12/07 $\quad B_{dR}^+ \quad Gr$

$E \supseteq O_E \supseteq \pi \quad \mathbb{F}_q, \ \overline{\mathbb{F}}_q, \ \breve{E}$

G/E reducitive gp

$G\text{-Isoc} =$ " isocrystals with G-struct "

$B(G) = G\text{-Isoc}/\cong$

$\nu : B(G) \longrightarrow (X_{\mathbb{Q}}^+)^{\Gamma} \qquad \nu \times \kappa$ is injective

$\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$

<u>Def'n</u> $\quad Bun_G \quad$ v-stack on $Perf_{\overline{\mathbb{F}}_q}$

<u>Note</u> $\quad G\text{-Isoc} \longrightarrow Bun_G(S)$

as pullback along $X_S = Y_S/\phi^{\mathbb{Z}} \longrightarrow Spa\,\breve{E}/\sigma^{\mathbb{Z}}$

Anschütz $\quad G\text{-Iso} \xrightarrow{\sim} \varprojlim_S Bun_G(S)$

Thm (Fargues, Anschütz) If $S = \mathrm{Spa}(C, C^+)$
$E|\mathbb{Q}_p$ general E (C complete alg closed)

then $B(G) \cong \mathrm{Bun}_G(S)$

$\rightsquigarrow$ $|\mathrm{Bun}_G| \cong B(G)$ (even as top spaces)

Thm 1) $\nu : |\mathrm{Bun}_G| \longrightarrow (X_{\mathbb{Q}}^{+})^{\Gamma}$ semi-cont

2) $\kappa : |\mathrm{Bun}_G| \longrightarrow \pi_1(G)_\Gamma$ local constant

$\rightsquigarrow$ $\kappa : \pi_0 \mathrm{Bun}_G \xrightarrow{\cong} \pi_1(G)_\Gamma$

Pf 1) know this for GL_n

Rapoport – Richartz : reduce to GL_n by some embedding
$G \hookrightarrow GL_n$

2) harder as the defn of κ is by
Induced tori $\rightsquigarrow$ tori $\rightsquigarrow$ gp with der simply connected
$\rightsquigarrow$ general gp
central ext

Lem $G' \twoheadrightarrow G$ extension by central torus

Then $\mathrm{Bun}_{G'} \longrightarrow \mathrm{Bun}_G$ is a surjective map
between v-stacks

Assume this lem, go through construction of κ

- If G prod of induced tori, then $\pi_1(G)_\Gamma$ torsion free, and

$$\begin{tikzcd} & (X^+_{\mathbb{Q}})^\Gamma \arrow[dr] & \\ B(G) \arrow[ur, "\nu"] \arrow[dr, "\kappa"'] & \circlearrowright & (\pi_1(G)_{\mathbb{Q}})^\Gamma \\ & \pi_1(G)_\Gamma \arrow[ur, "\text{average}"'] & \end{tikzcd}$$

$\Gamma \supseteq \Gamma'$ have same image

injective as $\pi_1(G)_\Gamma$ tor free

- $\forall$ general torus T,

$\exists$ surjection $\tilde{T} \longrightarrow T$

$\tilde{T}$: prod of induced tori

Lem: $\mathrm{Bun}_{\tilde{T}} \longrightarrow \mathrm{Bun}_T$ surj

$$\Rightarrow \begin{tikzcd} |\mathrm{Bun}_{\tilde{T}}| \arrow[r] \arrow[d, "\kappa_{\tilde{T}}"'] & |\mathrm{Bun}_T| \arrow[d, "\kappa_T"] \\ \pi_1(\tilde{T})_\Gamma \arrow[r] & \pi_1(T)_\Gamma \end{tikzcd}$$

quotient map ?

- G with G^{der} simply connected

$$\pi_1(G) \simeq \pi_1\left(\frac{G/[G,G]}{T}\right)$$

$$\begin{tikzcd} {|\mathrm{Bun}_G|} \arrow[r] \arrow[dr, "\kappa_G"'] & {|\mathrm{Bun}_T|} \arrow[d, "\kappa_T"] \\ & \pi_1(G)_\Gamma = \pi_1(T)_\Gamma \end{tikzcd}$$

- General G. Then $\exists\ G' \to G$ central ext by torus, G'^{der} simply connected $\square$

<u>Rek</u>

$$\begin{tikzcd} \mathrm{Bun}_{SL_n} \arrow[r] \arrow[d, "\kappa"] & \mathrm{Bun}_{PGL_n} \arrow[d, "\kappa"] \\ 0 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \end{tikzcd}$$

<u>not surjective</u> !

