12/04 G-bundle on FF curve

$E \supseteq \mathcal{O}_E \ni \pi$, $\mathbb{F}_q$, $\overline{\mathbb{F}_q}$

$\rightsquigarrow \breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}_q})[\frac{1}{\pi}] / E$

fix a connected reductive gp G/E

e.g. $G = GL_n$, Sp_{2n}, SL_n, U_n,
E_8, G_2, SO_n, $D^{\times}_{\frac{1}{n}}$ ...

<u>prop</u> $\forall$ X scheme$/E$ Then

1) "Geometric G-torsors"

$Y \longrightarrow X$ scheme with G-action $/X$

s.t étale / smooth / fppf / fpqc locally on X

$\exists$ G-equiv isom $Y \cong G \times X$

2) "Cohomological G-torsors"

sheaf $\mathcal{F}$ on X_{et} + action of G

s.t. locally on X_{et} $\mathcal{F} \simeq G$

G-equivariantly

3) " Tannaka G-torsor " exact $\otimes$-functors

$$\mathrm{Rep}_E(G) \longrightarrow VB(X)$$

then 1) $\Longleftrightarrow$ 2) $\Longleftrightarrow$ 3)

<u>Exa</u> 1) $G = GL_n$ $\Longleftrightarrow$ rk n vect bundles on X

2) $G = Sp_{2n}$ $\Longleftrightarrow$ rk $2n$ vect bundles + perfect alternating forms on E

<u>proof</u> 1) $\to$ 2) : Take sections of $Y \longrightarrow X$

2) $\to$ 3) $V \in \mathrm{Rep}_E G$ $\mathcal{F}$

then $V \overset{G}{\times} \mathcal{F}$ is an $\mathcal{O}_X$-mod on X_{et}

locally free of finite rk $\Rightarrow$ a vect bundle by étale descent

3) $\longrightarrow$ 1) Can consider

coordinate ring $\quad G \circlearrowright \mathcal{O}(G) \circlearrowleft G \quad \in \mathrm{Ind}\,\mathrm{Rep}(G)$

with G-action, in fact an algebraic obj

Apply $F : \mathrm{Rep}_E G \longrightarrow VB(X)$

$\rightsquigarrow \quad F(\mathcal{O}(G)) \in \mathrm{Alg}(\mathrm{Ind}\,VB(X))$

$$\longrightarrow \mathrm{Alg}(\mathrm{QCoh}(X))$$

with G-action

Take $Y = \underline{\mathrm{Spec}}\, F(\mathcal{O}(G))$

$\square$

<u>Rek</u> A similar discussion to G-torsor on adic spaces: (no good theory of QCoh

Most convient option is $\otimes$-exact functors

Cor G-torsor on X are classified by $H^1(X_{et}, G)$

G - Isocrystals (Kottwitz)

Recall $\mathrm{Isoc}_E \simeq \{ (V, \phi) \mid V$ f.d. $\breve{E}$-vs, $\phi : V \simeq V$ σ-linear $\}$

Def'n A G-isocrystal is an exact $\otimes$-functor $\mathrm{Rep}_E G \longrightarrow \mathrm{Isoc}_E$

Prop Any G-isocrystal is of the form

$$\begin{aligned} \mathrm{Rep}_E\, G &\longrightarrow \mathrm{Isoc}_E \\ V &\longmapsto (V \otimes_E \breve{E}, b\sigma) \end{aligned}$$

for some $b \in G(\breve{E})$

So $\{$ isom classes of G-isocrystals $\}$

$$\cong \quad G(\breve{E}) / \sigma\text{-conj}$$

$$b \rightsquigarrow g^{-1} b\, \sigma(g)$$

$$g \in G(\breve{E})$$

Q: how to understand $H^1(F, G) \hookrightarrow B(G)$ in this way ?

