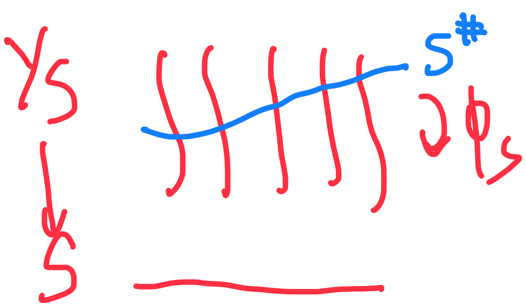
11/23 Banach - Colmez spaces

+ classification of vect bundles

Last Time $E \supset \mathcal{O}_E \ni \pi$ $\mathbb{F}_q$

$S \in \mathrm{Perf}_{\mathbb{F}_q} \rightsquigarrow$

$$\begin{array}{ccc} Y_{S,E} & = & Y_S \\ \downarrow & & \downarrow \\ X_{S,E} & = & X_S \end{array}$$



prop 1 The following sets are canonically in bijection

- sections of $Y_{S,E}^\diamond \to S$ (Frob does not act on $\mathrm{Spa}\, E$ but on $(\mathrm{Spa}\, E)^\diamond$)
- maps $S \longrightarrow (\mathrm{Spa}\, E)^\diamond$
- untilt $S^\# / E$ of S
- deg 1 closed Cartier divisors $D\ (\simeq S^\#) \subset Y_{S,E}$

$\rightsquigarrow$ moduli problem $\mathrm{Div}_Y^1 = (\mathrm{Spa}\, E)^\diamond$

prop 2 . . . . canonical in bijection !

$=$ maps $S \longrightarrow (\mathrm{Spa}\, E)^\diamond / \varphi^{\mathbb{Z}}$

$=$ deg 1 closed .... $D \subset X_S$

$\rightsquigarrow$ moduli problem $\mathrm{Div}^1_X = (\mathrm{Spa}\, E)^\diamond / \varphi^{\mathbb{Z}}$

often work over $\mathrm{Perf}_{\overline{\mathbb{F}_q}}$, get Weil gp of E:

$$\mathrm{Div}^1_Y = (\mathrm{Spa}\, \breve{E})^\diamond$$

$$\mathrm{Div}^1_X = (\mathrm{Spa}\, \breve{E})^\diamond / \varphi^{\mathbb{Z}}$$

$$= \underbrace{(\mathrm{Spa}\, C)^\diamond}_{\mathrm{Spa}\, C^b} / \underbrace{I_E \times \varphi^{\mathbb{Z}}}_{W_E}$$

$C = \hat{\overline{E}}$

$\rightsquigarrow$ $\pi_1(\mathrm{Div}^1_X) \cong W_E$

Abel-Jacobi map

$$\mathrm{Div}^1_X \longrightarrow \mathrm{Pic}^1_X$$

$$D \longmapsto \mathcal{O}(D)$$

moduli of line bundles of degree 1

Let $G / \mathcal{O}_{\breve{E}}$ Lubin–Tate gp with

$$\mathcal{O}_E \circlearrowright \qquad \log_G(X) = X + \frac{1}{\pi} X^q + \frac{1}{\pi^2} X^{q^2} + \cdots$$

(depends on choice of π)

$$\breve{E}_\infty = \breve{E}\widehat{(G[\pi^\infty])}$$

<u>prop 3</u> . . . . . . canonical bijection:

- maps $S \longrightarrow (\operatorname{Spa} \breve{E}_\infty)^\diamond$
$$= \operatorname{Spa} \breve{E}_\infty^\flat$$
$$\left(\breve{E}_\infty^\flat \simeq \overline{\mathbb{F}}_q ((X^{\frac{1}{p^\infty}})) \right)$$

- deg 1 closed Cartier div $D \subset X_S$
+ isom $\mathcal{O}(D) \cong \mathcal{O}(1)$

$$\begin{tikzcd}
\operatorname{Spa} \breve{E}_\infty \arrow[r] \arrow[d, "\underline{E}^\times\text{-torsor}"'] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d, "\text{given by } \mathcal{O}(1)"] \\
\mathrm{Div}^1_X \arrow[r] & \mathrm{Pic}^1_X
\end{tikzcd}$$

$$\mathrm{Pic}^1_X = [* / \underline{E}^\times]$$

$$AJ^1 : \mathrm{Div}^1_X \longrightarrow \mathrm{Pic}^1_X$$

$\rightsquigarrow$ map $W_E = \pi_1(\mathrm{Div}^1_X) \longrightarrow \pi_1(\mathrm{Pic}^1_X) = E^\times$

this is exactly the Artin reciprocity map in local class field theory

Fargues ('Simple connect des fibres d'un application d'Abel-Jacobi ...)

