02/08 L - parameter

Set up: E nonarch local field

G/E reductive

LLC:

$$\{\text{irr smooth } G(E)\text{-rep}\} \longrightarrow \{L\text{-parameter}\}$$

$$\pi \longmapsto \varphi_\pi$$

Usually, work with $\mathbb{C}$-coefficients

$\Rightarrow$ canonical $\sqrt{q} \in \mathbb{C}$

L-gp : $G/E \rightsquigarrow$ dual gp $\hat{G} / \mathbb{Z}$

$\circlearrowright$

$\mathrm{Gal}(\bar{E}/E) \twoheadrightarrow Q$

factor through finite quotient

Def'n $\ {}^L G = \hat{G} \rtimes Q$ alg gp$/_{\mathbb{Z}}$

(Take 1)

An L-parameter $/\mathbb{C}$ is a continuous map

$$\begin{tikzcd} W_E \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & {}^L G(\mathbb{C}) \arrow[dl] \\ & Q & \end{tikzcd}$$

equiv, a continuous cocycle

$$W_E \longrightarrow \hat{G}(\mathbb{C})$$

<u>Rek</u> Continunity $\Longleftrightarrow$ factors over a discrete quotient W_E / I'

$I' \subseteq I_E$ open finite index subgps

<u>Deligne</u>: It's better to keep track of monodromy operator

(Take 2) An L-parameter / $\mathbb{C}$ is a pair (φ, N), where

$\varphi: W_E \longrightarrow {}^L G(\mathbb{C})$ cont gp homo

$N \in \mathrm{Lie}\, \hat{g} \otimes \mathbb{C}$

s.t. $\forall w \in W_E$ $\mathrm{Ad}(\varphi(w))N = q^{|w|} N$

(or $q^{-|w|} N$?)

(For $G = GL_n$, these are so-called Weil-Deligne reps)

(Take 3, uncommon)

<u>Def'n</u> An L-parameter / $\mathbb{C}$ is a pair (φ, r), $\varphi: W_E \to {}^L G(\mathbb{C})$ cont gp homo

$r: SL_2 \to \hat{G} / \mathbb{C}$ alg rep s.t. r, φ commute $(W_E \times SL_2 \to {}^L G)$

Then $\varphi'(w) = \varphi(w)\, r\begin{pmatrix} q^{|w|/2} & \\ & q^{-|w|/2} \end{pmatrix}$

with $N = (\mathrm{Lie}\, r)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ gives an L-parameter in Take 2

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each Take 1, 2, 3 ~ a moduli of L-parameter

all distinct L-parameters in Take 2, Take 3 are (up to $\hat{G}(\mathbb{C})$-conj)

bijection

but scheme structure for 2, 3 are different

Reason: In Take 2, $N \neq 0$ degenerates to $N = 0$

we want this so 2 is the best

In Take 3, SL_2 has "rigid" rep theory

Deligne's motivition: Fix $\mathbb{C} \cong \overline{\mathbb{Q}}_l$

(Take 2') L-parameter$/\overline{\mathbb{Q}}_l$ is a continuous gp homo

$$\begin{tikzcd} \varphi : W_E \arrow[rr] \arrow[dr] & \circlearrowleft & {}^L G(\overline{\mathbb{Q}}_l) \arrow[dl] \\ & Q & \end{tikzcd}$$

Thm (Grothendieck–Deligne) Take 2 $\Leftrightarrow$ Take 2'
l-adic monodromy thm

Goal: Construct a moduli space of L-parameters i.e scheme locally of finite type

$$Z^1(W_E, \hat{G}) / \mathbb{Z}_\ell$$

s.t A-valued points (A any $\mathbb{Z}_\ell$-alg) are the continuous gp homo

$$\begin{tikzcd} \varphi : W_E \arrow[r] \arrow[dr] & {}^L G(A) \arrow[d] \\ & Q \end{tikzcd}$$

Dat-Helm-Kurinczuk-Moss, Zhu, i.e continuous 1-cocycles $W_E \to \hat{G}(A)$

Obvious question: What topology on A?

Construction: Any $\mathbb{Z}_\ell$-mod M can be endowed with the filt colimit topology

$$M = \varinjlim_{\substack{M' \subset M \\ \text{f.g} / \mathbb{Z}_\ell}} (M', \ell\text{-adic})$$

[In language of condensed math

$$\underline{M} = M_{disc} \otimes_{\mathbb{Z}_{\ell, disc}} \mathbb{Z}_\ell$$
]

<u>Rek</u> The moduli has no derived structure

<u>Thm</u> There is a scheme $Z^1(W_E, \hat{G})/_{\mathbb{Z}_\ell}$ param L-parameters for G

It's a disjoint union of affine schemes of finite type over $\mathbb{Z}_\ell$ that are flat, complete intersections, and of $\dim G = \dim \hat{G}$

<u>Note</u>: can divide by conjugation action of $\hat{G}$ to get an Artin stack
"$\mathrm{LocSys}_{\hat{G}}$"

<u>Rek</u> The natural ext to animated $\mathbb{Z}_\ell$-alg gives same moduli

<u>proof</u> (Sketch). Any cont 1-cycle

$$\varphi : W_E \longrightarrow \hat{G}(A)$$

is trivial on an open subgp P of wild inertia

$$Z^1(W_E, \hat{G}) = \bigcup_P Z^1(W_E/P, \hat{G})$$

transition maps are <u>open + closed</u>

enough: All $Z^1(W_E/P, \hat{G})$ are affine, flat, complete int of dim $= \dim \hat{G}$