Remark G-isocrystal

$=$ " G-torsors on $\operatorname{Spec} \breve{E} / \sigma^{\mathbb{Z}}$ "

Sketch enough to see that all G-torsor on $\operatorname{Spec} \breve{E}$ is trivial

Thm (Steinberg) $H^1_{et}(\operatorname{Spec} \breve{E}, G) = 0$

use $\breve{E}$ has cohom dim 1

Q: can define it for other G non-reductive

Def'n $B(G) = \{G\text{-isocrys}\} / \cong \; = G(\breve{E}) / \sigma\text{-conj}$

__Exa__ $G = GL_n$,

$B(GL_n)$ = Newton polygon of width n

(Kottwitz) Combinatorial description of $B(G)$ __for all G__

roughly Newton polygon + a finite amount extra data

ν κ

Newton pt:

__Note__: For any $(V, \phi) \in \mathrm{Isoc}_E$

V is naturally $\mathbb{Q}$-graded $V = \bigoplus_{\lambda \in \mathbb{Q}} V^\lambda$

$\Rightarrow$ a map $\mathbb{D} \longrightarrow GL_{\breve{E}}(V)$

$\mathbb{D}$ is the (pro-) torus with character gp $\mathbb{Q}$

$$\mathbb{D} = \varprojlim_{\times n} \mathbb{G}_m = \operatorname{Spec} E[T^{\mathbb{Q}}]$$

$\rightsquigarrow \operatorname{Rep}_E(\mathbb{D}) = \{ \mathbb{Q}\text{-graded } E\text{-vs} \}$

So $\forall\ F : \operatorname{Rep}_E G \longrightarrow \operatorname{Isoc}_E$

get compatible map $\mathbb{D} \longrightarrow GL_{\breve{E}}(F(V))$

Tannakian $\rightsquigarrow$ map $\mathbb{D} \longrightarrow G_{\breve{E}}$

well-defined up to conjugacy

This can be factored over a $\overset{\max}{\vee}$ torus

$X = X_*(T)$ $\quad T \subset B \subset G_{\breve{E}}$

$\circlearrowright$

$\Gamma = \operatorname{Gal}(\bar{E}|E)$

canonical, independent of T

$X^+ \subseteq X$ dominant coweights (defined by B)

$\rightsquigarrow$ get an element

$\nu(b) \in (X^+_{\mathbb{Q}})^{\Gamma}$ (why it's in X^+)

<u>Exa</u> $G = GL_n \supset B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \supset T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$

$X = X_*(T) = \mathbb{Z}^n \circlearrowleft \Gamma$ trivial

(G split)

$\cup\!|$

$X^+ = \{(m_1 \geq \cdots \geq m_n)\}$
$m_i \in \mathbb{Z}$

$X^+_{\mathbb{Q}} = \{(\lambda_1 \geq \cdots \geq \lambda_n)\}$
$\lambda_i \in \mathbb{Q}$

for rk n isocrystal this is just

$V = \bigoplus_{\lambda \in \mathbb{Q}} V^\lambda$, list λ with mult $= \dim V^\lambda$

For GL_n $\nu : B(G) \longrightarrow (X^+_{\mathbb{Q}})^{\Gamma}$ Q: for which G it's injective?

(not true for general G) is injective

Exa $\{\text{Tori } T \text{ over } E\} \cong \{X = X_*(T_{\bar{E}}) \circlearrowleft \Gamma\}$

equivalence of two categories

prop

$$B(T) \cong X_*(T)_\Gamma$$

coinvariants

pf:

$$B(T) = T(\breve{E}) / \sigma\text{-conj}$$
$$= T(\breve{E}) / (\sigma - 1)$$

$\square$

for T
$T = B$

Under this iso

$$v: B(T) \longrightarrow (X^+_{\mathbb{Q}})^\Gamma = X_{\mathbb{Q}}^\Gamma$$

is given by

average map

$$X_*(T)_\Gamma \longrightarrow X_*(T)^\Gamma_{\mathbb{Q}}$$
$$\gamma \longmapsto \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma$$

So not injective if $X_*(T)_\Gamma$ has torsion

e.g. $X_*(T) = \mathbb{Z} \circlearrowleft \Gamma$ by ± 1

replace Γ by finite quotient over which the action factors

Sketch

1) $T = \mathbb{G}_m$

$$\Gamma \circlearrowright \text{ trivial}$$

$$B(T) = \breve{E}^\times / (\sigma - 1) \longrightarrow \mathbb{Z} = X_*(T)$$

$$b \longmapsto \text{valuation of } b \quad v(b)$$

is an isom

(by classification of rk 1 isocrystals)

