11/02 The Fargues-Fontaine curve I

(start at 10:30 am $(+\varepsilon)$) next time

E non-arch local field

$\pi \in \mathcal{O}_E$ uniformizer

- $E \cong \mathbb{F}_q((t))$
- $[E:\mathbb{Q}_p] < +\infty$

Goal " Make Spec E geometric "

Note: $(\operatorname{Spec} E)_{\mathrm{Zar}} = *$ $\quad (\operatorname{Spec} E)_{\text{ét}} = \{\text{finite sep } E\text{-alg}\}^{op}$

$= B\operatorname{Gal}(\overline{E}/E) = \{\text{finite set w/ cont } \operatorname{Gal}(\overline{E}/E)\text{-action}\}$ groupoid

profinite situation:

$$0 \to \underset{\substack{\uparrow \\ \text{inertia}}}{I_E} \to \operatorname{Gal}(\overline{E}/E) \to \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1$$

$\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}} \ni \mathrm{Frob} = \mathrm{Frob}_q$

History: Galois gp of local field

$$0 \to \underset{\substack{\uparrow \\ \text{wild inertia, pro-}p}}{P_E} \to I_E \to \prod_{l \neq p} \mathbb{Z}_l \to 0$$

$\prod_{l \neq p} \mathbb{Z}_l = \hat{\mathbb{Z}}^p$

$\rightsquigarrow$ (local Tate duality) $\forall$ all torsion $\operatorname{Gal}(\overline{E}/E)$-rep M (prime-to-p if $E \simeq \mathbb{F}_q((t))$)

the pairing

$$H^i_{et}(\operatorname{Spec} E, M) \otimes H^{2-i}_{et}(\operatorname{Spec} E, M^*(1))$$

$$\xrightarrow{\cup} H^2_{et}(\operatorname{Spec} E, \mathbb{Q}/\mathbb{Z}) \underset{\text{Brauer gp}}{\cong} \mathbb{Q}/\mathbb{Z}$$ away from p if $E/\mathbb{F}_q$

is a <u>perfect pairing</u>

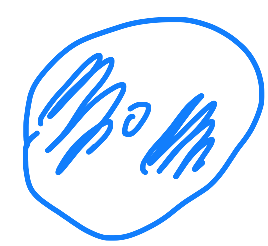
Here $M^* = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ Pontryagin dual

$\mathbb{Q}/\mathbb{Z}$ = Tate twist $= \bigcup_n \mu_n$

This looks like <u>Poincare duality</u> on a compact Riemann surface

<u>Ex</u> $E = \mathbb{F}_q((t))$ $\breve{E} = \overline{\mathbb{F}_q}((t)) \circlearrowleft \phi_{\overline{\mathbb{F}_q}}$

$\mathrm{Spec}\, \overline{\mathbb{F}_q}((t))$ "formal punctured open unit disc $/\overline{\mathbb{F}_q}$"



$\uparrow$

it's just a pt

$\rightsquigarrow$ make more spaces Fix $C/\overline{\mathbb{F}_q}$ complete alg closed nonarch field

(e.g $C = \widehat{\overline{\mathbb{F}_q(u)}}$ still alg closed by approximation)

Consider

$$\mathrm{Spa}\, C \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, \mathbb{F}_q((t)) = D^*_C \circlearrowleft \phi_C$$

Spa = adic spectrum

$\nwarrow$ punctured open unit disc $\{x \mid 0 < |x| < 1\}$ over C

Defn The FF curve (for E, C) is

$$X_{C,E} := \mathbb{D}_C^* / \phi_C^{\mathbb{Z}}$$ an adic space

shall have an absolute FF curve indep of C

over $\breve{E}^{\phi=1} = E$

(not finite type) dim $=1$

Thm (FF)

- $H^0(X_{C,E}, \mathcal{O}) = E$
- $\text{FÉt}(X_{C,E}) = (\operatorname{Spec} E)_{\text{ét}}$
- $H^i_{et}(X_{C,E}, M) = H^i_{et}(\operatorname{Spec} E, M)$

