12/14 <u>Etale cohomology of diamonds</u> Summary of the paper

<u>Goal</u> Set the foundation to define

$$D(\mathrm{Bun}_G, \mathbb{Z}_l) \qquad (l \neq p)$$

l-adic étale sheaves on general small v-stacks

<u>Recall</u> Perf = perfectoid spaces of char p with v-topology

<u>Small</u> v-sheaf :

$\Leftrightarrow$ a v-sheaf $\mathcal{F}$ s.t

$\exists$ surjection $X \twoheadrightarrow \mathcal{F}$ for some perfd space X

$\mathcal{F} = X/R$ $\quad R \subseteq X \times X$ equiv relation sub-v-sheaf

General Idea for def'n: descent to strict tot disc space

Caution: 6-functor is not compatible with locally profinite sets)

Λ be a ring killed by n

$(n,p)=1$

Goal: $\forall$ small v-stack X define triangulated Λ-linear category

$D_{ét}(X,\Lambda)$ + 6 functors

1) $f: Y \to X$ $\quad f^*$ $\quad Rf_*$

2) $- \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} -$ $\qquad f^*(- \overset{L}{\underset{\Lambda}{\otimes}} -) = f^*(-) \overset{L}{\underset{\Lambda}{\otimes}} f^*(-)$ ☆

$R\mathrm{Hom}_\Lambda(-,-)$

3) If $f: Y \to X$ reps in locally spatial diamonds, ☆compatifiable, dimtrg $f < +\infty$

geometric trans dim

∃ a functor

$$Rf_! : D_{ét}(Y,\Lambda) \longrightarrow D_{ét}(X,\Lambda)$$

and proj formula

$$Rf_!(A \overset{\mathbb{L}}{\otimes}_\Lambda f^* B) = Rf_! A \overset{\mathbb{L}}{\otimes}_\Lambda B \qquad \star$$

and base change

admitting a right adjoint

$$Rf^! : D_{ét}(X,\Lambda) \longrightarrow D_{ét}(Y,\Lambda)$$

Really only need 3: f^*, $- \overset{\mathbb{L}}{\otimes}_\Lambda -$, $Rf_!$

others are right adjoints

$(\star)$:
$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$
$Rf_!$ is defined

then (proper base change)

$$g^* Rf_! = Rf'_! \, g'^*$$

Actually, all $D_{\acute{e}t}(X,\Lambda)$ are homotopy categories of stable Λ-linear ∞-cat $\mathcal{D}_{\acute{e}t}(X,\Lambda)$, and all functors are defined on this level

<u>Descent</u>: $X \longmapsto \mathcal{D}_{\acute{e}t}(X,\Lambda)$ is a v-sheaf of ∞-cat.

(even a hyper-v-sheaf i.e. descent along hypercover)

In particular, if $Y \longrightarrow X$ v-cover

$$\mathcal{D}_{\acute{e}t}(X,\Lambda) \simeq \varprojlim \left(\mathcal{D}_{\acute{e}t}(Y,\Lambda) \rightrightarrows \mathcal{D}_{\acute{e}t}(Y\times_X Y,\Lambda) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{D}_{et}(Y\times_X Y\times_X Y) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

(descent only works on ∞-cat level not homotopy-cat level)

$\Rightarrow$ enough to define $\mathcal{D}_{et}(X,\Lambda)$ for strict tot disc X

Def $D_{et}(X,\Lambda) := D(X_{ét},\Lambda)$

derived ∞-cat of abelian cat of étale Λ-modules on X (in case X tot disc)

+ symmetric monoidal $- \overset{\mathbb{L}}{\otimes}_{\Lambda} -$

+ pull back functoriality

From here, already get well-defined $D_{ét}(X,\Lambda)$ for any small v-stack X

+ $- \overset{\mathbb{L}}{\otimes}_{\Lambda} -$ + f^* ($Rf_!$ is subtle and will be only defined in the homotopy cat $D_{ét}(X,\Lambda)$)

prop f^* adm a right adj Rf_*

$- \otimes^{\mathbb{L}}_{\Lambda} -$ adm a partial right adj $R\mathrm{Hom}_{\Lambda}(-,-)$

Pf: $D_{ét}(X,\Lambda)$ presentable stable ∞-cat

apply Lurie's adjoint functor thm

Q: What's this?