Using also $AJ^d : \mathrm{Div}^d_X \longrightarrow \mathrm{Pic}^d_X$ $(d \geq 1)$

$\mathrm{Pic}^d_X = [*/\underline{E}^\times]$

imitate Deligne's proof of geometric class field theory

$$\longrightarrow \quad W_E^{ab} \simeq E^\times$$

(Any 1-dim character of W_E induces 1-dim local system on $\mathrm{Div}^d_X = (\mathrm{Div}^1_X)^d / \Sigma_d$

but $\underline{\text{fibers of } AJ^d}$ $\underline{\text{simply connected}}$ for $d \geq 2$ (or 3)

(classically, they are proj spaces on a curve by RH ...)

hard

$\rightsquigarrow$ descends to $\mathrm{Pic}^d_X = [*/\underline{E}^\times]$

Banach-Colmez Spaces

Reference: A-C le Bras: Coherent Sheaves on the FF curve ...

Berkeley Lectures

$S \in \mathrm{Perf}_{\mathbb{F}_q} \rightsquigarrow X_S$ $\quad\mathcal{E}$ vect bundle on X_S

$=$ (locally free $\mathcal{O}_{X_S}$-mod) of finite rk

Thm (Kedlaya-Liu)

If $X = \mathrm{Spa}(A, A^+)$ affinoid analytic adic space (so $\mathcal{O}_X$ is sheaf)

$$VB(X) \xleftarrow{\sim} \{\text{finite projective } A\text{-modules}\}$$

$$M \otimes_A \mathcal{O}_X \longmapsfrom M$$

and $H^i(X, \mathcal{E}) = 0 \quad \forall\, i > 0$ (analogs of classical vanishing for affine schemes)

(also for H^i_{et} if $\mathcal{O}_X$ is an étale sheaf) (when it's in deg 0)

perfd

(a natural derived sheaf

Prop $\forall S$ affinoid

$$H^i(X_S, \mathcal{E}) = 0 \quad \forall i \geq 2$$

$$H^i(Y_S, \mathcal{E}) = 0 \quad \forall i \geq 1$$

Sketch ϖ pseudouniformizer

$\rightsquigarrow$ $\mathrm{rad} : Y_S \to (0, \infty)$ comparing $|[\varpi]|, |\pi|$

$Y_S \circlearrowleft \varphi_S$, $(0,\infty) \circlearrowleft \times q$

$\forall I = [a,b] \quad a, b \in \mathbb{Q}$ have rational subset

$$Y_{S,I} = \{ |[\varpi]|^b \leq |\pi| \leq |[\varpi]|^a \neq 0 \} \subset \underbrace{\mathrm{rad}^{-1}(I)}_{\text{closed}}$$

same rk 1 pts (remove some rk 2 pt to make it open)

$Y_{S,I} \subset Y_S$

affinoid, analytic

Then $X_S = Y_{S,[1,q]} \big/ \left(Y_{S,[1,1]} \overset{\varphi}{\cong} Y_{S,[q,q]} \right)$

$\rightsquigarrow$

Čech complex

$$R\Gamma(X_S, \mathcal{E}) \simeq [\, \mathcal{E}(Y_{S,[1,q]}) \xrightarrow{\varphi^{-1}} \mathcal{E}(Y_{S,[q,q]}) \,]$$

apply vanishing result for affinoid

$\Rightarrow$ vanishing in deg ≥ 2 for X_S

Y_S: $\quad Y_S = \bigcup_I Y_{S,I} \qquad \mathcal{O}(Y_{S,I}) \to \mathcal{O}(Y_S)$

dense image

$$R\Gamma(Y_S, \mathcal{E}) = R\varprojlim_I \mathcal{E}(Y_{S,I})$$

$\varprojlim^1 = 0$ by Mittag-Leffler

$\Rightarrow$ " $Y_{S,I}$ Stein "