(no definition of fund gp for general adic space, at finite cover the quotient of W_E and Gal_E is the same but you can see the diff in ∞-level)

Recollections on adic spaces

Roughly variants of schemes to topological rings (e.g. Banach algs)

quite general, specialization, no Noetherian assump

Defn A topological ring

1) A is adic if there is some ideal I s.t. $\{I^n\}_{n \geq 0}$ is a nbhd (neighborhood) basis of 0

Such an I is called <u>ideal of definition</u> (non-unique)
e.g $I \rightsquigarrow I^n$

2) A is <u>Huber</u> (= f-adic in Huber's papers)

if $\exists$ open subring $A_0 \subseteq A$

that is adic with a f.g ideal of defn

($A_0 \subseteq A$ "ring of defn")

<u>Rek</u> $A \rightsquigarrow$ completion $\hat{A}$ $\hat{A_0} \subseteq \hat{A}$ open

More important case (analytic)

<u>Defn</u> A is <u>Tate</u> if it contains

a <u>topo nilpotent unit</u> $\varpi \in A$

(pseudouniformizer) it may not generate the max ideal in A_0

<u>ex</u> $\mathbb{F}_q((t)) \ni t$ $\quad \mathbb{Q}_p \ni p$ $\quad E \ni \pi$

any non arch field, any Huber ring / a nonarch field

<u>Rek</u> K any nonarch field $\varpi \in K$ pseudounif

A/K complete Huber, then

A is naturally a Banach alg over K

with $\{f \in A \mid \|f\| \le 1\} = A_0$

"unit ball"

Any $A_0 \subseteq A$ ring of defn has ϖ-adic top

Construction: $\|\cdot\| : A \longrightarrow \mathbb{R}_{\geq 0}$

($\|\ \|$ is non-unique)

$$a \longmapsto \inf_{\{n \mid \varpi^n a \in A_0\}} 2^n$$

Thm

$$\left\{ \begin{array}{c} \text{Banana alg}/K \\ \text{w./ cont maps} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Tate-Huber} \\ \text{rings}/K \end{array} \right\}$$

(Huber ring is better)

- absolute no base
- no need of $\|\ \|$

Defn The valuation spectrum (before Grothendieck) of Huber ring A is

don't like valuation...?

$$\mathrm{Cont}(A) := \{ \underbrace{|\cdot| : A \longrightarrow \Gamma \cup \{0\}}_{\text{cont valuation}} \} / \simeq$$

with topology generated by opens

$\{ |f| \leq |g| \neq 0 \} \subseteq \mathrm{Cont}\, A$ for $f, g \in A$

Here: cont valuation:

- Γ totally ordered group (e.g $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\geq 0} \times \gamma^{\mathbb{Z}}$, $r > \gamma > 1$ $\forall r \in \mathbb{R}$, $r > 1$

- $|\cdot| : A \longrightarrow \Gamma \cup \{0\}$

$|a+b| \leq \max\{|a|, |b|\}$

$|0| = 0, \quad |1| = 1$

$\forall \gamma \in \Gamma, \ \{a \mid |a| < \gamma\} \subseteq A$ open ? 0 shall not be open

Two $||_1, ||_2$ are equiv if $|a|_1 \geq |b|_1 \iff |a|_2 \geq |b|_2$ what if "$\leq$"

if Γ_i are chosen to be minimal then equiv $\iff \exists\ \Gamma_1 \simeq \Gamma_2$

s.t

$$\begin{tikzcd}
 & \Gamma_1 \cup \{0\} \arrow[d, phantom, "\wr\|"] \\
A \arrow[ur, "|\cdot|_1"] \arrow[r, "|\cdot|_2"'] & \Gamma_2 \cup \{0\}
\end{tikzcd}$$