2) $T = \mathrm{Res}_{E'/E}\, \mathbb{G}_m$ $\quad E'/E$ finite separable

"Shapiro" : $B(E, \mathrm{Res}_{E'/E} G)$

$= B(E', G)$

$$\rightsquigarrow B(T) \cong B(E', \mathbb{G}_m) \cong \mathbb{Z}$$

$$X_*(T) = \mathrm{Ind}_{\Gamma_{E'}}^{\Gamma_E} \mathbb{Z} \qquad \Big\downarrow \Sigma$$

$$X_*(T)_{\Gamma_E} = \mathbb{Z}_{\Gamma_{E'}} = \mathbb{Z}$$

3) Resolve by induced tori

Any T admits an surj (resolve the abelian gp by choosing a basis)

$$\prod_i \mathrm{Res}_{E_i/E} \mathbb{G}_m \longrightarrow T \qquad \square$$

Back to general G, can define

$$\pi_1(G) := \pi_1(G_{\bar{E}})$$

$$= X_*(T) / \text{coroot lattice}$$

$\Gamma \circlearrowright$

" Borovoi fundamental gp " (really algebraic, not profinite étale fund)

$\forall$ $G/\mathbb{C}$ will recover usual π_1

<u>prop</u> There is a unique functorial (Kottwitz map) extension

$$K : B(G) \longrightarrow \pi_1(G)_\Gamma$$

of the above map

$$B(T) \xrightarrow{\simeq} X_*(T)_\Gamma = \pi_1(T)_\Gamma$$

for tori

<u>Sketch</u> 1) Tori ✓

2) G s.t G_{der} simply connected

$$1 \longrightarrow G_{der} \longrightarrow G \longrightarrow \underset{\substack{\uparrow \\ \text{torus}}}{D} \longrightarrow 1$$

$$\pi_1(G) \simeq \pi_1(D)$$

K must be defined by projecting to D

3) <u>General</u> $\exists$ central extension

$G' \longrightarrow G$ s.t G'_{der} is simply connected

$$\begin{tikzcd} B(G') \arrow[r, two heads] \arrow[d, "\kappa"] & B(G) \arrow[d, dashed, "\exists"] \\ \pi_1(G')_\Gamma \arrow[r] & \pi_1(G)_\Gamma \end{tikzcd}$$

$\square$

<u>Example</u> $\quad E = \mathbb{F}_q((t))$ $\quad$ Q: GL_n

$$\begin{tikzcd} B(G) \arrow[r] & \pi_1(G) & \text{"topological loops"} \\ & G(\overline{\mathbb{F}_q((t))}) \arrow[ul] \arrow[u] & \text{"loops in } G\text{"} \arrow[u, mapsto] \end{tikzcd}$$

$\left(GL_n \xrightarrow{\det} \mathbb{G}_m\right)$

<u>Exa</u> $\quad G = GL_n \quad \kappa : \{\lambda_1, \dots, \lambda_n\} \longmapsto \sum \lambda_1 + \dots + \lambda_n$

<u>Thm</u> (Kottwitz) $\quad \forall$ all G

$$(\nu, \kappa) : \quad B(G) \longrightarrow (X^+_{\mathbb{Q}})^\Gamma \times \pi_1(G)_\Gamma$$

is injective

$\rightsquigarrow$ partial order on $B(G)$

$b \leq b'$ if $\nu(b) \leq \nu(b')$ in dom orders

$+$ $\kappa(b) = \kappa(b')$

minimal elements in this order are called "basic" ("semi-stable G-torsors")

<u>Prop</u> $B(G)_{basic} \simeq \pi_1(G)_\Gamma$

(for tori, all elments are basic

because $b \leq b' \Rightarrow \kappa(b) = \kappa(b') \Rightarrow b = b'$)