($\ker = \mu_n$ not a torus)

<u>Cor</u> (slight strengthing of Rapoport - Richartz)

If S perfect scheme $/\overline{\mathbb{F}_q}$ $\mathcal{E}$ G-isocrystal $/S$

Then $\kappa : |S| \longrightarrow \pi_1(G)_\Gamma$ given by $\mathcal{E}$

is locally constant

<u>Sketch</u> $S = \operatorname{Spec} R$

$\forall$ S' perfd $/ \overline{\mathbb{F}}_q$ with $S' \longrightarrow \operatorname{Spa}(R, R)$

get map $S' \longrightarrow \operatorname{Bun}_G$

use κ is loc const for Bun_G $\square$

<u>Lem</u> $G' \longrightarrow G$ ext by a central

torus, then $\operatorname{Bun}_{G'} \longrightarrow \operatorname{Bun}_G$ surjective

<u>pf</u> Use BL uniformization

$\boxed{\operatorname{Gr}_G^{B_{dR}^+} \longrightarrow \operatorname{Bun}_G}$ surjective map of v-stacks

$+$ $\operatorname{Gr}_{G'}^{B_{dR}^+} \longrightarrow \operatorname{Gr}_G^{B_{dR}^+}$ surjective

B_{dR}^+ - affine Grasmannian

R any perf'd ring $/ O_E$ so we allow char p untilt

$\varpi \in R^b$ pseudo-unif

$\rightsquigarrow \quad \theta : W_{O_E}(R^{bo}) \twoheadrightarrow R^o$

$\ker \theta = (\xi) \qquad \xi = \pi + [\varpi] \cdot a$

$a \in W_{O_E}(R^{bo})$

$$\theta : W_{O_E}(R^{bo})\left[\frac{1}{[\varpi]}\right] \twoheadrightarrow R = R^o\left[\frac{1}{\varpi^\#}\right]$$

<u>Def'n</u> $B_{dR}^+(R) = \xi$-adic completion of

$$W_{O_E}(R^{bo})\left[\frac{1}{[\varpi]}\right]$$

<u>Note</u> - $B_{dR}^+(R)/(\xi) = R$

"1-parameter deformation" of R

- $B_{dR}^+(R) \to R$ <u>univ</u> pro-infinitesimal thickening in solid $\mathcal{O}_E$-algs

(condensed math)

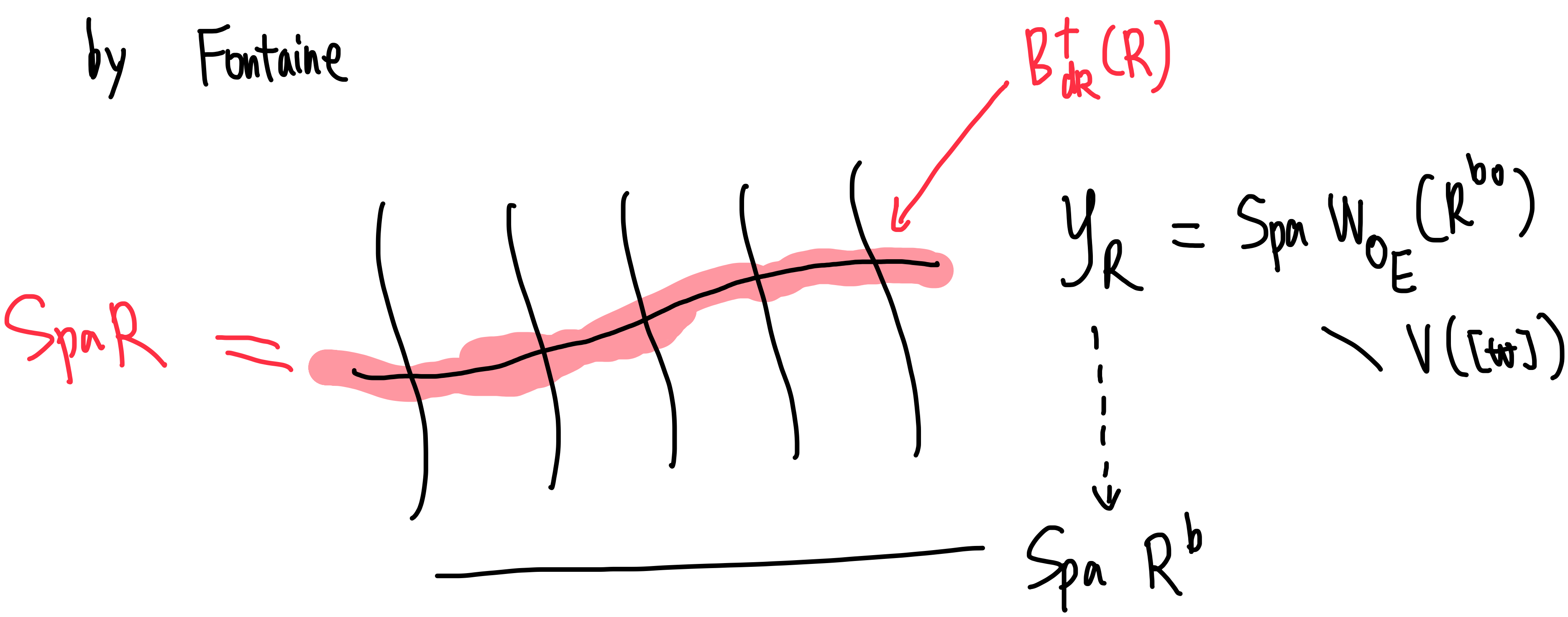
— $R / \mathbb{F}_q$ then $B^+_{dR}(R) = W_{\mathcal{O}_E}(R)$