<u>Trick</u>: Pick $W \subset W_E/P$ dense discrete subgp of following form:

pick generators $\sigma \in W_E$ Frob

$\tau \in I_E$ generator of tame inertia

Taking subgp generated by σ, τ, wild inertia

generated by σ

$$1 \to I \to W \to \mathbb{Z} \to 1$$

gen by τ

$$1 \to (\text{finite } p\text{-gp}) \to I \to \mathbb{Z}[\tfrac{1}{p}] \to 1$$

<u>Claim</u> $Z^1(W_E/P, \hat{G}) \longrightarrow Z^1(W, \hat{G})$

is an isomorphism

<u>proof</u> enough to show:

a cocycle $\varphi_0 : W \longrightarrow \hat{G}(A)$ extends uniquely

to a cont cocycle

$$\varphi : W_E/P \longrightarrow \hat{G}(A)$$

uniqueness: $W \subset W_E/P$ dense

existence: may enlarge E, need to see

for any $\tau^{\mathbb{Z}[\frac{1}{p}]} \rtimes \sigma^{\mathbb{Z}} \longrightarrow GL_n(A)$

$$\sigma^{-1}\tau\sigma = \tau^q$$

, the map $\mathbb{Z}[\frac{1}{p}] \longrightarrow GL_n(A)$

$n \longmapsto \mathrm{im}(\tau)^n$

extends continuously to $\prod_{l \neq p} \mathbb{Z}_l$

note: $\mathrm{im}(\tau)$ conj. to $\mathrm{im}(\tau)^q$

$\Rightarrow$ all eigenvalues are roots of unity of order prime-to-p.

$\Rightarrow$ some power is unipotent

But for unipotent matrices, all $\mathbb{Z}_l$-power are well-defined $\square$ (Claim) $\Rightarrow$ $Z^1(W_E|P, \hat{G})$ affine scheme finite type

can extend deform theory: "looks like complete intersection"

$W_E|P$ has cohom dim ≤ 2

to prove flat + correct dim, enough to bound the dim of special fibre.

Thm (Lusztig) There are finitely many unipotent conjugacy classes

$\mathcal{C} \rightsquigarrow$ stratify according to conj class of τ

$$\dim \underset{\text{conj}}{\hat{G} \cdot \tau} + \dim C_{\hat{G}}(\tau) = \dim \hat{G}.$$

A presentation of $\mathcal{O}(Z^1(W_E|P, \hat{G}))$

Fix discretization $W \subset W_E|P$

Then for any map $F_n \longrightarrow W$ from a free gp F_n

get map

$$Z^1(W_E|P, \hat{G}) = Z^1(W, \hat{G})$$

$$\longrightarrow Z^1(F_n, \hat{G}) = \hat{G}^n$$

<u>prop</u> $\underset{(n, F_n \to W)}{\mathrm{colim}} \mathcal{O}(\hat{G}^n) \xrightarrow{\sim} \mathcal{O}(Z^1(W_E/P, \hat{G}))$

shifted colimit (so agrees in mod/alg)

<u>Corollary</u> The map

$$\underset{(n, F_n \to W)}{\mathrm{colim}} \mathcal{O}(\hat{G}^n)^{\hat{G}} \longrightarrow \mathcal{O}(Z^1(W_E/P, \hat{G}))^{\hat{G}}$$

is a universal homeomorphism on spectra

and an isom after inverting l

global functions on stack of L-parameters "spectral Bernstein center"

(Use Haboush's thm on geom reductivity.

This will appear as "the algebra of excursion operators"

Thm the map

$$\operatorname*{colim}_{(n,\ F_n \to W)} \mathcal{O}(\hat{G}^n)^{\hat{G}} \longrightarrow \mathcal{O}(Z^1(W_E/P, \hat{G}))^{\hat{G}}$$

$\hat{G}$-action of simultaneous twisted conjugation

is an isomorphism if

$\hat{G}$ der simply connected $\Leftrightarrow$ $Z(G)$ connected

can get rid this assumption later

and ℓ is "not too small"

(ℓ good)

- all ℓ type A
- all $\ell \neq 2$ type ${}^2A_n, B_n, C_n, D_n, {}^2D_n$
- all $\ell \neq 2, 3$ type ${}^3D_4, {}^6D_4, E_6, E_7, F_4, G_2$
- all $\ell \neq 2, 3, 5$ type E_8

Q. (Dat) last thm $Z(G)$ non-connected

Q. (Zhiyu) Richardson's thm on closed orbits

semisimple

Q. (Yujie)

deformation theory (Tate duality)

$\mathcal{O}(Z^1(W_E|P, \hat{G}))$ has a good

$\Rightarrow H^i(\hat{G}, \mathcal{O}(Z^1)) = 0 \;\; \forall i > 0$ | $\hat{G}$-fil

$\Rightarrow$ formalutation of

$\hat{G}$-invariant commutes with

any base change

Q. (Le Bras) $Z(G)$ connected

$\Rightarrow$ all inn form from basic class

not related

Q. (Le Bras) colim in derived cat

thm is still true $D(\mathbb{Z}_\ell)$