Def (Huber pair) A Huber

(A^+, A) $A^+ \subseteq A$ open int closed power bd subring

subring of power bounded elements

$$A^\circ := \bigcup_{\substack{A_0 \subseteq A \\ \text{ring of defn}}} A_0$$

subspace top

Def'n 1) $\mathrm{Spa}(A, A^+) = \{ |\cdot| \mid |A^+| \le 1 \} \subseteq \mathrm{Cont}(A)$

2) $\mathrm{Spa}(A) = \mathrm{Spa}(A, A^\circ)$

Can endow $\mathrm{Spa}(A)$ with presheaf

$\mathcal{O}_{\mathrm{Spa}\,A}$ of Huber rings

$\cup\!|$ open subsets in $\mathrm{Spa}(A, A^+)$

$\mathcal{O}^+_{\mathrm{Spa}\,A}$ basis of $\vee$ rational subsets

$$U\left(\frac{f_1, \dots, f_n}{g}\right) = \{ |f_i| \le |g| \ne 0 \}$$

where (f_i, g) generates an open ideal

$$\mathcal{O}_{\mathrm{Spa}A}\left(U\left(\frac{f_1 \cdots f_n}{g}\right)\right) := A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$$

"allow conv series in $\frac{f_i}{g}$'s"

(Similarly $\mathcal{O}^+$ = minimal one that contains $A^+, \frac{f_i}{g}$)

Thm (Huber, ...)

In "all practical cases" $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is a sheaf

But not always !

the non-sheaf problem (can be solved in general if one uses condensed math and derived algs)

(not related to this course! everything will be sheafy in practice)

Rek Bambozzi - Kremnizer

Clausen - S

Non-sheafy can be corrected by allowing "derived Huber rings"

benefits: univ property by rational open hence gluing uniquely

Defn An adic space is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$

top space

sheaf of complete top rings

subsheaf of $\mathcal{O}_X$

that is locally of the form

$$\left(\mathrm{Spa}(A, A^+), \mathcal{O}, \mathcal{O}^+ \right)$$

(correct analogue of "locally ringed")

Exa $\mathbb{D}_C^* =$ " $\{ x \mid 0 < |x| < 1 \}$ " $\quad$ C nonarch field

punctured disc

$\rightsquigarrow$ Tate alg

$$C\langle T \rangle = \left\{ \sum_{n \geq 0} a_n T^n \;\middle|\; a_n \in C \quad a_n \to 0 \right\}$$

$\forall\, x \in C$, $|x| \leq 1$

$\rightsquigarrow$ a map $\quad C\langle T \rangle \longrightarrow C$

$$\sum a_n T^n \longmapsto \sum a_n x^n$$

So $\quad \mathrm{Spa}\, C\langle T \rangle = \mathbb{B}_C =$ " $\{ x \mid 0 \leq |x| \leq 1 \}$ "

"closed unit disc"

$$\mathbb{B}_C^* = \mathbb{B}_C \setminus \{0\}$$

$$= \bigcup_{\varepsilon > 0} \mathbb{A}(\varepsilon, 1)$$

annulus $\{x \mid \varepsilon \le |x| \le 1\}$

$\varepsilon \in |C|$

$$\mathbb{A}(\varepsilon, 1) = \mathrm{Spa}\, A_\varepsilon$$

$$A_\varepsilon := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \;\middle|\; a_n \in C \quad \underset{n \to +\infty}{|a_n| \to 0}, \quad \underset{n \to -\infty}{\varepsilon^n |a_n| \to 0} \right\}$$

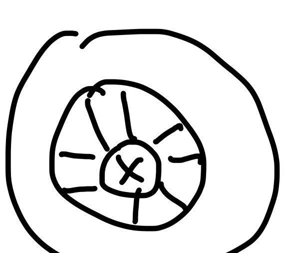
So $\mathbb{A}(\varepsilon, 1) \subseteq \mathbb{B}_C$ is a rational open subset $\left(\begin{matrix} a \in C \\ |a| = \varepsilon \end{matrix}\right)$

$\{|a| \le |T| \ne 0\}$

but $\mathbb{B}_C^*$ is not quasi-cpt

Similarly $\mathbb{D}_C^* = \bigcup_{\substack{\varepsilon > 0 \\ r < 1}} \underbrace{\mathbb{A}(\varepsilon, r)}_{= \{\varepsilon \le |T| \le r\}}$

$\ne 0$



Beware: $\mathbb{D}^*_C \subseteq B_C$ open

but $\neq \{0<|T|<1\}$ loc closed (not open!)

