Final

Construction of L-parameters

E nonarch local field $\qquad G/E$ reductive gp

$\ell \neq p$ $\qquad \hat{G} / \mathbb{Z}_\ell$ $\qquad$ Fix $\sqrt{q}$

$\circlearrowleft$ W_E $\qquad (\mathbb{Z}_\ell \rightsquigarrow \mathbb{Z}_\ell[\sqrt{q}])$

Representation theory side $\qquad$ **Galois side** (spectral side)

$D(G(E), \mathbb{Z}_\ell)$ (derived) cat of smooth $G(E)$-rep $\qquad$ Artin stack

$$\downarrow \qquad\qquad Z^1(W_E, \hat{G})/\hat{G}$$

$D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ $\quad \overset{\text{local}}{\underset{\text{Langlands}}{\longleftrightarrow}}$ $\quad$ of L-parameter

(use "solid 6-functor formalism")

irreducible obj $\longmapsto$ point

e.g π irr smooth rep of $G(E)$ $\longmapsto$ L-parameter φ_π

" $\pi \longmapsto \varphi_\pi$ shall vary algebraically "

<u>Def'n</u> The Bernstein center $Z(G)$ of G

$$= \mathrm{End}\left(\mathrm{id}_{\mathrm{Rep}(G(E))^{sm}}\right)$$

i.e $\forall$ each π, give $f(\pi) : \pi \to \pi$

comm. with all $\pi \to \pi'$

In particular if $f \in Z(G)$, $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$

$(\rightsquigarrow \mathrm{End}(\pi) = \overline{\mathbb{Q}}_\ell)$

get scalar $f(\pi) \in \overline{\mathbb{Q}}_\ell$.

$$Z(G)_{\overline{\mathbb{Q}}_\ell} \hookrightarrow \{\text{functions on } \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)\}$$

shall be thought as " the alg function on the set $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$ "

Q: φ_π closed pt ?

want: for any $f \in \mathcal{O}(Z^1(W_E, \hat{G}))^{\hat{G}}$

the map $\pi \longmapsto f(\varphi_\pi)$ should be algebraic i.e in $Z(G)_{\overline{\mathbb{Q}}_\ell}$

Spectral Bernstein center

$$Z^{spec}(G) := \mathcal{O}(Z^1(W_E, \hat{G}))^{\hat{G}}$$

Also consider

$$Z^{geo}(G) := \mathrm{End}(\mathrm{id}_{D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)})$$

$$\longrightarrow Z(G)$$

Main thm

(FS, small assumption on ℓ)

all ℓ OK for GL_n

$\exists$ canonical map

$\psi : Z^{spec}(G) \longrightarrow Z^{geom}(G)$

over $\mathbb{Z}_\ell$

In particular, $\forall$ each $A \in D_{lis}(\mathrm{Bun}_G, L)$

$$\mathrm{End}(A) = L$$

$L / \mathbb{Z}_\ell$ alg closed field

$\exists !$ (up to $\hat{G}(L)$-conj)

$$\varphi_A : W_E \longrightarrow \hat{G}(L)$$

, "semisimple"

($\Leftrightarrow$ closed orbit)

s.t. $\forall f \in Z^{spec}(G)$,

$$f(\varphi_A) = \psi(f)(A) \in L$$

<u>properties of this correspondence</u>

<u>prop</u> The map $\pi \mapsto \varphi_\pi$ has following prop:

(i) for tori, it agrees with usual LLC

(ii) compatible with twisting, central characters

(iii) compatible with duals

(iv) if $G' \to G$ map inducing isom of adjoint gps, π irr repr of $G(E)$

π' irr constitent of $\pi|_{G'(E)}$, then

$$\varphi_{\pi'} = \text{image of } \varphi_\pi \text{ under } \hat{G} \to \hat{G}'$$

(v) compatible with prod

(vi) compatible with Weil restriction of scalars

(vii) compatible with parabolic induction

(viii) agree with LLC for GL_n

[Only place where global method is used]
(Harris-Taylor)

(ix) comp with Hecke functor on Bun_G

(x) comp with cohomology of moduli spaces of local Shtukas e.g RZ spaces

Q: local function field LLC ?