Assume X locally spatial diamond

$\leadsto$ has site $X_{\acute{e}t}$

($f: Y \to X$ étale iff f locally separated

and $\forall$ perf'd space $X' \to X$

$Y' = Y \times_X X' \longrightarrow X'$

is rep by perfd space

and étale

Thm There is a natural functor

$$\mathcal{D}(X_{\acute{e}t}, \Lambda) \longrightarrow \mathcal{D}_{\acute{e}t}(X, \Lambda)$$

def derived ∞-cat of

abelian cat of

étale Λ-mod on X

induce an equiv $\mathcal{D}^+(X_{\acute{e}t}, \Lambda) \cong \mathcal{D}^+_{\acute{e}t}(X, \Lambda)$

and $\mathcal{D}_{et}(X, \Lambda)$ is the left completion i.e $\quad D_{et}(X,\Lambda) \simeq \varprojlim_n D^{\geq n}_{ét}(X,\Lambda)$

$\simeq \varprojlim_n D^{\geq n}(X_{ét},\Lambda)$

In particular, if $D(X_{ét},\Lambda)$ is left-complete

e.g "$X_{ét}$ is finite cohomological dim"

we get $D(X_{ét},\Lambda) \simeq D_{ét}(X,\Lambda)$

<u>Sketch of proof</u>. $D_{ét}(X,\Lambda)$ always has natural t-structure, left-complete.

(reduce to strt. tot. disc X where coh dim $= 0$)

$\Rightarrow$ left complete

$\leadsto$ enough to show

$$D^{\geq 0}(X_{ét},\Lambda) \simeq D^{\geq 0}_{ét}(X,\Lambda) \subseteq D^{\geq 0}(X_v,\Lambda)$$

λ^*

v-topology with set-theoretic issue so cut off $\leq w$

$\lambda : X_v \longrightarrow X_{ét}$

Key λ^* fully faithful i.e

$\forall$ étale Λ-module $\mathcal{F}$

$$\mathcal{F} \cong \lambda_* \lambda^* \mathcal{F}$$

$$R^i \lambda_* \lambda^* \mathcal{F} = 0 \quad \text{for } i > 0$$

" invariance of cohomology under passage from étale site to v-site "

(like one can compute étale coh using fppf cover for schemes)

Sketch 1) étale $\rightsquigarrow$ pro-étale :

write pro-étale cover as filtered limits of étale covers

filtered limits on H^* are exact ✓

2) pro-étale $\rightsquigarrow$ v : By pro-étale descent can assume X strt. tot. disc

Trick: If $Y \longrightarrow X$ v-cover we can

aff'd perf'd

write $Y = \lim\limits_{\leftarrow i} Y_i \longrightarrow X$

each $Y_i \longrightarrow X$ is (open in some finite dim ball over X)

(in particular smooth over X !!)

why v-top is well-behaved

e.g $X = \mathrm{Spa}(R, R^+)$

$Y = \mathrm{Spa}(S, S^+)$

$\exists \quad R\langle T_i^{\frac{1}{p^\infty}} \mid i \in I \rangle \twoheadrightarrow S$ for some set I

$$Y = V(f_j \mid f_j \in J) \subseteq \mathbb{B}_X^I$$

This is a limit of

$$\{ |f_j| \le |\varepsilon| ,\ j \in J' \subseteq J \} \subseteq \mathbb{B}_X^{I'}$$

take as Y_i

$I' \subset I$ finite

$J' \subset J$ finite

(like the proof any Zariski closed imm is pro-étale) descent for $Y \to X$ reduces to

descent for $Y_i \to X$ but this has

a section $\Rightarrow$ descent automatic

$(Y_i \to X)$ $\square$

smooth cover

(not just local spatial diamonds)

prop $\forall$ all small v-stack X

$$D_{et}(X, \Lambda) \hookrightarrow D(X_v, \Lambda)$$

and $A \in D(X_v, \Lambda)$ in the image

$\Leftrightarrow$ $H^i(A)$ in the image

and this can be checked after pullback

to any locally spatial diamond $Y \to X$

then $\Leftrightarrow$ comes from étale Λ-mod on Y

Base change

Thm (qcqs base change)

