10/30 Geometrization of local Langlands ~~correspondence~~

Taker: Zhiyu

all mistakes are mine

Fargues - Scholze

(long project ...)

History: MSRI 2014 lecture on p-adic geome~~try~~

Fargues: Idea of geometrization using FF curve

"$X_{FF} \approx \mathbb{P}^1$"

Set up

E non arch local field

- $E \simeq \mathbb{F}_q((t))$
- or E finite ext of $\mathbb{Q}_p$

Notation $\mathbb{F}_q$ residue field $\pi \in \mathcal{O}_E$

G/E reductive gp

e.g. $G = GL_n, Sp_{2n}, SO_{2n}, U_n$

$\ldots E_8, G_2$ tori $\mathbb{G}_m$

but not $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$, $\mathbb{G}_a$

Recall **Definition** Let Γ locally profinite gp

L field, a smooth rep of Γ over L

locally constant analogue to smooth rep of real gp

is an L-vector space V

+ map $\Gamma \longrightarrow GL(V)$

s.t $\forall\ v \in V$ $\quad Stab(v) \subseteq \Gamma$ is an open subgp

Ex. 1) $K \subseteq \Gamma$ open cpt subgp

$K \twoheadrightarrow \overline{K}$ finite quotient

$\rho : \overline{K} \longrightarrow \underset{= GL(V_0)}{GL_n(L)}$ rep of $\overline{K}$

w) $c\text{-}Ind_K^\Gamma \rho = \{ f : \Gamma \longrightarrow V_0 \mid f$ locally const, cpt support $\}$ $\forall g \in K$ $f(gx) = \rho(g) f(x)$

These reps form compact projective generators of all smooth rep (at least if char $L = 0$ or not p)

e.g $K = GL_n(O_E) \subseteq \Gamma = GL_n(E)$

$K \twoheadrightarrow \overline{K} = GL_n(\mathbb{F}_q)$

ρ supercuspidal rep of $GL_n(\mathbb{F}_q)$

$\rightsquigarrow$ (up to center) $c\text{-}\mathrm{Ind}_K^{P} \rho$ is

"irreducible supercuspidal rep"

2) If $P \subseteq G$ parabolic with Levi L

(V_0, φ_L) smooth rep of $L(E)$

then $\mathrm{Ind}_{P(E)}^{G(E)} V_0 = \Big\{ f : G(E) \to V_0 \;\Big|\; f(\gamma p) = \varphi_L(\gamma) p \;\; \forall \gamma \in P(E), \; f \text{ locally const} \Big\}$

smooth rep of $G(E)$

In some (vague) sense, all irr smooth rep of $G(E)$ are built from parabolic induction of supercuspidal of Levi subgp

Goal of this course: Only for supercuspidal

Don't know parabolic induction now

3) if $\mathbb{E}$ global field

G reductive on $\mathbb{E}$

s.t. (G, E) is some localization of $(G, \mathbb{E})$

space of automorphic forms

$$\phi\left(G(\mathbb{E}) \backslash G(\mathbb{A}_{\mathbb{E}}), \mathbb{C}\right) \circlearrowleft \quad \begin{array}{c} G(\mathbb{A}_E) \\ \cup\!\!| \\ G(E) \end{array}$$

is a smooth rep of $G(E)$

local $\rightsquigarrow$ global

Conjecture (Langlands) (for simplicity assume G split)

$L = \mathbb{C}$

There is a natural map

$$\mathrm{Irr}(G(E))/\cong \longrightarrow \mathrm{Hom}(W_E, \hat{G}(\mathbb{C}))/\hat{G}(\mathbb{C})$$

"L-parameters"

(not precise, finite-to-one in general)

Auto $\longrightarrow$ Galois $\pi \mapsto \varphi_\pi$

where $\hat{G}$ = Langlands dual group

$$\begin{array}{ccc} W_E = \text{Weil gp of } E & \twoheadrightarrow & \mathbb{Z} \\ \cap \text{ dense} & & \cap \\ \mathrm{Gal}(\bar{E}/E) & \twoheadrightarrow & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}} \ni \mathrm{Frob} \end{array}$$

topology on $\mathbb{Z}$ is discrete not induced from $\hat{\mathbb{Z}}$

$x \mapsto x^q$

+ description of the fiber (L-packets)

Recent improvement (Hellman, Zhu, Chen-...-Nadler ...)

can actually describe whole category $\mathrm{Rep}(G(E))$

in terms of $\underbrace{\mathrm{Hom}(W_E, \hat{G})/\hat{G}}_{\text{Artin stack}}$

<u>Question</u> 1) How does W_E relate to $\mathrm{Rep}(G(E))$?

