\rightarrow$ arises uniquely via base change from a torus T/E

$X^*(T_{\bar{E}}) \circlearrowleft \mathrm{Gal}(\bar{E}/E)$ finite free $\mathbb{Z}$-mod

$$\begin{tikzcd} X^* \arrow[r, phantom, "\supseteq"] & X^*_+ \\ \mathrm{Gal}(\bar{E}|E) \arrow[u, bend left] \end{tikzcd}$$

dominant cocharacter (fix a B)

$\mathrm{Gal}(\bar{E}|E)$ + Weyl gp W acts on X^*

also get set of simple reflections

$$\begin{tikzcd} S \arrow[r, phantom, "\subset"] & W \\ & \mathrm{Gal}(\bar{E}|E) \arrow[u, bend left] \end{tikzcd}$$

and for any simple reflection $s \in S$

have simple roots $\alpha_s \in X^*$

root space $\mathcal{U}_{\alpha_s} \subseteq G_{\bar{E}}$

+ simple coroot $\alpha_s^\vee \in X_* := (X^*)^\vee$

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, phantom, "\subseteq"] \arrow[drr] & SL_2 \arrow[r] & G_{\bar{E}} \\ & & T_{\bar{E}} \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

→ root datum $\circlearrowright \mathrm{Gal}(\bar{E}|E)$

$$(X^*, X_*, \Phi, \Phi^\vee, X^*_+, X^+_*)$$

(only depends on inner twist of G)

<u>Obs</u> Exchanging $X^* \leftrightarrow X_*$ also a root datum

and (reductive gp) ⟶ (root data)

has a canonical splitting by

Chevalley gp scheme

$$\hat{G} / \mathbb{Z} \circlearrowleft \mathrm{Gal}(\bar{E}|E)$$

the "dual gp" of G

$\hat{G}$ comes with $\hat{T} \subset \hat{B} \subset \hat{G}$

($\mathrm{Gal}\,\bar{E}/E$ stable)

s.t. $X^*(\hat{T}) = X_*$

+ for any simple reflection $s \in S$

$$\operatorname{Lie} \hat{U}_{\alpha_s^\vee} \underset{\psi_s}{\cong} \mathbb{Z}$$

such that ψ_s are $\operatorname{Gal}(\bar{E}/E)$-inv

cyclotomic twist : will work over $\mathbb{Z}_l$ or $\mathbb{Z}/l^n$

replace ψ_s with isom

$$\operatorname{Lie} \hat{U}_{\alpha_s^\vee} \underset{\psi'_s}{\cong} \underbrace{\mathbb{Z}_l(1)}$$

↖ Tate-twist rep of $\operatorname{Gal}(\bar{E}/E)$

w/ change the $\operatorname{Gal}(\bar{E}/E) \supseteq W_E$ -action on $\hat{G}$

Q: exa? unitary gp?

To prove Thm, need to find

$$\check{T} \subseteq \check{B} \subseteq \check{G} \qquad W_E\text{-stable}$$

(geom. constructed)

- $X^*(\check{T}) = X_*$
- $\check{G}$ reductive of correct type
- isom Lie $\check{U}_{\check{\alpha}_s} \cong \mathbb{Z}/l^n\mathbb{Z}(1)$

(Slightly finer than [MV]
pinning gives can isom)

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proof of thm: Isom canonical, so may extend E to assume G split

fix a splitting of G

$$T \subseteq B \subseteq G$$

(indep of this choice as $\mathcal{F}l$ is simply connected.)

$$CT_B : Sat_G \longrightarrow Sat_T$$

sym monoidal & comm. with

Q: for any $T<G$ there is CT_B even G is non-quasi-split ✓

Q: where is it used?

$$Sat_T = Perv_{\substack{ULA \\ flat}}(Hck_T) = \bigoplus_{X_*(T)} Rep_{W_E}(\Lambda)$$

disj pts indexed by $X_*(T)$

i.e $Hck_T = \coprod_{X_*(T)} [Div^1_X / L^+T]$

trivial

$Rep(\check{T})$

this CT_B. by Tanakian formalism correspond to a map (alg)

$$\begin{tikzcd} \check{T} \arrow[rr] & \cup & \check{G} \\ & \mathbb{G}_m \arrow[ul] \arrow[ur] & \end{tikzcd}$$