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

all locally spatial diamonds

f qcqs

$$\Rightarrow \quad Rf'_* g'^* \cong g^* Rf_*$$

on $D^+_{et}(Y, \Lambda) \longrightarrow D^+_{et}(X', \Lambda)$

if f has finite cohom dim even on the whole $D_{et}(Y, \Lambda) \to D_{et}(X, \Lambda)$

no properness needed here

But for strictly local space $\mathrm{Spa}(C, C^+)$

don't have base change wrt.

$$\{s\} \hookrightarrow \mathrm{Spa}(C, C^+)$$
$$\| \qquad\qquad \|$$
$$X' \qquad\qquad X$$

(not an adic space)

pf: If $X' \to X$ pro-étale

descent (sheaf prop) automatic ✓

$Y' \to Y$.. ✓

$\Rightarrow$ assume X', X strict local

Y strict tot disc, even strict local

$X = \mathrm{Spa}(C, C^+)$ $\qquad X' = \mathrm{Spa}(C', C'^+)$

$Y = \mathrm{Spa}(\tilde{C}, \tilde{C}^+) \Rightarrow Y' = \mathrm{Spa}(R, R^+)$

$$R = C' \hat{\otimes}_C \tilde{C} \supseteq R^+$$

<u>Lem</u> $H^i(Y', \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$

Pf: "invariance of coh under base change of alg closed field" by Huber 12

Thm If f is addition proper (proper base change) also have the base change

(Used to define $Rf_!$ later)

Pf. reduce to proper base change of schemes by similar reductions, ☐

$Rf_!$ If $f: Y \to X$ rep in spatial diamonds, compactifiable, $\dim \operatorname{trg} f < +\infty$

$$\begin{tikzcd} j : Y \arrow[r, hook, "\text{open}"] \arrow[dr, "f"'] & \overline{Y}^{/X} \arrow[d] \\ & X \end{tikzcd}$$

$$\begin{matrix} \mathrm{Spa}(R,R^\circ) \\ \cup \\ \mathrm{Spa}(R,R^+) \end{matrix}$$

where ☆ $\underline{\overline{Y}^{/X}(R,R^+)} :\overset{\text{def}}{=} X(R,R^+) \times_{X(R,R^\circ)} Y(R,R^\circ)$

☆ (definition of the compactification)

compactifiable $\Leftrightarrow$ j is open immersion

<u>recall</u> $Y \to X$ proper iff

$$\begin{tikzcd} \mathrm{Spa}(R,R^\circ) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spa}(R,R^+) \arrow[r] \arrow[ur, dashed, "\exists !"] & X \end{tikzcd}$$

