1106

FF curve II

E non arch local field

π , $\mathbb{F}_q$

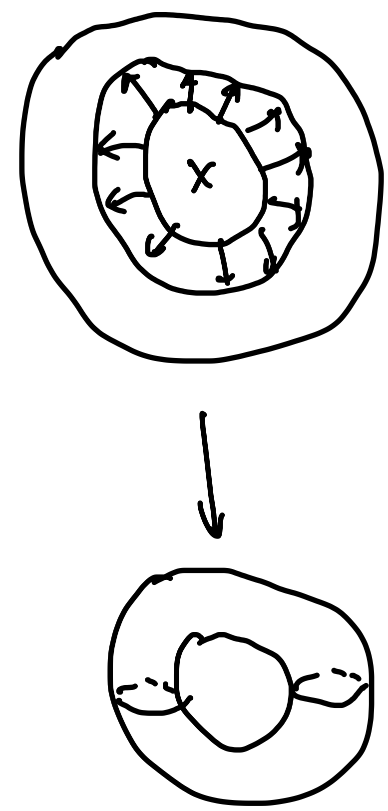
$E = \mathbb{F}_q((t))$

$C / \mathbb{F}_q$ complete alg closed

$$\operatorname{Spa} E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C = \mathbb{D}_C^*$$

punctured open unit disc

<u>Def</u> $X_{C,E} = \mathbb{D}_C^* / \phi_C^{\mathbb{Z}}$



$\mathbb{D}_C^*$

$X_{C,E}$

<u>Classical Points</u>

Tate 70's

$$\left\{ \begin{matrix} \text{rigid analytic} \\ \text{var} / C \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{adic spaces} \\ \text{"loc finite type"} \\ \text{over } \operatorname{Spa} C \end{matrix} \right\}$$

$$X(C) \longleftarrow\!\shortmid \; X$$

$X(C) \subset |X|$ "classical pts"

locally $X = \operatorname{Spa} A$, $A = C\langle T_1, \cdots, T_n \rangle / I$

$$\begin{matrix} X \\ \downarrow \\ \operatorname{Sp} A \end{matrix}$$

$$|\operatorname{Sp} A| = X(C) = \operatorname{Spm} A$$

$$\left\{ (x_1, \cdots, x_n) \in C^n \;\middle|\; \begin{matrix} |x_i| \le 1 \\ f(x_1, \cdots, x_n) \end{matrix} \; \forall f \in I \right\}$$

<u>Thm</u> (Huber) $(\mathrm{Sp}\,A)_{\mathrm{rig}}$ Grothendieck top on $\mathrm{Sp}\,A$

$\cong$ $\mathrm{Spa}\,A$ qc adm opens $=$ qc opens of $\mathrm{Spa}\,A$; adm cover $=$ cover

adic space

For $\mathbb{D}^*_C$ classical pts are

$\phi_C \circlearrowright \mathbb{D}^*_C \supseteq \{x \in C \mid 0<|x|<1\} \circlearrowleft$

$x \mapsto x^{1/q}$

For any connected affinoid $\mathrm{Spa}\,A \subseteq \mathbb{D}^*_C$

A is a PID (1-dim regular)

max ideal cor to x is gen by $T-x$

By descent, can also define classical pts

of $X_{C,E} \supseteq \{0<|x|<1\}/\phi = X^{cl}_{C,E}$

Again, any conn aff subset of $X_{C,E}$

is Spa(a PID)

Now $E/\mathbb{Q}_p$ still $C/\mathbb{F}_q$

<u>Question</u> "$\mathrm{Spa}\, E \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\, C$" $= ?$

(however, E is not over $\mathbb{F}_q$! no prod)

<u>Idea</u> In char p, deform any $\mathbb{F}_q$-alg R to $\mathbb{F}_q[[t]]$ by taking $R[[t]]$

<u>Note</u> If R <u>perfect</u> $\mathbb{F}_q$-alg, then $\exists!$ a unique (up to unique isom) lift

$\tilde{R}/\mathcal{O}_E$ flat, π-adic complete

with $\tilde{R}/\pi = R$

One choice : $\tilde{R} = W_{\mathcal{O}_E}(R) = W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$

using p-typical Witt vector

"ramified Witt vectors"