← comes from $F = \bigoplus_i R^i\pi_{G*}$

can check $\check{T} \hookrightarrow \check{G}$ as one computing CT_B of std to show it's "ess surj"

$\mathbb{G}_m$ defines attracting "parabolic" (dynamic method)

$$\check{T} \subseteq \underset{\pi}{\underline{\check{B}}} \subseteq \check{G}$$

Q: how to make sure it's not the opposite Borel

$$X^*(\check{T}) = X_*(T)$$

Case of rk 1 gp

$$G \twoheadrightarrow G_{ad} \cong PGL_2$$

$$Gr_G = Gr_{G_{ad}} \times_{\underbrace{\pi_0(Gr_{G_{ad}})}_{\cong \mathbb{Z}/2}} \pi_0(Gr_G)$$

$$\Rightarrow \check{G} \cong \check{G}_{ad} \times_{\mu_2} \check{Z}$$

$$X^*(\check{Z}) = \pi_1(G)$$

reduce to adjoint case

$G = PGL_2$

$$Gr_{PGL_2} \overset{i_\mu}{\supseteq} Gr_{PGL_2,\mu} \cong (\mathbb{P}^1)$$

$\uparrow$ unique minuscule cochar

$$A = i_{\mu *}\Lambda[1] \in \mathrm{Sat}_G(\Lambda) = \mathrm{Rep}(\check{G})$$

$$F(A) = H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^2)$$

$$= \Lambda \oplus \Lambda(-1)$$

therefore $\check{G} \longrightarrow \mathrm{Aut}(F(A)) = GL_2 / \Lambda$

(and we know Galois action by Tate twist.)

Know: over $\mathbb{Q}_\ell$ + $\mathbb{F}_\ell$

$$\mathrm{Ind}\,\mathrm{Rep}(\check{G}) \longleftarrow X_*^+ = \mathbb{Z}_{\geq 0}$$

$$\mathcal{I}C_\lambda \longleftarrow\!\!\shortmid \lambda$$

$$\mathrm{Sat}_G(\mathbb{Z}_\ell) = \boxed{\varprojlim_n} \mathrm{Sat}_G(\mathbb{Z}/\ell^n\mathbb{Z})$$

$\to \mathbb{Q}_\ell$?

$$\mathrm{Sat}_G(\mathbb{Q}_\ell) = \mathrm{Sat}_G(\mathbb{Z}_\ell)[\tfrac{1}{\ell}]$$

Fact: $\mathrm{Sat}_G(\mathbb{Q}_\ell)$ is semi-simple

all objs are direct sum of $\mathcal{I}C_\lambda$

via cmp to Witt Gr

decomp thm Lie

$\Rightarrow \check{G}_{\mathbb{Q}_\ell}$ reductive + connected

semi-simple of reps

$rk = 1$

no proper stable subcat

$$\begin{tikzcd}
\rightsquigarrow \check{G} \arrow[rr] & & SL_2 \subseteq GL_2 / \mathbb{Z}_\ell \\
& \mathbb{G}_m \arrow[ul] \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

must be an iso / $\mathbb{Q}_\ell$

Over $\mathbb{F}_\ell$, $\check{G}_{\mathbb{F}_\ell} \to SL_{2,\mathbb{F}_\ell}$ must be surj, as otherwise image in torus

or Borel

or normalized of torus

This would lead to too many irreducible rep of $\check{G}_{\mathbb{F}_\ell}$ $\otimes$

Thus $\mathcal{O}(SL_2) \longrightarrow \mathcal{O}(\check{G})/\mathbb{Z}_\ell$

isom in char 0 both flat $\mathbb{Z}_\ell$

injective mod ℓ Q: ?