$\Leftrightarrow \quad Y \simeq \overline{Y}^{/X}$ iso

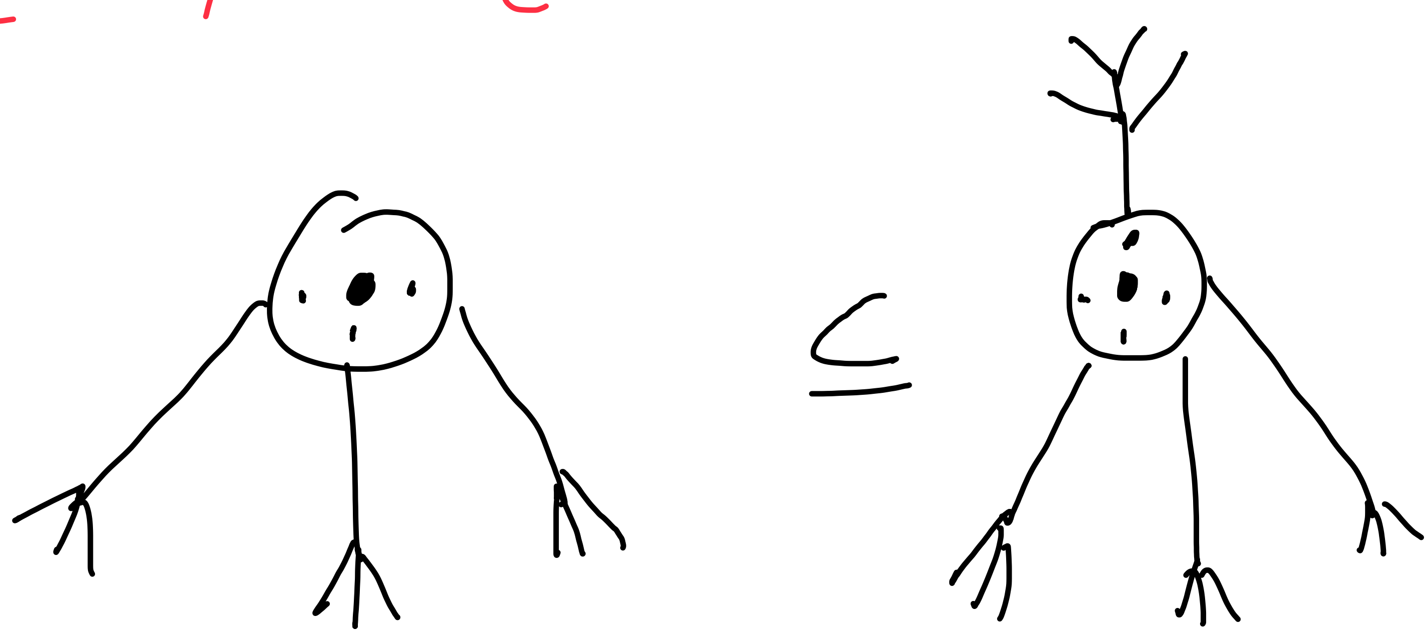
For general $Y \to X$

$\overline{Y}/X$ is proper and is the initial proper diamond $/X$ with a map from Y

Def'n $Rf_! = Rf_*^{\overline{Y}/X} \circ j_!$

($j_! :=$ ext by zero $=$ left adj of j^*)

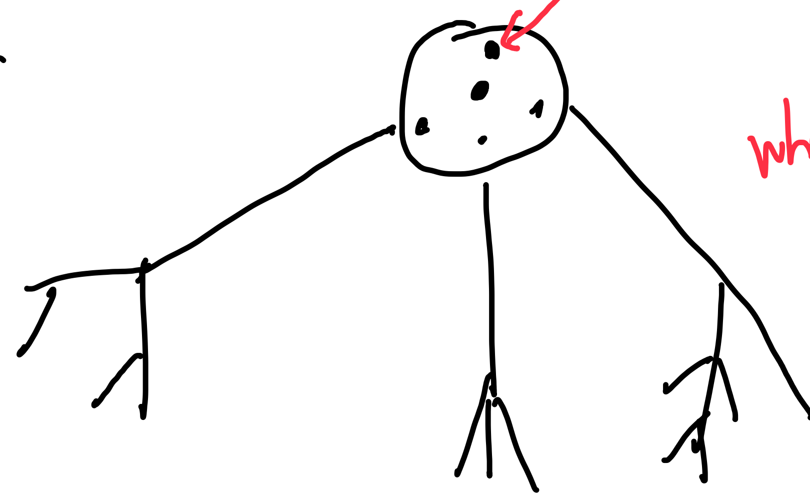
Exa $Y = B_C \longrightarrow X = \mathrm{Spa}\, C$



$\mathrm{Spa}(C\langle T\rangle, \mathcal{O}_C\langle T\rangle) = B_C \longrightarrow \mathbb{P}^1_C$

$B_C \subseteq \overline{B_C}/C \subseteq \mathbb{P}^1_C$

so $\overline{B}^{|C}_{C}$ =



adding extra rk 2 pt where $|T| > 1$ infinitesimally so

$\mathrm{Spa}(C\langle T\rangle, \mathcal{O}_C + m_C\langle T\rangle)$

more generally, can be defined if f is only repr. in locally spatial diamonds

(+ compactifiable, $\dim \mathrm{trg} f < \infty$)

check using this exa

always OK

Thm. $Rf_!$ satisfy base change + proj. formula + composes

Pf. easy from proper base change

(comp of $j_!$ and Rf_* for proper f)

Thm. $Rf_!$ has a right adjoint $Rf^!$.