Teichmüller map $[\cdot] : R \longrightarrow \tilde{R}$

$$x \longmapsto \lim_{n\to\infty} \tilde{x}_n^{p^n}$$

$\tilde{x}_n \in \tilde{R}$ any lift of $x^{\frac{1}{p^n}}$

multiplicative

not add

(one is char $= p$
target is char $= 0$)

Any element of $\tilde{R}$ admits a unique expression as $\sum_{n\geq 0} \pi^n [r_n]$ $r_n \in R$

analogue of $\mathrm{Spa}\, \mathbb{F}_q[[t]] \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, \mathcal{O}_C \overset{?}{=} \mathrm{Spa}\, \mathcal{O}_C[[t]]$

$\mathcal{O}_C \otimes_{\mathbb{F}_q} \mathbb{F}_q[[t]] \cancel{\neq} \mathcal{O}_C[[t]]$

(true after $\wedge$)

in mix char

is "$\mathrm{Spa}\, \mathcal{O}_E \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, \mathcal{O}_C$" $:= \mathrm{Spa}\, W_{\mathcal{O}_E}(\mathcal{O}_C)$

and $\mathrm{Spa}\, \mathbb{F}_q((t)) \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, C = \mathbb{D}^*_C$

$\frac{1}{\pi}$ $\frac{1}{[\varpi]}$

mixed char case "$\mathrm{Spa}\, E \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, C$" $:= Y_{C,E} := \{\pi \neq 0, [\varpi] \neq 0\}$

$$Y_{C,E} \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(\mathcal{O}_C) \circlearrowleft \phi_C$$

<u>Def</u> FF curve

$$X_{C,E} := Y_{C,E} / \phi_C^{\mathbb{Z}}$$ over $\underbrace{\mathrm{Spa}\, E}_{\text{char} = 0}$

Thm (FF, Kedlaya)

There is a notion of classical pts

$$Y_{C,E}^{cl} \subseteq Y_{C,E} \quad \text{s.t}$$

$Y_{C,E}$ is the "mixed punct open disc"

π is the mixed T

(1) $\forall$ any connected affinoid $\mathrm{Spa}\, A \subseteq Y_{C,E}$,

A is a PID and

$$\mathrm{Spm}\, A \xrightarrow{\cong} \mathrm{Spa}\, A \cap Y_{C,E}^{cl} \subseteq Y_{C,E}$$

(2) $\forall\, y \in Y_{C,E}^{cl}$, $\exists\, x \in C$ $0<|x|<1$

use C alg closed

s.t $y = V(\pi - [x])$

$V(\pi - [x_1])$
$= V(\pi - [x_2])$

$\nRightarrow$ This x is not unique!

(equal char unique, set $t = x$)

Q: how to determine all x?

(3) $\forall\, y \in Y_{C,E}^{cl}$, the complete residue field at y is a complete alg closed nonarch field $C(y)$ with a distinguished iso $C(y)^{b} \cong C$

tilting

which gives bijection

$$Y^{cl}_{C,E} = \{ \text{untilts } C^\# / E \text{ of } C \}.$$

Tilting $\forall$ comp alg closed nonarch field s.t $|p|_K < 1$ (can be weaken to pefd alg)

one can define a comp alg closed field

$$K^\flat := \varprojlim_{x \mapsto x^p} K \quad \text{(as top mult monoid)}$$

of char p $\cup\!|$ $\qquad$ $\cup\!|$

$$\mathcal{O}_{K^\flat} := \varprojlim_{x \mapsto x^p} \mathcal{O}_K \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_K / p$$

cf. def of Teichmüller map

proof of the thm ($\theta : W(R^\flat) \twoheadrightarrow R$, $\ker\theta$)

Step 1 Construct inj map

$$\{ C^\# / E \text{ until of } C \} \longrightarrow |Y_{C,E}|$$

$$\mathcal{O}_C \cong \varprojlim_{x \mapsto x^q} \mathcal{O}_{C^\#} \longrightarrow \mathcal{O}_{C^\#}$$

$$x \longmapsto x^\#$$

(can $\otimes \mathcal{O}_E$ as $C^\#$ is E-alg)