2) Where does $\hat{G}$ come from ?

$\hat{G}$ split gps over any field are classified by root data

$$(\underset{\substack{\text{char}\\\text{lattice}}}{X}, \underset{\text{root}}{\Phi}, \underset{\substack{\text{cochar}\\\text{lattice}}}{X^*}, \underset{\text{co roots}}{\Phi^\vee})$$

dual gp

$$(X, \Phi) \xleftrightarrow{\text{switch}} (X^*, \Phi^\vee) \rightsquigarrow G \longleftrightarrow G^*$$

<u>Ex</u> $\widehat{GL_n} = GL_n$ $\qquad \widehat{SL_n} = PGL_n$

type A

simply-conn type $\qquad$ adjoint type

type C $\qquad$ type A

$\widehat{Sp_{2n}} = SO_{2n+1}$ $\qquad \widehat{SO_{2n}} = SO_{2n}$ ← type D where does $\hat{G}$ occure in geometry of G?

geo Satake

Exa of local langlands

1) $G = \mathbb{G}_m$, $G(E) = E^\times$

$$\mathrm{Irr}(E^\times) = \{\chi : E^\times \longrightarrow \mathbb{C}^\times\}$$

characters

$$\hat{G} = \mathbb{G}_m$$

$$\mathrm{Hom}(W_E, \hat{G}(\mathbb{C})) = \mathrm{Hom}(W_E^{ab}, \mathbb{C}^\times)$$

local langlands $\Longleftrightarrow$ local class field theory

explain why we replace Gal_E by W_E

$$W_E^{ab} \cong E^\times \cong \mathcal{O}_E^\times \times \mathbb{Z}$$

(and compatible to finite field Langlands

i.e $W_{\mathbb{F}_q} \simeq \mathbb{Z}$)

2) $G = GL_n$, $\hat{G} = GL_n$

<u>Thm</u> (Langlands–Rapoport–Stuhler $E = \mathbb{F}_q((t))$

Harris–Taylor, Henniart for $E = \mathbb{Q}_p$)

$L = \mathbb{C}$

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{supercuspidal irr rep} \\ \text{of } GL_n(E)\end{array}\right\} & \xleftrightarrow{\ \cong\ } & \left\{\begin{array}{c}\text{irr } n\text{-dim} \\ \text{rep} \\ W_E \to GL_n(\mathbb{C})\end{array}\right\} \\
\cap & \swarrow \text{Bernstein-Zelevinsky classification (and Involution)} & \cap \\
\mathrm{Irr}\, G(E)/\simeq & \text{-------------} & \mathrm{Hom}(W_E, \hat{G}(\mathbb{C}))
\end{array}$$

and pin down the LLC by ε-factors and axioms

(not done in this course)

<u>Example</u> E'/E $\deg = n$ extension

$\chi' : W_{E'} \twoheadrightarrow W_{E'}^{ab} \simeq E'^{\times} \longrightarrow \mathbb{C}^{\times}$ "<u>generic</u>" character

$\rightsquigarrow \quad \rho = \mathrm{Ind}_{W_{E'}}^{W_E} \chi' \quad$ irr n-dim rep of W_E

$\downarrow$ automorphic induction

a supercuspidal rep $\pi_\rho = \pi_{\chi'}$ of $GL_n(E)$

mysterious, don't know how to construct π_ρ from χ' directly

(the direct induction from $E'^\times \hookrightarrow GL_n(E)$ does not work.)

for explicit LLC for GL_n, see type theory

Goal of Course:

- Give construction of the map

$$\pi \longmapsto \varphi_\pi$$

irr rep $\qquad$ L-para $\qquad$ uniformly for any G

- purely local (no global method as before)
  (no edoscopic transfer trick to GL_n char relation)

- "explain" where W_E and $\hat{G}$ come from

- Formulate a form of LLC as equivalence of categories and construct the functor in one direction

- Extend everything from char 0 to coefficient Λ s.t $p \in \Lambda^\times$

mod p
p-adic
is **hard**

Idea Develop the geometric Langlands program on the Fargues-Fontaine curve using geometry of perfectoid spaces / diamonds

Basic References ① Berkeley Lectures on p-adic geometry

② "Montreal" notes

the original place is not ...

