11/20 Units, $\mathcal{O}(1)$, Lubin-Tate theory

E nonarch local field, $\mathcal{O}_E \ni \pi$, $\mathbb{F}_q \subseteq \overline{\mathbb{F}_q}$

$\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}_q})[\frac{1}{\pi}]$ $\quad S \in \mathrm{Perf}_{\mathbb{F}_q}$

$\rightsquigarrow$

$$\begin{array}{c} Y_{S,E} \\ \downarrow \\ X_{S,E} = Y_{S,E}/\phi_S^{\mathbb{Z}} \end{array} \qquad Y_{S,E}^{\diamond} = S \times (\mathrm{Spa}\, E)^{\diamond}$$

" Untilts$/E$ = deg 1 Cartier divisor on $Y_{S,E}$ "

Given an untilt $S^\#/E$ of S

locally $S = \mathrm{Spa}(R, R^+)$, $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$

$\rightsquigarrow$ $\theta : W_{\mathcal{O}_E}(R^+) \twoheadrightarrow R^{\#,+}$

$$\left(R^+ \xrightarrow{pr} R^{\#+}/\pi \right)$$

$$= \varprojlim R^{\#+}/\pi$$

$0<|x|<1$

(primitive deg=1, $\xi = \pi - [x]$)

$\ker\theta = (\xi)$ $\quad \xi$ non-zero divisor in $W_{\mathcal{O}_E}(R^+)$

(general structure result on $\underbrace{\text{integral perfectoid rings}}_{=\text{perfect prisms}}$)

$$\rightsquigarrow \quad S^{\#} \hookrightarrow \operatorname{Spa} W_{\mathcal{O}_E}(R^+) \setminus \{\pi = 0 \text{ or } [\varpi] = 0\}$$

$$S^{\#} = V(\xi)$$

Def'n X uniform analytic adic space

A closed Cartier divisor on X is an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, locally free $rk = 1$

s.t $\forall\ \mathcal{U} \subseteq X$ affinoid

$\mathcal{I}(\mathcal{U}) \longrightarrow \mathcal{O}_X(\mathcal{U})$ has closed images

$\rightsquigarrow Z = (V(\mathcal{I}), \mathcal{O}_X/\mathcal{I}, \text{valuations})$ is an adic space inside X

prop'n 1) $V(\xi) = S^{\#} \hookrightarrow Y_{S,E}$ is a closed Cartier divisor

2) $S^{\#} \hookrightarrow Y_{S,E} \hookrightarrow X_{S,E}$ is ...

...

See prop 11.3.1 in Berkeley notes

Def'n Div^1_Y, $\mathrm{Div}^1_X : \mathrm{Perf}_{\overline{\mathbb{F}_q}} \longrightarrow \mathrm{Sets}$

be the functor $S \longmapsto$ closed Cartier divisor on $Y_{S,E}$ / $X_{S,E}$ that locally on S arise as $S^\# \hookrightarrow Y_{S,E}$ ($S^\# \hookrightarrow X_{S,E}$)

"moduli of deg 1 Cart divisor" for untilt $S^\#/E$

prop'n 1) $\mathrm{Div}^1_Y = (\mathrm{Spa}\,\breve{E})^\diamond$

2) $\mathrm{Div}^1_X = \mathrm{Div}^1_Y / \phi_E^{\mathbb{Z}} = (\mathrm{Spa}\,\breve{E})^\diamond / \phi_E^{\mathbb{Z}}$

Any diamond has a Frobenius! (as $\mathrm{Perf}_{\mathbb{F}_q}$)

$$(\mathrm{Spa}\,\breve{E})^\diamond = (\mathrm{Spa}\,E)^\diamond \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$$

ϕ_E

$S \circlearrowleft \phi_S$

this is the Frob

Proof 1) By def'n, $(\mathrm{Spa}\,\breve{E})^\diamond \twoheadrightarrow \mathrm{Div}^1_Y$

Conversely, any closed Cart div $Z \subseteq Y_{S,E}$ locally on S is an untilt of S

$\rightsquigarrow$ $Z/\breve{E}$ is an untilt of S

2) Take quotient by Frob $\square$

(note the abs Frob acts trivially)

$\rightsquigarrow$ $$X_{S,E}^\diamond = S/\phi_S^{\mathbb{Z}} \times (\mathrm{Spa}\,E)^\diamond$$

$$\mathrm{Div}^1_X = (\mathrm{Spa}\,\breve{E})^\diamond / \phi_{E}^{\mathbb{Z}}$$

"mirror curve" only a diamond

not quasiseparated, not locally spatial

"moduli of deg 1 Cartier divisor" on the curve

is not the curve!