$\rightsquigarrow \quad \theta : W(\mathcal{O}_C) \longrightarrow \mathcal{O}_{C^\#}$ "Fontaine's map"

$$\sum_{n \geq 0} [x_n] \pi^n \longmapsto \sum_{n \geq 0} x_n^\# \pi^n$$ (θ is surjective)

$$\rightsquigarrow \quad \begin{array}{ccc} \mathrm{Spa}\, \mathcal{O}_{C^\#} & \hookrightarrow & \mathrm{Spa}\, W_{\mathcal{O}_E}(\mathcal{O}_C) \\ \cup & & \cup \\ \mathrm{Spa}\, C^\# & \hookrightarrow & Y_{C,E} \end{array}$$

only 2 pts

generic pt

image $y \in Y_{C,E}$

complete residue field at y is $C^\#$ (by almost def)

$\rightsquigarrow$ injection (as the comp residue field of y recover the field)

$$\{ C^\# / E \text{ untilt of } C \} \hookrightarrow Y_{C,E} \qquad \mathrm{Image} \overset{\mathrm{def}}{=} Y_{C,E}^{cl}$$

<u>Aside</u> If (A, A^+) Huber pair $x \in \mathrm{Spa}(A, A^+)$

$$\rightsquigarrow \quad |\ |_x : A \longrightarrow \Gamma_x \cup \{0\}$$

$R_x = \{ f \in A \mid |f|_x = 0 \} \subset A$ prime idea

the complete residue field $K(x) := \widehat{\mathrm{Frac}\,(A/R_x)}$

2) Tilting for the whole disc $Y_{C,E}$

~~do it without bare change~~

Q: how about other perfd field

Let $E_\infty = E(\pi^{1/p^\infty})^\wedge$

$= \left(\bigcup_n E(\pi^{1/p^n}) \right)^\wedge$

it's a "perfectoid field"

$\mathcal{O}_{E_\infty}/p \circlearrowleft x \mapsto x^p$ is surjective

$\rightsquigarrow$ tilt $E_\infty^\flat \cong \mathbb{F}_q((t^{\frac{1}{p^\infty}}))$

$\quad \| $

$\varprojlim_{x \mapsto x^p} E_\infty \ni (\pi, \pi^{\frac{1}{p}}, \pi^{\frac{1}{p^2}}, \dots) = t$

Claim $\underbrace{(Y_{C,E} \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty)}_{\text{a perfd space}}{}^\flat$

$\cong \mathbb{D}_C^* \times_{\mathrm{Spa}\, \mathbb{F}_q((t))} \mathrm{Spa}\, \mathbb{F}_q((t^{1/p^\infty}))$

Moreover, classical pts are in bijection under this $\cong$

Cor $|\mathbb{D}_C^*| = |\mathbb{D}_C^* \times_{\mathbb{F}_q((t))} \mathbb{F}_q((t^{\frac{1}{p^\infty}}))| \overset{\text{Tilting}}{\cong} |Y_{C,E} \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty|$

perfection doesn't change Top $\longrightarrow |Y_{C,E}|$

Q: does this extend to $\mathbb{D}_C$?

$$\begin{array}{ccc} \cup\!\! | & & \cup\!\! | \\ \mathbb{D}^{*,cl}_{C} & \twoheadrightarrow & Y^{cl}_{C,E} \\ \wr\!\wr & & \\ \{0<|x|<1,\ x\in C\} & \longmapsto & V(\pi-[x]) \end{array}$$

Aside   perfd Spaces

Def 1) A **perfectoid Tate ring** is

a complete Tate ring A ($\exists\ \varpi \in A$ top nilp unit)

$\exists\ A_0 \subset A$ open ϖ-adic, $A = A_0[\frac{1}{\varpi}]$

if $\exists\ \varpi$ s.t. power b.d rngs

$\varpi^p \mid p$ in A°, A° ϖ-adic

$\Downarrow$

p is top nilpotent

$x \mapsto x^p \circlearrowright A^\circ/p$ is surjective

2) A perfectoid space is an adic space X covered by $\mathrm{Spa}(A, A^+)$ with A perfectoid Tate