Big picture

" Space Spec E " (long long ago known just the scheme Spec E)

functions on it

$$\mathrm{Spec}\, E$$

coherent $\swarrow$ $\searrow$ étale

vector bundles on it

$=$ E-vector spaces

$\pi_1^{et}(\mathrm{Spec}\, E)$
$= \mathrm{Gal}(\bar{E}/E)$

need to change

G-torsors

$$\underbrace{[*/G]}_{\text{quotient stack}}(\mathrm{Spec}\, E) = \coprod_{\substack{\alpha \in H^1(E,G) \\ \text{classifying } G\text{-torsors}}} [*/G_\alpha(E)]$$

$$[*/G_\alpha(E)] \supseteq [*/G(E)]$$

$$\rightsquigarrow \mathrm{Rep}(G(E)) = \mathrm{Shv}([*/G(E)]) \subseteq \mathrm{Shv}\left([*/G](\mathrm{Spec}\, E)\right)$$

Bernstein, Vogan

(LLC using stacks)   (Vogan L-packet) view from rep theory

want to change $\mathrm{Gal}(\overline{E}/E)$ to W_E

- For scheme $X/\mathbb{F}_q$, replace X by $X_{\overline{\mathbb{F}}_q} \circlearrowleft \mathrm{Frob}$

then $\pi_1(X_{\overline{\mathbb{F}}_q} / \mathrm{Frob})$

$$= \underbrace{\pi_1^{et}(X_{\overline{\mathbb{F}}_q}) \times \mathbb{Z}}_{} \quad \text{Weil gp}$$

$$\cap$$

$$\pi_1^{et}(X_{\overline{\mathbb{F}}_q}) \times \hat{\mathbb{Z}}$$

Suggests replacing

$\mathrm{Spec}\, \mathbb{F}_q((t))$ by $\mathrm{Spec}\, \overline{\mathbb{F}}_q((t)) / \mathrm{Frob}_q$

E by $\breve{E} / \mathrm{Frob}_q$

max unram ext of E

$$\begin{tikzcd} & \operatorname{Spec} \breve{E}/\mathrm{Frob} \arrow[dl, squiggly, "\text{coherent}"'] \arrow[dr, squiggly, "\text{etale}"] & \\ \mathrm{Iso}_E & & \pi_1 = W_E \end{tikzcd}$$

Gal_E

$\cup_{(*)}$

$\mathrm{Iso}_E = \{ \breve{E}\text{-vector spaces } V + \text{Frob-linear iso } \phi^* V \simeq V \}$

$\cup (*)$

$\{ E\text{-vector spaces} \}$

$\|$ Dieudonne-Manin classification

$\bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Iso}^\lambda_E$

iso crystal "pure of slope λ"

$\mathrm{Iso}^\lambda_E = D_\lambda$-vector space

D_λ / E is the division alg of invariant $\lambda \in \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ $\simeq \mathrm{Br}(E)$

Motivated by question on special fiber of Shimura Variety, Kottwitz w/ G-torsor in Iso_E

$$G\text{-Isoc} \cong \coprod_{b \in B(E,G)} [* / G_b(E)]$$

$\leftarrow$ Kottwitz Set

G_b Inner form of Levi subgp of G

$$H^1(E,G) \overset{\text{inj}}{\hookrightarrow} B(E,G)$$

$$\alpha \longmapsto b \qquad G_b = G_\alpha$$

Kottiwitz, Kaletha view from rep theory

LLC for G_b together

need to be more geometric on coherent side

(not just rep theory of G_b)

want a geometric stack of

G-Isocrystals

Several ways to define a stack of G-Isocrystals

a) Iso_E E-linear category

for any E-algebra A

can consider G-torsors on $\mathrm{Iso}_E \otimes_E A$

$\rightsquigarrow$ Artin stack $/E$

$$\| $$

$$\coprod_{b \in B(E,G)} [* / G_b]$$

$\uparrow$ alg gp $/E$

not E-pts

wrong idea!

Better:

1) replace $\overline{\mathbb{F}_q}$ by any perfect $\overline{\mathbb{F}_q}$-alg R

$$\operatorname{Spec} \overline{\mathbb{F}}_q((t)) / \mathrm{Frob} \rightsquigarrow \operatorname{Spec} R((t)) / \mathrm{Frob}$$

shall think as small punctured disc

$$\operatorname{Spec} \breve{E} / \mathrm{Frob} \rightsquigarrow \operatorname{Spec} \left(W(R) \otimes_{W(\mathbb{F}_q)} E \right) / \mathrm{Frob}$$

$$\breve{E} = W(\overline{\mathbb{F}}_q) \otimes_{W(\mathbb{F}_q)} E$$

or

define stack on perfect $\overline{\mathbb{F}}_q$-alg

$$R \longmapsto \{ G\text{-torsors on } \}$$

G-Isoc

<u>Thm</u> (Rapport-Kottimtz, Canari-Scholze, ...)