$\square$

prop. $T \in \mathrm{Perf}/_S \mapsto H^0(X_T, \mathcal{E}|_{X_T})$

$\mapsto R\Gamma(X_T, \mathcal{E}|_{X_T})$

are v-sheaves

In particular, if $H^0(X_T, \mathcal{E}|_{X_T}) = 0 \ \forall T$

Then $T \mapsto H^1(X_T, \mathcal{E}|_{X_T})$ is a v-sheaf

Sketch WLOG $- \hat{\otimes}_E E_\infty$ (as $E \to E_\infty$ splits)

$X_S \times_E E_\infty$ is perfectoid

v-covers on S induces v-covers

use v-sheaf + acyclicity for general perfectoid spaces $\square$

Def'n 1) $BC(\mathcal{E}) : \mathrm{Perf}/S \to \mathrm{Sets}$

$T \mapsto H^0(X_T, \mathcal{E}|_{X_T})$

Banach-Colmez space of $\mathcal{E}$

(much interesting than just an affine space in classical picture)

2) If $BC(\mathcal{E}) = 0$

$$BC(\mathcal{E}[1]) : T \longmapsto H^1(X_T, \mathcal{E}|_{X_T})$$

"negative Banach - Colmez space"

<u>prop</u> 1) $BC(\mathcal{E})$, $BC(\mathcal{E}[1])$

locally spatial diamonds

2) $E \cong \mathbb{F}_q((t))$, $BC(\mathcal{E})$ represented by perfd space

(Rek $\forall \lambda \overset{\text{mix dim}}{>0}$, $BC(\mathcal{E})$ is not represented by perfd $\otimes$ trick!)

3) $\mathcal{E} = \mathcal{O}(\lambda)$ $\quad 0 < \lambda \leq [E : \mathbb{Q}_p]$

$\lambda = \frac{r}{s}$ $\quad (s,r)=1$, $r,s>0$ $\quad (0<\lambda)$ if $E \simeq \mathbb{F}_q((t))$

then $BC(\mathcal{E}) \cong \tilde{D}^r_S$ r-dim open perf'd unit disc $/S$

$+ \forall S$ affinoid $H^1(X_S, \mathcal{E}) = 0$

4) $R\Gamma(X_S, \mathcal{O}_{X_S}) \cong R\Gamma_{\text{proét}}(S, \underline{E})$

In particular, $\forall\ S = \mathrm{Spa}\, C$

$$R\Gamma(X_C, \mathcal{O}) = E[0]$$

<u>Sketch</u> 3) Similar to

$$BC(\mathcal{O}(1)) \cong \widetilde{D}_S$$

$$BC(\mathcal{O}(\lambda)) \cong \widetilde{G}_S$$

$\widetilde{G}$ = univ cover of p-div gp $G / \overline{\mathbb{F}_q}$ with Dieudonne module $= D_{-\lambda}$

vanishing of H^1: direct computation using $X_S = Y_{S,[1,q]} / \cdots$

4) Use

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1) \to \mathcal{O}_{S^\#} \to 0$$

$$\rightsquigarrow \quad 0 \to H^0(\mathcal{O}) \to \underset{\widetilde{G}(S^\#)}{\overset{\shortparallel}{H^0(\mathcal{O}(1))}} \xrightarrow{\log_{\widetilde{G}}} R^\# \to$$

$$H^1(\mathcal{O}) \to 0$$

$\log_{\widehat{G}}$ is pro-étale locally surjective

then $\operatorname{Ker} = \underline{E}$

1) + 2) : Bootstrap from 3) using exact seq as above

e.g. $BC(\mathcal{O}(-1)[1]) \cong (\mathbb{A}^1_E)^\diamond / \underline{E}$

$\forall S / (\operatorname{Spa} E_\infty^{LT})^\diamond$

use $0 \to \mathcal{O}_{X_S}(-1) \to \mathcal{O}_{X_S} \to \mathcal{O}_{S^\#} \to 0$

$$\to 0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}_{S^\#}) \to H^1(\mathcal{O}(-1)) \to H^1(\mathcal{O}) \to 0$$

$H^0(\mathcal{O}) = \underline{E}(S)$, $H^0(\mathcal{O}_{S^\#}) = (\mathbb{A}^1_E)^\diamond(S)$

$H^1(\mathcal{O})$: pro-étale locally on S $\simeq 0$

Classification of Vector Bundle

Back to $S = \mathrm{Spa}\, C$ generic pt

Thm $\mathrm{Iso}_E/\cong \longrightarrow VB(X_C)/\cong$

Any $\mathcal{E} \in VB(X_C)$ is isom to

$$\bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}_{X_C}(\lambda)^{n_\lambda} \quad \text{for unique } n_\lambda$$