<u>Prop</u> b basic $\Longleftrightarrow$ $\nu(b)$ central

<u>Prop</u> $\forall$ $b \in B(G)$

$J_b :=$ σ - centralizer of b

$=$ auto gp of the corr $\otimes$-funct $\mathrm{Rep}_E(G) \to \mathrm{Isoc}_E$

$\rightsquigarrow$ J_b connected reductive gp $/E$

$\forall$ b basic, J_b is inner form of G

In general, $\forall$ G quasi-split,

J_b is inner form of

a Levi subgp of G

(C = centralizer of $\nu(b)$)

usual notation J_b, here we use G_b

$b=1$ $\rightsquigarrow$ $J_1 = G$

c.f discussion last time for $\underset{GL_n}{\overset{G}{\shortparallel}}$

Back to FF curve

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\qquad X_S = X_{S,E}$

<u>Def</u> A G-torsor on X_S is

an exact $\otimes$-functor

$$\mathcal{E} : \mathrm{Rep}_E(G) \longrightarrow VB(X_S)$$

<u>Def'n</u> Bun_G is the v-stack on

$$\mathrm{Perf}_{\overline{\mathbb{F}}_q} : S \longmapsto \{G\text{-bundle on } X_S\}$$

groupoid

Stack of G-bundle on FF curve

<u>well-defined</u> (the curve is defined using a base S)

Thm If $S = \mathrm{Spa}(C, C^+)$ C complete alg closed

(Fargues if $E/\mathbb{Q}_p$ func field is harder
Anschütz in general)

the functor

$$\begin{tikzcd}
G\text{-}\mathrm{Isoc} \arrow[r] \arrow[d, equal] & \mathrm{Bun}_G(S) \arrow[d, equal] \\
G\text{-torsors on } \mathrm{Spa}\,\breve{E}/\sigma^{\mathbb{Z}} & G\text{-torsors on } Y_S/\phi^{\mathbb{Z}} = X_S \arrow[l]
\end{tikzcd}$$

induces a bijection on isom classes

$\rightsquigarrow$ $\mathrm{Bun}_G(S)/\cong \;\cong\; B(G)$

$\rightsquigarrow$ $|\mathrm{Bun}_G| \cong B(G)$

Sketch $\forall\, V \in \mathrm{Rep}_E G$

$\mathcal{E}(V)$ has HN filtration

So get

$$\begin{tikzcd}
 & VB(X_C) & \\
\mathrm{Rep}_E(G) \arrow[ur] \arrow[rr] \arrow[r, phantom, "\circlearrowright"] & & \mathbb{Q}\text{-Fil } VB(X_C)^{HN} \arrow[ul]
\end{tikzcd}$$

$:= \{ \mathcal{E}$ $\mathbb{Q}$-filt VB on X_C

$\mathcal{E} \supseteq \mathcal{E}^{\geq \lambda}$ s.t all $\mathcal{E}^{\lambda}$ semi-stable of slope λ $\}$

it's ① $\otimes$-functor: HN filtration is compatible with $\otimes$-product

by classification thm

(Usually hard, p-adic Simpson)

② exact-functor : trivial if $E/\mathbb{Q}_p$ as $\mathrm{Rep}_E G$ semi-simple

exact $\Leftrightarrow$ additivity $\Leftarrow$ trivial hard if E in char p

as not semi-simple

have to use thm of Haboush

on geometric reductivity.

$\rightsquigarrow$ can project to

$$\mathbb{Q}-\underline{\underline{Gr}}\ VB(X_C)^{HN} = Isoc_E$$

$\rightsquigarrow$ get candidate G-isocrystal

need to <u>split filtration</u>

Use $\boxed{H^1(X_C, \mathcal{O}(\lambda)) = 0 \quad \forall \lambda > 0}$

$\square$

<u>Cor</u> $b \in B(G)_{basic}$

$\iff \mathcal{E}_b \in Bun_G(X_C)$ semi-stable

in the sense of Atiyah-Bott

Thm $|\mathrm{Bun}_G| \longrightarrow B(G)$ is

continuous i.e

- ν : $|\mathrm{Bun}_G| \longrightarrow |X^+_{\mathbb{Q}}|^{\Gamma}$ is

semi-continuous

- κ : $|\mathrm{Bun}_G| \longrightarrow \pi_1(G)_{\Gamma}$ locally const

In fact : $\kappa : \pi_0 \mathrm{Bun}_G \cong \pi_1(G)_{\Gamma}$ ☆

(sign)

Q : all inner form of G occur as J_b ?

No, SL_n center not connected !

But Kaletha can modify the gp st center connected

So it's enough but not all to consider J_b

Q: $H^1(E, G) \hookrightarrow B(G, E)$

Q: Bun_G is "Smooth"

$Bun_G^{\leq \lambda}$