- G-Isoc is a stack for a very strong topology (v-topology, arc-topology)

For any $b \in B(E, G)$, get locally closed substack

$$G\text{-}\mathrm{Isoc}^b \subseteq G\text{-}\mathrm{Isoc}$$

$$\wr\!\wr$$

$$[* / G_b(E)]$$

$$G\text{-}\mathrm{Isoc} \overset{\star}{=} LG /_{G\text{-conj}} LG$$

loop gp

Gaisgory, Genenestr-V.Laffgre Zhu ...

Xiao-Zhu, H...)

They $\vee$ define

$$\boxed{D(G\text{-}\mathrm{Isoc}, \overline{\mathbb{Q}}_l)}$$

will not work on it, but a variation in this course

$$\text{\rotatebox{90}{$\subseteq$}} \qquad \overline{\mathbb{Q}}_l \simeq \mathbb{C}$$

$$D(G_b(E), \overline{\mathbb{Q}}_l)$$

Rep$(G_b(E))$ derived

(need analytic version using perf space)

hence more well-behaved

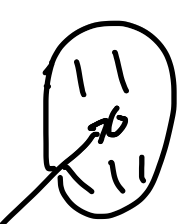
How to get a relation W_E & $\hat{G}$?

Answer: Hecke operators

These are related to modification of G-torsors on small & punctured disc $\left(W(R) \otimes_{W(\mathbb{F}_q)} E\right) / \mathrm{Frob}$

$\mathcal{E}_1, \mathcal{E}_2$ G-torsor

$+$ iso $\mathcal{E}_1 \simeq \mathcal{E}_2$ on



the point x

need sections of $\mathcal{E}$ $\Rightarrow$ work analytically

take R Banach alg

2) To any perfectoid affinoid alg $(R, R^+) / \mathbb{F}_q$

Can associate FF curve

$$= D^{\times}_{\mathrm{Spa}(R,R^+)} / \mathrm{Frob}$$

$\rightsquigarrow$ moduli space of G-tor on FF curve

$\underline{\underline{\mathrm{Bun}_G}}$ $\boxed{D^b(\mathrm{Bun}_G, \Lambda)}$ $\overset{\text{conj}}{\simeq}$ $\underline{\underline{\text{LLC}}}$ $D(*/G, \Lambda)$

$\Lambda = \overline{\mathbb{Q}}_\ell, \cdots$

now have Hecke operators

(better)

$$\mathrm{Hecke}_G = \{ \mathcal{E}_1, \mathcal{E}_2 + \text{iso on the disc} \}$$

$$\begin{tikzcd}
 & \mathrm{Hecke}_G \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times \boxed{\mathrm{Spa}\, \breve{E}/\phi}
\end{tikzcd}$$

↑ the curve

$$T_\mu = p_{2!} * p_1^* \qquad D(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$$

$$\downarrow$$

Geometric Satake

$$\mathrm{Rep}(\hat{G}) \qquad D(\mathrm{Bun}_G \times \mathrm{Spa}\, \breve{E}/\phi, \overline{\mathbb{Q}_\ell})$$

$$\| $$

$$D(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})^{W_E}$$

This Categorical structure is exactly precisely what is needed to define L-parameters

excursion operators V. Lafforgue's criterion

acts on the category ≈) L-parameter

V. Lafforgue's lem to construct L-parameter from these actions (Spetral action)

Q: in equal local char.

how to compare two LLCs

FS $\longleftrightarrow$ Genestier, V. Lafforgue

Q: Use of Condensed math?

no need for mod l, but to describe spectral action

Q, parallel to Geometric Langlands

Q, will be recorded since next time

discussion hour

Q. Isocrystals.

modular curve ss locus | different Iso crystals
ordinary locus

Q. Adelic uniformization
geometric Langlands

Q. Categorification and L-parameter

Stacks of L-parameters

a L-parameter ⇝ a hyperskyscraper sheaf on it

Q. Whittaker

Q. functoriality ?

Now we have

- center change $GL_n \rightsquigarrow PGL_n$

but endoscopic not known ...