$\mathcal{O}(1)$ + Lubin-Tate theory

Recall: $\mathcal{O}_{X_{S,E}}(1)$ is the line bundle on $X_{S,E}$ corr to isocrystal $(\breve{E}, \pi^{-1}\sigma)$

$$\rightsquigarrow \quad H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) = H^0(Y_{S,E}, \mathcal{O}_Y)^{\phi_S = \pi}$$

Goal: $\forall$ $S^\#$ untilt $/E$ of S

then $I_{S^\#} \subseteq \mathcal{O}_{X_{S,E}}$ is

(at least after pro-étale localization on S)

necessary in general the torsor can be non-trivial)

isomorphic to $\mathcal{O}_{X_{S,E}}(-1)$

(This explains $S^\# \hookrightarrow X_{S,E}$ deg = 1)

So need to construct maps

$\mathcal{O}(-1) \cong I_{S^\#}$, we embedd $I_{S^\#} \hookrightarrow \mathcal{O}$

$\cong$ maps $\mathcal{O} \longrightarrow \mathcal{O}(1)$ that vanishes at $S^\#$

So we will give a formula for $H^0(X, \mathcal{O}(1))$

in terms of a Lubin-Tate formal gp

<u>Recall</u> A Lubin-Tate gp is a

1-dim formal gp $G/\mathcal{O}_{\breve{E}}$ with

an action of $\mathcal{O}_E$ s.t two induced actions on $\mathrm{Lie}\, G$ agree

and <u>height = 1</u>

<u>Ex</u> $E = \mathbb{Q}_p$, G = formal mult group $\cong \mathrm{Spf}\, \breve{\mathbb{Z}}_p[[X]]$

$$X +_G Y = (1+X)(1+Y) - 1$$

$$[\pi]_G(X) = \pi X + a_2 X^2 + \cdots$$

mod π : first $\neq 0$ coeff is $a_{q^h} X^{q^h}$

h is <u>called the height</u>

Generic fiber : $G \times_{\mathcal{O}_E} \breve{E} \overset{\log_G}{\cong} \mathbb{G}_{a, \breve{E}}$ additive gp

One can choose $G \cong \mathrm{Spf}\, \mathcal{O}_{\breve{E}}[[X]]$ so that $\log_G(X) = X + \frac{1}{\pi} X^q + \frac{1}{\pi^2} X^{q^2} + \cdots$

(can use this to define a LT formal group)

$\rightsquigarrow \quad X +_G Y = \exp_G(\log_G(X) + \log_G(Y))$

$\in \mathcal{O}_{\breve{E}}[[X,Y]]$ (non-trivial)

Connection to local class field theory

$\forall\ n \geq 1 \quad G[\pi^n] \subseteq G$ Kernel of $\times[\pi^n]$ on G

$\simeq \mathrm{Spf}\ \mathcal{O}_{\breve{E}}[[X]] / ([\pi^n]_G(X))$

($E = \mathbb{Q}_p$ get μ_{p^n} = gp of p^n-th roots of unity)

$$G[\pi^n] \times_{\mathcal{O}_{\breve{E}}} \breve{E} = \coprod_{i=0}^{n} \underline{\mathrm{Spec}\, \breve{E}_i}$$

$(\mathcal{O}_E/\pi^n)^\times$

$\rightsquigarrow \quad \breve{E}_0 = \breve{E}, \quad \breve{E}_0 \subset \breve{E}_1 \subset \cdots \subset \breve{E}_n \subset \cdots$

adjoin a primitive π^n-th root

Thm max abelian ext of E (not $\breve{E}$!) is $\bigcup \breve{E}_n$ (up to completion issues)

$$\mathrm{Gal}(\breve{E}_n/\breve{E}) = (\mathcal{O}_E/\pi^n)^\times$$

Also, $\breve{E}_\infty$ = completion of $\bigcup \breve{E}_n$ is perfectoid $\square$