— $R = C / E$ complete alg closed

$B^+_{dR}(R) \simeq C[[\xi]]$ as abstract E-alg

ring of formal power series of 1-var

— name: "ring of p-adic de Rham periods" by Fontaine



Def'n $\mathrm{Gr}_G^{B_{dR}^+}$ is the étale sheafification functor on

$\mathrm{Perf}/(\mathrm{Spa}\,E)^\diamond \cong \mathrm{Perfd}/E$ taking

$\mathrm{Spa}(R, R^+)/E$

$$\longmapsto G(B_{dR}(R)) / G(B_{dR}^+(R))$$

where $B_{dR}(R) = B_{dR}^+(R)[\frac{1}{\xi}]$

equiv, this is classifying G-bundles on $\mathrm{Spec}\, B_{dR}^+(R)$ with a trivialization over $\mathrm{Spec}\, B_{dR}(R)$

Note $\mathrm{Gr}_G^{B_{dR}^+}$ is a p-adic version of usual affine Grassmanian

$$R/\mathbb{C} \longmapsto G(R((t))) / G(R[[t]])$$

Note If $E = \mathbb{F}_q((t))$ it's literally

$$R / \mathbb{F}_q((t)) \longmapsto G(R((t-s))) / G(R[[t-s]])$$

R is already Banach

$\rightsquigarrow$ different top on B^+_{dR}

$$W_{O_E}(R^\circ) \qquad \xi = t - s$$

$= R^\circ[[t]]$ $\rightsquigarrow$ get usual affine Grassmannian!

Prop'n $$Gr_G^{B^+_{dR}}(C) = \coprod_{\mu \in X^+} G(B^+_{dR}(C))[\mu(\xi)]$$

C complete alg closed field

$\mu(\xi) \in G(B_{dR}(C))$

Ex $G = GL_n$

$\mu = (a_1 \geq \cdots \geq a_n)$

$$\begin{pmatrix} \xi^{a_1} & & \\ & \ddots & \\ & & \xi^{a_n} \end{pmatrix} = \mu(\xi)$$

$Gr_{G_{E'}} = Gr_G \times_E E'$

finite étale

do a descent to reduce to split one

proof $B^+_{dR}(C) \cong C[[\xi]]$ abstractly

so use usual Cartan decomp

$$G(\mathbb{C}[[t]]) \backslash G(\mathbb{C}((t))) / G(\mathbb{C}[[t]]) \overset{G \text{ split}}{=} X^+ \qquad \square$$

<u>Def'n</u> $\forall \mu \in X^+$, let

$$\mathrm{Gr}^{B^+_{dR}}_{G, \le \mu} \subseteq \mathrm{Gr}^{B^+_{dR}}_{G}$$

" Schubert variety " subfunctor of all

$S \longrightarrow \mathrm{Gr}^{B^+_{dR}}_{G}$ s.t. at all

geometric pts, the point lies in $G(B^+_{dR})$-orbit

of $\mu'(\xi)$, where $\mu' \le \mu$

i.e $\mu - \mu'$ sum of pos coroots.

<u>Remark</u> the defn is strange (pt by pt)

but every test obj is <u>reduced</u> in Perf_E

so point-wise def'n works

Main thm in Berkeley note

Thm $\mathrm{Gr}^{B_{dR}^+}_{G,\le\mu} \longrightarrow (\mathrm{Spa}\,E)^\diamond$ proper

and $\mathrm{Gr}^{B_{dR}^+}_{G,\le\mu}$ spatial diamonds

(can work $/\mathcal{O}_E$
the proof over $\mathcal{O}_E$ is harder
while there is a new pf $/E$ using moduli of Sht by Ben Heuer Master thesis)

$$\mathrm{Gr}^{B_{dR}^+}_{G} = \bigcup_\mu \mathrm{Gr}^{B_{dR}^+}_{G\le\mu}$$

transition maps are closed immersions

If μ minuscule, then

$$\mathrm{Gr}^{B_{dR}^+}_{G,\le\mu} \cong (G/P_\mu)^\diamond$$

(If $E/\mathbb{F}_q((t))$, $\mathrm{Gr}_{G,\le\mu}$ is just proj Schubert varieties, but $E/\mathbb{Q}_p$ get diamonds)

proof: a variant of Artin's criterion for alg stacks see Chapter 98 of Stacks Project