Ex $A = E_\infty, C, \mathbb{F}_q((t^{\frac{1}{p^\infty}})), C\langle T^{1/p^\infty}\rangle$

if $A/\mathbb{F}_p$ Tate ring, then

$$A \text{ perfd} \Leftrightarrow A \text{ perfect}$$

Tilting holds for $\underbrace{\text{perfd rings}}$

$$A \longmapsto A^\flat = \varprojlim_{x \mapsto x^p} A$$

Ex $E_\infty\langle T^{1/p^\infty}\rangle^\flat = E_\infty^\flat \langle T^{1/p^\infty}\rangle$

and holds for perfd spaces

$$X \longmapsto X^\flat$$

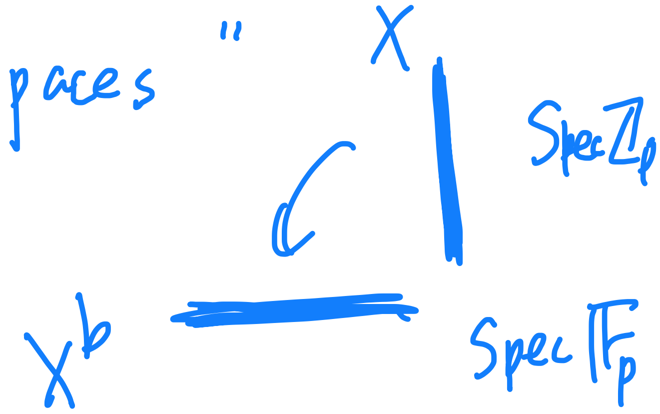
$$\mathrm{Spa}(A, A^+) \longmapsto \mathrm{Spa}(A^\flat, A^{\flat +})$$

Thm 1) $|X| \cong |X^\flat|$

Q: mixed open unit disc

$$x \longmapsto x^\flat : \quad |f(x^\flat)| := |f^\#(x)|$$

"Tilting preserves top spaces"



2) Given perfd space X

$$\{\text{perfd } Y/X\} \xrightarrow{\simeq} \{\text{perfd } Y'/X^\flat\}$$

$$Y \longmapsto Y^\flat$$

(hence fix a perfd base, untilt is unique!)

not if no base see $Y^{cl}_{C,E}$

3) (Bhatt-S) If $X = \mathrm{Spa}(A, A^+)$

$X^\flat = \mathrm{Spa}(A^\flat, A^{\flat +})$

Then Zariski closed subsets of X and $X^\flat$ correspond

originally thought wrong (strongly Zar closed)

$\Downarrow$ easy $\Uparrow$

$Z \subseteq |X|$ Zariski closed (:= vanishing locus of some ideal)

$\Updownarrow$ $\|$

$Z^\flat \subseteq |X^\flat|$ Zariski closed

Challenge: $X = \mathrm{Spa}\, C^{\#}\langle T^{\frac{1}{p^\infty}}\rangle$
(tricky)

$\supseteq Z = V(T-1)$

Show that Z^b Zar closed in $\mathrm{Spa}\,C\langle T^{\frac{1}{p^\infty}}\rangle$

(extremely hard, done in 70s)

pf of thm tilting

Q: true that $Y X_E^{E'}$ is perfd for any perfd E'

Q: " mixed open disc " dim > 1

diamonds

Q: proof the claim

Q: Why $(Y_{C,E} \times \mathrm{Spa}\,E$ $\mathrm{Spa}\,E_\infty)$ perfd (the adic

Because $Y_{C,E}$ is Noetherian ($\mathrm{Spa}\,A$ A [illegible]) $\rightsquigarrow$ $Y_{C,E}$ is not perfd

Note if R perf $\mathbb{F}_q$-alg $\mathbb{L}_{R/\mathbb{F}_q} = 0$

Q: global section of perfd space is perfd ring? open (unsolved) shall be false

Q:

Next time:

?