"Universal covers" $\tilde{G}$ of G

Def'n $\tilde{G} = \varprojlim_{[\pi]_G} G$

Prop'n $\tilde{G} \simeq \mathrm{Spf}\, \mathcal{O}_{\breve{E}}[[\tilde{X}^{\frac{1}{p^\infty}}]]$

pf only need to prove

$$\tilde{G} \times_{\mathcal{O}_{\breve{E}}} \overline{\mathbb{F}}_q \simeq \mathrm{Spf}\, \overline{\mathbb{F}}_q[[\tilde{X}^{\frac{1}{p^\infty}}]]$$

But $[\pi]_G(X) \equiv X^q \pmod{\pi}$ $\square$

have maps $\tilde{G} \xrightarrow{f_n} G$ projection map

$$\mathcal{O}_{\breve{E}}[[\tilde{X}^{\frac{1}{p^\infty}}]] \longleftarrow \mathcal{O}_{\breve{E}}[[X_n]]$$

$$\tilde{X} = \lim_{n\to\infty} X_n^{q^n}$$

adic

generic $\rightsquigarrow$ fiber

$$\underbrace{\tilde{G}^{ad} \times_{\mathcal{O}_E} \breve{E}}_{\text{infinite cover of open unit disc}} \longrightarrow \underbrace{G^{ad} \times_{\mathcal{O}_{\breve{E}}} \breve{E}}_{\text{open unit disc}} \xrightarrow{\log_G} \mathbb{G}_a$$

given by $\underbrace{\sum_{i\in\mathbb{Z}}}_{\text{why ??}} \pi^i \tilde{X}^{q^{-i}} \in \mathcal{O}(\tilde{G}^{ad} \times_{\mathcal{O}_{\breve{E}}} \breve{E})$ ($\varprojlim$ write it down)

prop Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\overline{\mathbb{F}_q}}$

$\rightsquigarrow$ $X_{S,E}$, $\mathcal{O}_{X_{S,E}}(1)$

Let $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ be untilt of S

Then $\tilde{G}^{ad}(S^\#) \cong \tilde{G}(R^{\#+}) \cong R^{\circ\circ}$

top nilpotent elements of R

$$\begin{tikzcd} R^{\circ\circ} \arrow[d] & \tilde{X} \arrow[d, mapsto] \\ H^0(Y, \mathcal{O}) \ni & \sum_{i \in \mathbb{Z}} \pi^i [\tilde{X}^{q^{-i}}] \end{tikzcd}$$

concrete power series

- Induces an isomorphism

$$\tilde{G}^{ad}(S^\#) \overset{\text{Key}}{\cong} H^0(X, \mathcal{O}(1)) \overset{\phi_s = \pi}{}$$
$$\cong H^0(Y, \mathcal{O})^{\phi_s = \pi}$$

- Under this iso, the evaluation

$$H^0(Y, \mathcal{O}) \longrightarrow R^\# \quad \text{at} \quad S^\# \hookrightarrow Y_{S,E}$$

is just

the logarithm map ☆

$$\begin{tikzcd} \tilde{G}(S^{\#}) \arrow[r, "\log \tilde{G}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & R^{\#} \\ \tilde{G}(R^{\#+}) \arrow[ur] & \end{tikzcd}$$

In particular

$$H^0(X, \mathcal{O}(1)) = \mathrm{Hom}(S, \mathrm{Spa}\, \overline{\mathbb{F}}_q[[\tilde{X}^{\frac{1}{p^\infty}}]])$$

<u>Rek</u> If $\underline{\underline{n \leq [E:\mathbb{Q}_p]}}$ (resp all n if E equal char)

Then $H^0(X, \mathcal{O}(n))$

$$\cong \mathrm{Hom}(S, \mathrm{Spa}\, \overline{\mathbb{F}}_q[[X_1^{\frac{1}{p^\infty}}, \dots, X_n^{\frac{1}{p^\infty}}]])$$

<u>Conclusion</u> $S \longmapsto H^0(X_{S,E}, \mathcal{O}(n))$

is represented by $\mathrm{Spa}\, \overline{\mathbb{F}}_q[[X_1^{\frac{1}{p^\infty}}, \dots, X_n^{\frac{1}{p^\infty}}]]$

a n-dim perfectoid open unit disc

if $n > [E : \mathbb{Q}_p]$ not representable

$\rightsquigarrow$ Banach-Colmez Spaces

Interesting examples of diamonds

Proof - Computation of $\log \tilde{a}$

use the formula

- clear one gets map to

$$H^0(X, \mathcal{O}(1)) = H^0(Y, \mathcal{O})^{\phi_s = \pi}$$

$E = \mathbb{F}_q((t))$,

$H^0(Y, \mathcal{O}) =$ certain power series

$t = \pi$ $\qquad \sum_{n \in \mathbb{Z}} t^n \cdot r_n \qquad r_n \in R$

subject to convergence on punctured open discs

Condition $\phi_s = \pi$, $r_{n+1}^q = r_n$, $r_0 = r \in R$

$\iff \sum_{n\in\mathbb{Z}} \pi^n r^{\frac{1}{q^n}}$ converges

This happens $\iff r \in R^{\circ\circ}$ top nilpotents

$E/\mathbb{Q}_p$ $\rightsquigarrow$ Dieudonne theory in [SW13]