Step 1 : $\mathcal{O}(1)$ is ample : can build alg FF curve

$\forall$ any $\mathcal{E}$ $\forall n \gg 0$

Kedlaya - Liu $\mathcal{E}(n)$ is globally generated

$+ \; H^1(X_C, \mathcal{E}(n)) = 0$

(work for any affinoid S)

(uses $X_S \simeq Y_{S,[1,q]} \Big/ \left(Y_{S,[1,1]} \overset{\varphi^{-1}}{\sim} Y_{S,[q,q]} \right)$

+ explicit estimates

Step 2 : $\mathrm{Pic}(X_C) \cong \mathbb{Z}$

$$\mathcal{O}_{X_C}(n) \longleftarrow\!\shortmid\; n$$

Step 1 $\Rightarrow$ any $\mathcal{L} \in \mathrm{Pic}(X_C)$ is ~~geometrically trivial~~

on the schematic curve

GAGA

$\rightsquigarrow$ $\mathcal{L} \cong \mathcal{O}(D)$ some divisors D on schematic curve

All closed pt in schematic curve are untilts

$\mathcal{O}(\text{untilt}) \cong \mathcal{O}(1)$ by Last lecture

$\rightsquigarrow$ $\mathcal{O}(D) \cong \mathcal{O}(\deg D)$

$\Rightarrow$ $\mathbb{Z} \twoheadrightarrow \mathrm{Pic}(X_C)$

but $H^0(\mathcal{O}(-n)) = 0$ $n>0$ injective

$\neq 0$ $n=0$

Step 3 : Build HN formalism

rk , deg : $VB(X_C) \longrightarrow \mathbb{Z}$ $\qquad \mu = \frac{\deg}{rk}$ slope

$\rightsquigarrow$ HN filtration

Using $H^1(X_C, \mathcal{O}_{X_C}(\lambda)) = 0 \quad \forall \lambda \geq 0$

reduce to the case of semistable $\mathcal{E}$

+ $\mathcal{E}$: semistable of slope 0

Step 4 : Any ss slope 0 $\mathcal{E} \cong \mathcal{O}_{X_C}^n$

v-descent : can enlarge C

($\mathrm{Spa}\, C' \longrightarrow \mathrm{Spa}\, C$ v-cover)

(torsor of isom $\mathcal{E} \cong \mathcal{O}^n$

is a $\underline{GL_n(E)}$ v-torsor over $\mathrm{Spa}\, C$

Any such torsor is split)

Assume by induction, true for $n' < n$

Consider minimal $d \geq 0$, $d \in \mathbb{Z}$ s.t there exist an injection

$$0 \longrightarrow \mathcal{O}_{X_C}(-d) \hookrightarrow \mathcal{E} \longrightarrow \overline{\mathcal{E}} \longrightarrow 0$$

$d=0$: Then $\overline{\mathcal{E}}$ ss slope 0

induction $\Rightarrow$ $\overline{\mathcal{E}} \simeq \mathcal{O}_{X_C}^{n-1}$

$H^1(X_C, \mathcal{O}_C) = 0 \Rightarrow$ ext splits

$\mathcal{E} \simeq \mathcal{O}^n$

$d \geq 2$: simple contradication

Key case $d=1$:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{E} \longrightarrow \overline{\mathcal{E}} \longrightarrow 0$$

$rk = n-1$, deg 1

slope ≥ 0

induction $\Rightarrow$ $\overline{\mathcal{E}} \simeq \mathcal{O}^i \oplus \mathcal{O}\left(\frac{1}{n-1-i}\right)$

Key case $\quad \overline{\mathcal{E}} \simeq \mathcal{O}\left(\frac{1}{n-1}\right) \quad (i=0)$

key

Lem $\mathcal{E}$ be an extension

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E} \to \mathcal{O}_X\left(\tfrac{1}{n}\right) \to 0$$

Then after enlarging C

$$H^0(X_C, \mathcal{E}) \neq 0$$

(so $\exists\ \mathcal{O} \overset{\neq 0}{\hookrightarrow} \mathcal{E}$, contradiction to $d=1$ is minimal)