Cor (Thm of Helm-Moss) For $G = GL_n$

$$Z^{spec}(G)_{\mathbb{Q}_\ell} \longrightarrow Z(G)_{\mathbb{Q}_\ell} \quad \text{defined by usual LLC}$$

is defined integrally i.e

induces map $Z^{spec}(G) \longrightarrow Z(G)$

" compatiblity of LLC with ℓ-adic congruence)

Construction of $\Psi : Z^{spec}(G) \to Z^{geom}(G)$

Have following : $\forall$ any finite set I

(∞)-cat $D_{lis}(Bun_G, \mathbb{Z}_\ell) = \mathcal{C}$

an exact monoidal functor

$(W_E \twoheadrightarrow Q \curvearrowright \hat{G})$

finite

$$: \mathrm{Rep}_{\mathbb{Z}_\ell}(\hat{G} \rtimes Q)^I \longrightarrow \mathrm{End}(\mathcal{C})^{W_E^I}$$

W_E^I-equiv endofunctor

linear over $\mathrm{Rep}_{\mathbb{Z}_\ell}(Q)^I$, functorially in I

We will only need this kind of abstract data.

<u>Prop</u> $\exists$ P open in wild inertia s.t.

W_E^I-action on $T_\ell(A)$ factors through W_E/P

$\rightsquigarrow$ can replace W_E by W_E/P

(discretization) ...

<u>Last time:</u>

<u>Thus</u>

$$\begin{tikzcd} \operatorname*{colim}_{(n,\, F_n \to W)} \mathcal{O}(\hat{G}^n)^{\hat{G}} \arrow[rr, "\sim"] \arrow[dr, dashed] & & \mathcal{O}(Z^1(W, \hat{G}))^{\hat{G}} \arrow[dl, dashed] \\ & Z^{geom}(G) = \mathrm{End}(\mathrm{id}_G) & \end{tikzcd}$$

enough to produce these maps

will be done by "excursion operators" (V. Lafforgue)

<u>Def'n</u> 1) An <u>excursion datum</u> is a tuple

$(I, V, \alpha: \mathbb{1} \to V|_{\Delta(\hat{G})}, \gamma_i \in W)$

finite set $\quad V \in \mathrm{Rep}(\hat{G} \rtimes Q)^I \quad \beta: V|_{\Delta(\hat{G})} \to \mathbb{1}, (\gamma_i)_{i \in I}$

2) the exc operator is the element in $\mathrm{End}(\mathcal{C})$

$\forall A \in \mathcal{C}$

$$A = T_1(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_1(A) = A$$

Prop These excursion operator define a map

$$\begin{tikzcd}
\operatorname*{colim}\limits_{(n,\, F_n \to W)} \mathcal{O}(\hat{G}^n)^{\hat{G}} \arrow[r] \arrow[dr, "\wr"'] & \mathrm{End}(\mathrm{id}_{\mathcal{C}}) \\
& \mathcal{O}(Z^1(W,\hat{G}))^{\hat{G}} \arrow[u]
\end{tikzcd}$$

Cor The L-parameter φ_A for A s.t. $\mathrm{End}(A) = L$ is characterized as follows

$\forall$ all exc data, the scalar

$$L \xrightarrow{\alpha} V \xrightarrow{(\varphi_A(\gamma_i))_{i\in I}} V \xrightarrow{\beta} L$$

agrees with scalar

$$A \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} A,$$

The spectral action

Thm (Nadler-Yun, Gaitsgory-Kazhdan-R-V, FS)

The ∞-cat data from above are equivalent to an action of

$$\mathrm{Perf}(Z^1(W_E, \hat{G})/\hat{G}) \text{ on } D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$$

$$\text{:} \quad \mathrm{Rep}(\hat{G} \times Q)^I \xrightarrow[\text{+ tensor}]{\text{pull back}} \mathrm{Perf}(Z^1(W_E, \hat{G})/\hat{G})^{W_E^I}$$

$$\downarrow$$

$$\mathrm{End}(\mathcal{C})^{W_E^I}$$

What does this mean for "elliptic" L-parameters

Assume for simplicity G semisimple, coeff $= \overline{\mathbb{Q}}_\ell$

Say φ elliptic if it defines an isolated comp of $Z^1(W_E, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}$

$$[*/S_\varphi] \xrightarrow[\text{open + closed}]{} [Z^1(W_E, \hat{G}) / \hat{G}]$$