Cor $G' \longrightarrow G$ z-ext

$\Longrightarrow$ $\mathrm{Gr}^{B_{dR}^+}_{G'} \longrightarrow \mathrm{Gr}^{B_{dR}^+}_{G}$ is v-cover pf: only needs to show

$\leq \mu' \longrightarrow \ \leq \mu$ is v-cover

$\forall\ \mu' \to \mu$

(here we use $X'^{+} \twoheadrightarrow X^{+}$ surjective not true for $SL_n \to PGL_n$)

These are spatial diamonds in particular qcqs

So surjectivity can be checked at geometric pts

Now it follows from Cartan decomposition

$+\ G'(B^+_{dR}(C)) \to G(B^+_{dR}(C))$ $\square$

(G general, Beauville - Laszlo unif)

<u>Def'n</u> $Gr^{B^+_{dR}}_{G} \longrightarrow Bun_G$ is the following

map:

$$S = Spa(R, R^+) / E$$

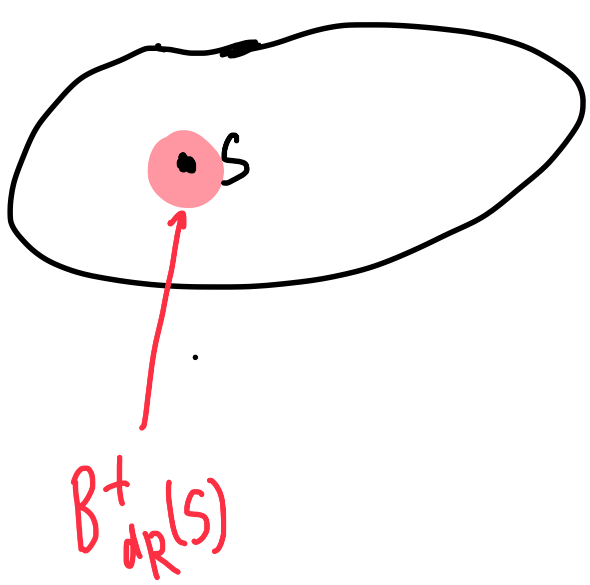
G - torsor $\mathcal{E}_0$ over $B^+_{dR}(R)$

trivialized over $B_{dR}(R)$

Can glue trivial G-torsor on

$X^{alg}_{S^b} \setminus \mathrm{Spec}(R)$ with $\mathcal{E}_0 / B^+_{dR}(R)$

along identification over $B_{dR}(R)$



$X^{alg}_{S^b}$

<u>Lemma</u> (BL gluing lem)

X scheme / E $\quad Z \hookrightarrow X$ Cartier div

Z affine

G - torsor over / X

$\Longleftrightarrow$ $\begin{cases} G\text{-torsor} / X \setminus Z \\ + G\text{-torsor} / X^{\wedge}_{Z} \\ + \text{identification} / X^{\wedge}_{Z} \setminus Z \end{cases}$

Exa $G = GL_2$ restrict to

$\mu = (1,0)$ get $Gr^{B^+_{dR}}_{GL_2, \leq \mu} \cong (\mathbb{P}^1_E)^{\diamond}$

recover previous example

Thm $Gr^{B^+_{dR}}_{G} \longrightarrow Bun_G$ is surjective

for all reductive G

Rek

Analogs of a result of Drinfeld - Simpson

(only true if G semi-simple)
not torus

like $\mathbb{P}^1$ - some pts

Key : Picard gp of the punctured FF curves is trivial, so here it also works for line bundles

Sketch. On geometric pts, due to Fargues | Anschütz, using classification of G-bundle

In general, let $S \longrightarrow Bun_G$ strictly tot disconnected

At $s \in S$ geo pt, can lift to $Gr_G^{B^+_{dR}}$

so get modification $\mathcal{E}'_s$ of $\mathcal{E}_s$

Pick a modif $\mathcal{E}'$ of $\mathcal{E}$ get $\mathcal{E}'_s$

enough: $\mathcal{E}'$ is trivial in a neighborhood of s

Remains to see: $\forall\ \mathcal{E}'/X_{S'}$ the locus where $\mathcal{E}'$ is trivial is open in S

Q: $Gr_G^{B_{dR}^+} \overset{\text{deform}}{\rightsquigarrow} (Gr_G^{Witt})^\lozenge$