$\wr$ E p-adic, one can't describe

$H^0(Y, \mathcal{O})$ as certain sums

$$\sum_{n\in\mathbb{Z}} \pi^n [r_n] \qquad r_n \in R$$

$\square$

Recall

$$0 \to \bigcup_n G[\pi^n] \to G^{ad}_{\times \breve{E}} \xrightarrow{\log_G} G_{a,\breve{E}} \to 0$$

as etale sheaves

$$\rightsquigarrow \quad 0 \to V_\pi G^{ad}_{\breve{E}} \to \tilde{G}^{ad}_{\breve{E}} \xrightarrow{\log_G} G_{a,\breve{E}} \to 0$$

$\wr\!\wr$ relative π-adic Tate module

E on geometric pts

$$V_\pi G^{ad}_{\breve E} \setminus \{0\} \cong \bigsqcup_{n \in \mathbb{Z}} \mathrm{Spa}\, \breve E_\infty$$

In particular, given an untilt $S^\# / \breve E_\infty$ get a distinguished section (choose one $\mathrm{Spa}\, \breve E_\infty$)

$$\begin{tikzcd} s \arrow[d, mapsto] & \in & \tilde G^{ad}(S^\#) \arrow[d, "\log_G"] \\ 0 & \in & G_{a,\breve E}(S^\#) \end{tikzcd}$$

corr to compatible π^n-torsion pts

Thus, under the iso, get

$$\begin{tikzcd} \mathcal{O}_{X,S} \arrow[r] \arrow[dr, "=0"'] & \mathcal{O}_{X,S}(1) \arrow[d] \\ & \mathcal{O}_{S^\#} \end{tikzcd}$$

evaluate at the chosen untilt

w/ a map $\mathcal{O}_{X,S} \longrightarrow I_{S^\#}(1)$

prop This map $\mathcal{O}_{X,S} \longrightarrow I_{S^\#}(1)$ is an isomorphism

pf: after confusing identifications follows from

$$\mathrm{Ker}(\log_{\tilde G}) \setminus \{0\} \simeq \coprod_{n \in \mathbb{Z}} \mathrm{Spa}\, \check{E}_\infty$$

reduce to universal case $S^\# = \mathrm{Spa}\, \check{E}_\infty$

Then $Y_{S,E} = \tilde{G}^{ad}_{\check{E}} \setminus \{0\}$

as both are $\mathrm{Spa}\, \mathcal{O}_{\check{E}}[[X^{\frac{1}{p^\infty}}]] \setminus \{\pi = 0 \text{ or } \varpi = 0\}$ $\square$

Next time:

BC spaces

$G/\overline{\mathbb{F}}_q$ any π-div O_E-mod

$\cong$

<u>Dieudonne module</u> (V, ϕ_V)

<u>Thm</u> $\tilde{G}(S) \cong H^0(X_{S,E}, \mathcal{E}(V))$

$n \leq [F : \mathbb{Q}_p] \iff$ slope ≤ 1

$\iff$ the isocrystal is from a p-div gp

<u>Thm</u> $Y_{S,E} \subseteq \mathcal{Y}_{S,E}$ $\quad S = \mathrm{Spa}(C)$

$\simeq \{\pi \neq 0\}$

How about VB on $\mathcal{Y}_{S,E}$

Prop (Kedlaya-Lin)

ϕ-equiv VB on $Y_{S,E}$

are equivalent to $\mathbb{Z}_p$-local systems on S

$$\mathbb{L} \longmapsto (\mathbb{L} \otimes \mathcal{O}_{Y_{S,E}}, \mathrm{id} \otimes \phi)$$

$$\mathcal{E}^{\phi=1} \longleftarrow (\mathcal{E}, \phi)$$

can be used to find integral structures (like the proof for Isocrystals) to prove classification of VB on FF $\square$

Bye ~