Q! Is the closed disc B_C "really closed" in $\mathbb{A}^1$?

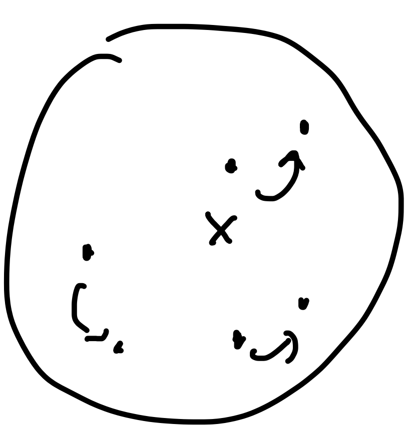
Problem There is one point $x \in B_C = \mathrm{Spa}(C\langle T\rangle)$

s.t. $r < |T(x)| < 1 \quad \forall r \in |C| \quad r<1$

$\mathrm{rk}\, T_x = 2$! ($\mathbb{D}^*_C$ doesn't contain it)

In fact, there is

$$\begin{array}{ccc} \mathbb{B}_C & \xrightarrow{\mathrm{rad}} & [0,1] \\ \cup\!| & & \cup\!| \\ \mathbb{D}^*_C & \xrightarrow{\mathrm{rad}} & (0,1) \\ \rotatebox{90}{$\in$} & & \\ x & \longmapsto & |T(x)| \end{array}$$



$\mathbb{D}^*_C \circlearrowleft \phi_C$

$\mathrm{rad} \circ \phi_C = \mathrm{rad}^{\frac{1}{q}}$

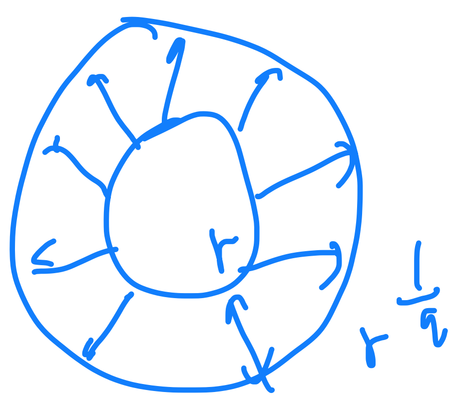
(do we need to take a the 1st generalization first)