exclude the case of an open subset $\subsetneq$

$\Rightarrow$ must be an isomorphism

$$\check{G} \cong SL_2 = SL(F(A))$$

$$= SL(\Lambda \oplus \Lambda(-1))$$

$\rightsquigarrow$ cyclotomic twist in root gp

Back to general G

CT_P for any P exists

$\uparrow$ parabolic $\quad P \twoheadrightarrow M$ Levi

$$\begin{tikzcd}
\Rightarrow \quad \mathrm{Sat}_G \arrow[r, "CT_P"] \arrow[rr, bend left, "CT_B"] & \mathrm{Sat}_M \arrow[r, "CT_{B_M}"] & \mathrm{Sat}_T
\end{tikzcd}$$

$$\begin{tikzcd}
W & \check{T} \arrow[r, hook] \arrow[dr, equal] & \check{M} \arrow[r, hook] \arrow[dr, equal] & \check{G} \\
& & \hat{T} \arrow[r, hook] & \hat{M}
\end{tikzcd}$$

use minimal parabolics of M is rank 1

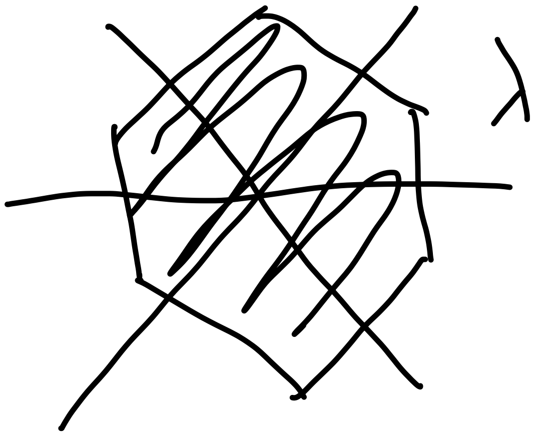
In particular, for any simple reflection s, get

$$\begin{tikzcd}
G_m \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend right] & SL_2 \arrow[r] & \hat{M} \arrow[r] & \check{G}
\end{tikzcd}$$

$$\cong \alpha_s \in X^* = X_*(\check{T})$$

Known $\check{G}_{Q_l}$ connected + reductive, and has at least as many roots / coroots as $\hat{G}$.

Doesn't have more: looking at weights appearing in $F(IC_\lambda)$



combinatoric argument

$$\Rightarrow \check{G}_{\mathbb{Q}_\ell} \cong \hat{G}_{\mathbb{Q}_\ell}$$

canonical, as most subgps are pinned.

also know $\hat{M} \hookrightarrow \check{G}$ defined integrally!

In particular

$$\begin{tikzcd} & \check{G}(\check{\mathbb{Z}}_\ell) \subseteq \check{G}(\check{\mathbb{Q}}_\ell) \\ & \cong \hat{G}(\check{\mathbb{Q}}_\ell) \\ \text{contains } \hat{M}(\check{\mathbb{Z}}_\ell) \arrow[ur, hook] & \end{tikzcd}$$

for any minimal Levi

$$\Rightarrow \underline{\hat{G}(\check{\mathbb{Z}}_\ell)} \subseteq \underline{\check{G}(\check{\mathbb{Z}}_\ell)} \subseteq \check{G}(\check{\mathbb{Q}}_\ell)$$

hyperspecial still bd subgp

note $\hat{G}$ reductive $/\mathbb{Z}$

BT theory $\Rightarrow$ $\check{G}(\mathbb{Z}_\ell) = \hat{G}(\mathbb{Z}_\ell)$

↑

This is why we work / $\mathbb{Z}_\ell$ and use ℓ-adic int

$\xRightarrow{\text{tg}}$ $\check{G} = \hat{G}$ as int model of $\check{G}_{\mathbb{Q}_\ell} \cong \hat{G}_{\mathbb{Q}_\ell}$

This finishes the proof of Geom Satake

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Q. (Zhiyu) Last step, don't know $\underline{\check{G} \text{ is finite type}}$