Remark History

Reduction to this lem goes back to

Hartl - Pink '04

Kedlaya - Liu Robba rings

Fargues - Fontaine period map LT Dr

proof of the Lem : Assume contrary, then

Then $\forall$ $S \in \mathrm{Perf}/C$

$$H^0(X_S, \mathcal{O}_{X_S}(\tfrac{1}{n})) \hookrightarrow H^1(X_S, \mathcal{O}(-1))$$

i.e

$$\begin{array}{ccc} BC(\mathcal{O}(\tfrac{1}{n})) & \hookrightarrow & BC(\mathcal{O}(-1)[1]) \\ \wr\!\wr & & \wr\!\wr \\ \tilde{\mathbb{D}}_C & & (\mathbb{A}^1_{C^\#})^\diamond / E \end{array}$$

perf'd open unit disc

also must be surjective : image cannot be contained in classical pt (totally disconnectedness)

$\Rightarrow$ Contain some non-classical pt

$\Rightarrow$ after enlarging C, image contains $\neq \phi$ open subset) translation

$\Longrightarrow$ contains an open subset $U \ni 0$

$\Longrightarrow$ image contains everything
action of $\times \pi$

(bijective on all R-pts)

$\Longrightarrow \widetilde{\mathbb{D}}_C \overset{?}{\cong} (\mathbb{A}^1_{C^\#})^\diamond / E$

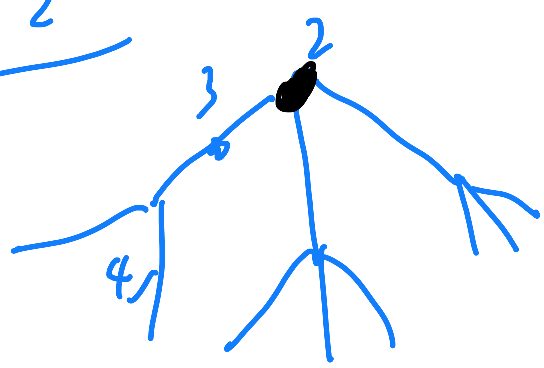
absurd, because the RHS is not perfectoid!
Stacky quotient

$E / \mathbb{Q}_p$ $\diamond$

Q: why enlarging C $\rightsquigarrow$ open subset

$x \in |(\mathbb{A}^1_C)^{ad}|$ $C(x)$

type 2:



$2 \simeq$ generic pt of disc

$B(x,r)$ $\quad$ $C(x)$

$$\tilde{x} \in (\mathbb{A}^1_{C(x)})^{ad} = (\mathbb{A}^1_C)^{ad} \times_{\mathrm{Spa}\, C} \mathrm{Spa}\, C(x)$$

$B(\tilde{x}, <r) \subset$ in the preimage of $\{x\} \subseteq |(\mathbb{A}^1_C)^{ad}|$

type 3

Q: Inertial LCC

$$\begin{tikzcd} W_E^{ab} \arrow[r, phantom, "\simeq"] \arrow[d] & E^\times \arrow[d] \\ \mathbb{Z} \arrow[r, phantom, "\simeq"] & \mathbb{Z} \end{tikzcd}$$

generally em

simply connected

Q: Abel–Jacobi map

fiber . analogs of $\mathbb{A}^n \setminus 0$

$BC(\mathcal{L}) \setminus \{0\}$

Q: E'/E deg d

$$\pi : X_{S,E'} \longrightarrow X_{S,E}$$

then
$$\pi_* \mathcal{O}(1)_{S,E'} \simeq \mathcal{O}\left(\tfrac{1}{d}\right)_{X_{S,E}}$$

compute
$$H^1\left(X, \mathcal{O}\left(\tfrac{1}{d}\right)\right)$$

Q:

$$\begin{tikzcd}
\pi_{HT} : \mathcal{M}_{ell,\infty} \arrow[r] & \mathbb{P}^1_{\mathbb{C}_p} \\
\mathcal{M}_{ell,\infty}(\varepsilon) \arrow[u, phantom, "\cup" rotate=90] \arrow[r] \arrow[ur, phantom, "\llcorner"] & \mathbb{B}(0,\varepsilon) \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

as adic spaces

anticanonical locus

affinoid perfectoid

$$f \in \mathcal{O}(\mathcal{M}^{*}_{ell,\infty,\mathrm{antican}}(\varepsilon))$$

generates a closed ideal

but not any more after passing to the ordinary locus

global function generates a closed ideal

after localization not a closed ideal