<u>ϕ_C action</u> free properly discontinuous

$\Rightarrow \quad X_{C,E} = \mathbb{D}^*_C / \phi_C^{\mathbb{Z}}$

((well-defined adic spaces

$\mathbb{A}(r, r^{\frac{1}{q}}) \big/$ identify boundary annuli



$\phi : \mathbb{A}(r,r) \cong \mathbb{A}(r^{\frac{1}{q}}, r^{\frac{1}{q}})$

looks like

complex torus

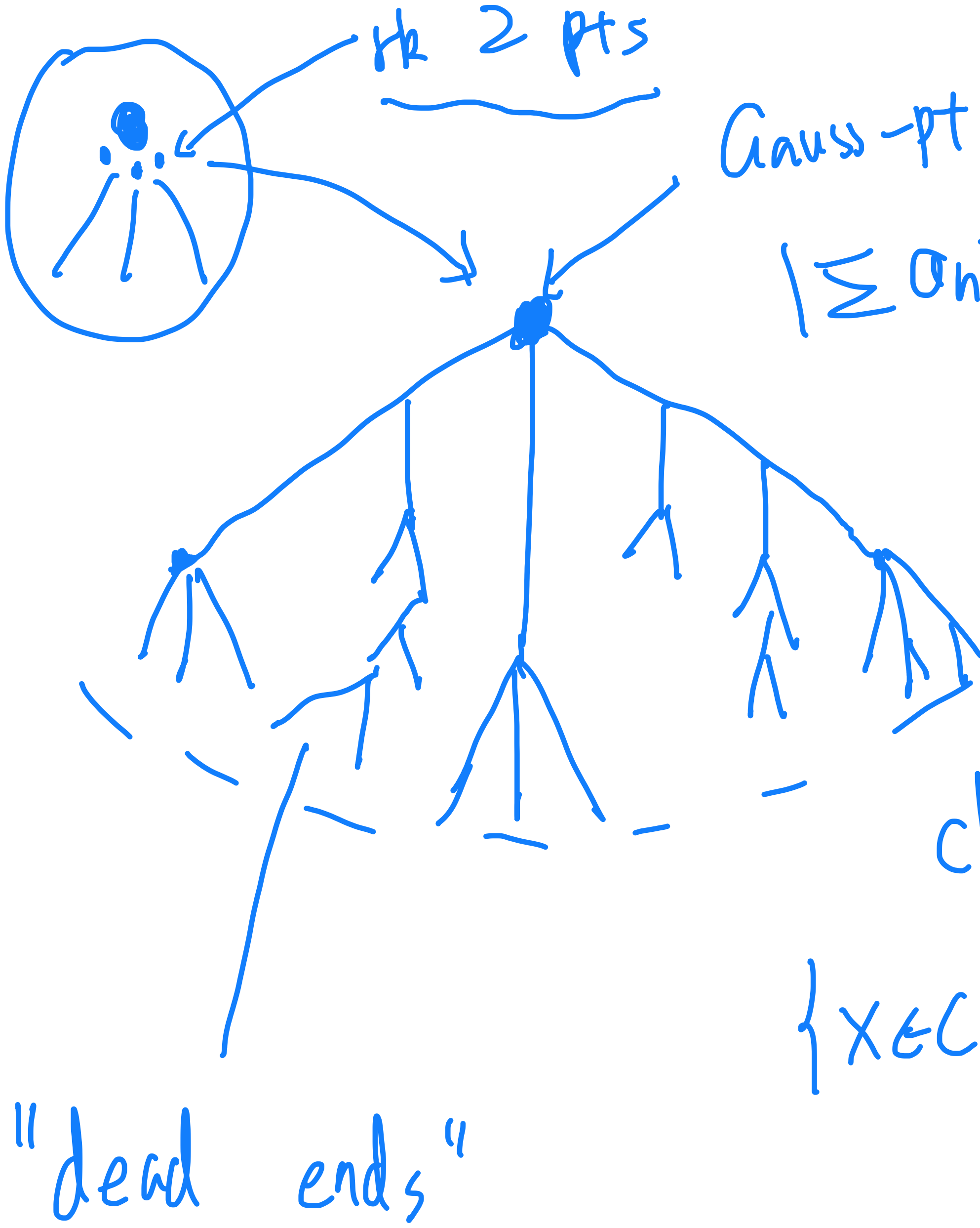
intuitively speaking

Q: other interesting autos on B_C

e.g. $x \mapsto x+1$

$|\mathbb{B}_C|$:

5 type pts



$$|\sum a_n T^n| = \sup |a_n| = \sup_{\substack{x \in \mathbb{B}_C \\ \text{classical}}} |f(x)|$$

classical pts

$\{x \in C \mid 0 \le |x| \le 1\}$

type 4

Can get rk $= n+1$ pt

in kin $= n$

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Bun$_G$ Bun_G

bad pts $\overset{SP}{\rightsquigarrow}$ good pts

Discussion : 8:15 am - 10:00 am

Q: Hecke eigensheaf loc const in curve direction

A: No idea of ramified Langlands

(level structure)

Q: What is

$$\pi_1(X_{FF} - \underline{pt})\,?$$

(no known. shall be large)

Conj by Fargues

Q: Space

relative geo Langlands

Banach-Colmez space

section of vect bundles

Next time:

mixed FF curve