⊳ Lem (Prasad-Yu) $H = \hat{G}$ reductive / $\mathbb{Z}_\ell$

H' affine flat gp scheme of finite type

s.t. $\rho : H \longrightarrow H'$ is a closed imm after $\otimes\, \mathbb{Q}_\ell$

Assume $\ell \neq 2$ and no almost simple factor of $H_{\overline{\mathbb{Q}}_\ell}$

isom to SO_{2n+1} (e.g derived gp of H simply connected)

Then ρ is a closed imm

How to apply? Can assume G adjoint hence $\hat{G}$ simply connected

Pick a repr. $\check{G} \to GL_N$ that is a closed imm on generic fibre.

$$\Rightarrow \hat{G}_{\mathbb{Q}_l} \cong \check{G}_{\mathbb{Q}_l} \hookrightarrow GL_{N,\mathbb{Q}_l}$$

and $\hat{G}(\check{\mathbb{Z}}_l) \cong \check{G}(\check{\mathbb{Z}}_l) \subseteq GL_N(\check{\mathbb{Z}}_l)$

as $\hat{G}$, GL_N smooth. this gives map

$$\hat{G} \longrightarrow GL_N / \mathbb{Z}_l$$

Lemma ∃ closed imm

$$\begin{tikzcd} \check{G} \arrow[rr] \arrow[dr, dashed, "\exists"'] & & GL_N \\ & \hat{G} \arrow[ur, hook] & \end{tikzcd} \quad / \mathbb{Z}_l$$

Q. (Yun) no know of finiteness

how to show $\hat{G}(\breve{\mathbb{Z}}_l)$ bd,

$\hookrightarrow GL_N$

Q. (Zhiyu) flat over Λ

(by def'n)

Q. (Zhiyu) flat and ULA condition

↑

☆ necessary

Q. (Trakin) many use of hyperbolic loc

IC sheaves of spherical Schubert

why use flags

Q. (Tony) not simply laced

line bundle chern class

on BD Gr.

Q. (Zhiyu) attracting parabolic B

or B^{op} ?

Q (Le Bras) see all $\mathbb{Q}_\ell$-coef rep

of W_E

$\hat{\rho} : \mathbb{G}_m \longrightarrow \hat{G}_{ad}$

$$\begin{tikzcd}
 & & \hat{G}(\mathbb{Z}_\ell) \arrow[d] \\
W_E \arrow[r] & \mathbb{Z}_\ell^\times \arrow[ru, "\text{lifts}"] \arrow[r, "\hat{\rho}"] & \hat{G}_{ad}(\mathbb{Z}_\ell) \\
w \arrow[r, mapsto] & q^{|w|} \arrow[r] & \mathrm{Aut}(\hat{G}_{\mathbb{Z}_\ell})
\end{tikzcd}$$

cycl twist = twist of the usual action by this

can be lifted to

$$\begin{tikzcd}
W_E \arrow[r, "2\hat{\rho}(\sqrt{q}^{|w|})"'] & \hat{G}(\mathbb{Z}_\ell[\sqrt{q}])
\end{tikzcd}$$

So over $\mathbb{Z}_\ell[\sqrt{q}]$, can undo the twist

Drinfeld lem

Q. (Dat) Why $\mathrm{LocSys}(\mathrm{Div}(X)^{I}) = \mathrm{Rep}(W_E^{1})$?

Q. $\mathrm{LocSys}((\mathrm{Div}^1_X)^I, \Lambda) \simeq \mathrm{Rep}_{W_E^I}(\Lambda)$.

$\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$: ue

$$\pi_1^{\acute{e}t}((\mathrm{Div}^1_X)^I) = \mathrm{Gal}(\bar{E}/E)^I$$

Q. (Zhiyun) $\mathrm{Sat}(\mathbb{Z}_\ell) = \lim \mathrm{Sat}(\mathbb{Z}/\ell^n)$

naive

Condensed math for Bun_G

Q. (Travkin)

$$\pi_1 \; D_{et}(\mathrm{Bun}_G(X_{E_v})) \rightsquigarrow \text{global}$$

✓ ✓