Verdier duality (cohomological smoothness)

Def'n $f: Y \to X$ (repr in loc. spat. diam, compactif. dim trg $f < +\infty$)

is (Λ-) cohomological smooth

if $Rf^! A \cong f^* A \otimes \mathbb{D}_f$

the degree encodes the dim

for some invertible $\mathbb{D}_f \in D_{ét}(Y, \Lambda)$

(+ after any base change)

no tangent space in perf'd world

So we really use " a thm in classical world " as definition

Non-trivial Task: Find examples that are ah smooth

next time

X usual qcqs rigid space $/C$

prop $\quad |X^{ad}| \overset{(Sp_{\mathfrak{X}})_{\mathfrak{X}}}{\cong} \varprojlim_{\mathfrak{X}} |\mathfrak{X}|$

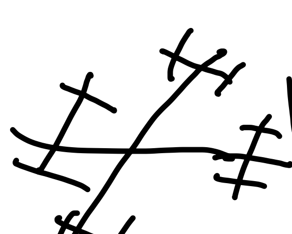
$\mathfrak{X}$ formal model of X

$$|\overline{X}^{/C}| \xrightarrow{\cong} \varprojlim_{\mathfrak{X}} |\overline{\mathfrak{X}}_s|$$

compactification

$\mathfrak{X}_s \hookrightarrow \overline{\mathfrak{X}}_s$ proper

$X = \mathbb{B}_C$



$\mathbb{A}^1$

Q: pro-étale not smooth

$$f : \underline{S} \times \mathrm{Spa}\, C \xrightarrow[\text{pro-étale}]{} \mathrm{Spa}\, C$$

$$\| $$

$$\mathrm{Spa}\, \mathrm{Cont}(S, C)$$

If S is infinite, f not cohomological smooth

$$R\Gamma(S, Rf^!\Lambda) = R\mathrm{Hom}(Rf_!\Lambda, \Lambda)$$

$$= R\mathrm{Hom}(\underbrace{Rf_*\Lambda}_{\mathrm{Cont}(S,\Lambda)}, \Lambda)$$

So $\|$

Λ-valued measures on S

$$(Rf^! \Lambda)_s = \varinjlim_{U \ni s} M(U, \Lambda)$$

if $\underline{s \text{ is not an isolate pt}}$

this is $\neq \Lambda$

- $[* / \underline{G}] \longrightarrow *$ $\underline{G}$ locally prop

is coh smooth

(once the notion is extended to stacks

Q : $\underline{\underline{\dim}}$: $\underline{\text{use uniformization}}$

Q : nearby cycle / vanishing cycle

Λ not tors for rigid spaces

nearby cycle preserving perverseness

(Artin vanishing ✓)

but ULA

some cases of perverse sheaf

gen smooth

Q : $\mathrm{CHaus} \hookrightarrow \mathrm{diamonds}\,|\,\mathrm{Spa}\,C$

$X \longmapsto \underline{X}$

prop $D^{+}_{et}(\underline{X}, \Lambda) = D^{+}(X, \Lambda)$

Q : dim trg f (open question)

Q : ∞ - cat $\xleftrightarrow{\text{motivation}}$ descent !!!

Q: torsion sheaf. (bad)

$R^i \nu_* \nu^* {}^o\mathcal{F} = 0 \quad \forall i > 0 \quad (l \neq p)$

? ?? $(l = p)$

Q: $\mathrm{Spa}(R, R^+) \subseteq \mathrm{Spa}(R, R'^+)$

$\qquad\quad\; \| \qquad\qquad\qquad\quad \|$

$\qquad\quad\; X \qquad\qquad\qquad\quad X'$

${}^o\mathcal{F}$ étale sheaf on X'

then $H^i_{et}(X', {}^o\mathcal{F}) \cong H^i_{et}(X, {}^o\mathcal{F}|_X)$

also $\forall\, \mathcal{G}$ on X

$$H^i_{et}(X, \mathcal{G}) \cong H^i_{et}(X', i_* \mathcal{G})$$

But this fails completely for

cohomology with compact supp coh

$$X = \mathbb{B}_{\mathbb{C}} \overset{\text{open}}{\subset} X' = \overline{\mathbb{B}}_{\mathbb{C}}$$

$$H^i_c(X, \Lambda) \longrightarrow H^i_c(X', \Lambda)$$

$\|$ dual of H^* and H^*_c $\qquad\qquad \|$

$$\begin{cases} 0 & i \neq 2 \\ \Lambda(-1) & i = 2 \end{cases} \qquad\qquad H^i(X', \Lambda) = \begin{cases} \Lambda & i = 0 \\ 0 & i > 0 \end{cases}$$