$S_\varphi \subset \hat{G}$ centralizer

finite

$\rightsquigarrow$ get corresponding

$$D^{\varphi}_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \underset{\oplus}{\subseteq} D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

$\mathrm{Rep}(S_\varphi)$

Spectral action

Compatibility of spectral action + Hecke action

Given $V \in \mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(\hat{G} \rtimes Q)$

$$V|_{S_\varphi \times W_E} = \bigoplus_{i=1}^{m} W_i \boxtimes r_i$$

$$T_V(A) = \bigoplus_{i=1}^{m} \mathrm{Act}_{W_i}(A) \otimes r_i \in (D^{\varphi}_{lis})^{W_E}$$

$$W_i \in \mathrm{IrRep}\, S_\varphi, \quad r_i \in \mathrm{Rep}_{GL_{n_i}}(W_E)$$

$$A \in D^{\varphi}_{lis}$$

This "is" the Kottwitz conj

<u>Prop'n</u> All $A \in D^{\varphi}_{lis}$ are concentrated on (semi-stable locus)

and corr. to supercuspidal reps

$$\Rightarrow \quad D^{\varphi}_{lis} = \bigoplus_{b \in B(G)_{basic}} \bigoplus_{\substack{\pi \text{ irr supercuspidal} \\ \text{rep of } G_b(E) \\ \varphi_\pi = \varphi}} D(\overline{\mathbb{Q}}_\ell)[\pi]$$

$\circlearrowright$

$\mathrm{Rep}(S_\varphi)$

abelian gp for classical gp

If S_φ abelian, character of S_φ will

p

Also $T_V([\pi]) = \pi$ - isotypic comp of cohom of some moduli space of local Sht

$$\Vert$$

$$\bigoplus_{i=1}^{m} \mathrm{Act}_{w_i}([\pi]) \otimes r_i$$

Expect param of all π with $\varphi_\pi = \varphi$:

<u>Conj</u> Assume G quasi-split ^ Whittaker data. Then

there is a unique <u>generic</u> π with

$\varphi_{\pi_0} = \varphi$, and

$$\mathrm{Perf}([*/S_\varphi]) \xrightarrow{\sim} (CD^{\varphi}_{lis})^{\omega}$$

$$= \bigoplus_{G} \bigoplus_{\pi} \mathrm{Perf}(\overline{\mathbb{Q}}_\ell)[\pi]$$

$$W \longmapsto \mathrm{Act}_W([\pi_0])$$

is an equiv of cat

$\Rightarrow$ $\mathrm{Irr}\ \mathrm{Rep}\ S_\varphi \xrightarrow{\sim} \{\pi\}$

(This is Kaletha/ Kottwitz formulation of LLC using $B(G)_{basic}$)

Back to all of D_{lis}:

Main conj (Categorical geom Langlands)

Fix Whitt data: $B \subset G$, $U \subset B$ (with $\mathcal{U}$), $\psi : U(E) \to \check{\mathbb{Z}}_\ell^\times$ nondeg char

$\rightsquigarrow$ $c\text{-}\mathrm{Ind}_{U(E)}^{G(E)} \psi$ smooth rep of $G(E)$

There is an equiv

$$D^b_{coh}(Z^1(W_E, \hat{G})_{\check{\mathbb{Q}}_\ell} / \hat{G})$$

stack is not smooth

$$\cong D_{lis}(\mathrm{Bun}_G, \check{\mathbb{Q}}_\ell)^\omega$$

linear over $\mathrm{Perf}(Z^1(W_E, \hat{G})_{\check{\mathbb{Q}}_\ell} / \hat{G})$

taking $\mathcal{O}_{[Z^1(W_E,\hat{G})/\hat{G}]}$

to $[c\text{-}\mathrm{Ind}_{U(E)}^{G(E)}\psi)$

Integrally:

$$D^b_{coh,\mathrm{Nilp}}(Z^1(W_E,\hat{G})_{\check{\mathbb{Z}}_\ell}/\hat{G})$$

$$\cong D_{lis}(\mathrm{Bun}_G,\check{\mathbb{Z}}_\ell)^\omega$$

nilpotent

Dream: Can one compare t-structures?

Thank you! (The End?