12/18 Smoothness

Last time Λ ring killed by n

$(n,p)=1$

X small v-stack

$\longmapsto D_{ét}(X,\Lambda)$

$(\cong D(X_{ét},\Lambda)$ if

X locally spatial

$+ \dim \mathrm{trg}\, X < +\infty$)

closed symmetric monoidal trian category

$R\mathrm{Hom}_\Lambda(-,-)$ $\quad - \overset{L}{\otimes}_\Lambda -$

$f: Y \to X \rightsquigarrow f^* \quad Rf_*$

??

If f rep in locally spatial diamonds

compatifiable, (locally) dimtrg $f < +\infty$

$\rightsquigarrow \quad Rf_! : D_{ét}(Y, \Lambda) \longrightarrow D_{ét}(X, \Lambda)$

with a right adjoint $Rf^!$

?? : these conditions makes

$Rf_!$ commutes with all direct sums

(so $\overset{\text{adjoint functor}}{\underset{\text{thm}}{\Longleftrightarrow}}$ it has a right adjoint)

If $A_n \in D_{ét}(Y, \Lambda)$ conc in deg 0 $(n \geq 0)$

then $\bigoplus A_n[n] \simeq \prod A_n[n]$ (left completeness)

$$Rf_!(\bigoplus A_n[n]) \cong Rf_!(\prod_n A_n[n])$$

$$\cong \prod_n (Rf_! A_n)[n]$$

$$\overset{!}{\cong} \bigoplus_n Rf_! A_n[n]$$

$!$ can only be true if $Rf_!$ has finite cohomological dim

Key:

Thm (Scheiderer '94) (No noetherian assumption)
Grothendieck

$\forall\ T$ any spectral topological space

we have $\text{coh.}\dim T \leq \dim T$
(Krull dim)

This is not true for compt Hausdorff spaces

$\left(\prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z} \quad \begin{array}{l} \text{Krull dim} = 0 \\ \text{coh dim} = \infty \end{array} \right)$

So need "locally spatial" assumption $\Rightarrow$ locally spatial

to control cohomological dim.

Now we have 6-functor formalism

want to understand $Rf^!$

<u>Def'n</u> Fix $l \neq p$, $f: Y \to X$ as above

is l-cohomological smooth if after any base change

$$Rf^! \simeq \mathbb{D}_f \otimes f^*$$

$$D_{\acute{e}t}(X, \mathbb{F}_l) \longrightarrow D_{\acute{e}t}(Y, \mathbb{F}_l)$$

and $\mathbb{D}_f$ is locally isom to $\mathbb{F}_l[n]$ $n \in \mathbb{Z}$

<u>Remark</u> So $\mathbb{D}_f = Rf^! \mathbb{F}_l$ by definition

and commutes with any base change

Conversely, if $\begin{cases} Rf^! \mathbb{F}_l \text{ is invertible} \\ Rf^! \mathbb{F}_l \text{ commutes with base change } Y \to X \end{cases}$ only this base change

☆ (easy criterion)

then f $\boxed{l\text{-cohomological smooth}}$

Def'n cohomological smooth if ℓ-cohom smooth $\forall\ \ell \neq p$

Q: independent of ℓ?

then $Rf^! \Lambda \cong Rf^! \mathbb{Z}/n \otimes^{\mathbb{L}}_{\mathbb{Z}/n} \Lambda$

locally isom to $\Lambda[n]$

Examples

The starting pt of the proof

1) $\mathbb{B}^\diamond_C \longrightarrow (\mathrm{Spa}\, C)^\diamond$ is coho smooth

(follows from Huber)

2) If $f: Y \longrightarrow X$ smooth map

locally étale over a ball

of analytic adic spaces $/\mathbb{Z}_p$

then $f^\diamond: Y^\diamond \longrightarrow X^\diamond$ is cohom smooth

3) $(\mathrm{Spa}\, E)^\diamond \longrightarrow (\mathrm{Spa}\, \mathbb{F}_q)^\diamond$ is cohom smooth

$(\mathrm{Div}^1)^d/\Sigma_d = \mathrm{Div}^d \longrightarrow *$ is cohom smooth

$(\mathrm{Spa}\, \mathcal{O}_E)^\diamond \longrightarrow (\mathrm{Spa}\, \mathbb{F}_q)^\diamond$ cohom smooth

4) If $Y \longrightarrow X$ coh smooth

second step of the proof

G pro-p-gp acting freely on $Y \to X$

then $f/\underline{G} : Y/\underline{G} \longrightarrow X$ is still coh smooth

Converse not true!

ex $\quad Y = \mathrm{Spa}\, C \times \underline{G} \longrightarrow X = \mathrm{Spa}\, C$

not cohom smooth

but $\quad Y/\underline{G} \simeq \mathrm{Spa}\, C = \mathrm{Spa}\, C$

5) $L^+G/(L^+G)_\mu \simeq \mathrm{Gr}_{G,\mu} \longrightarrow (\mathrm{Spa}\, E)^\diamond$

stablizer of $[\mu] \in \mathrm{Gr}_G$

open Schubert cell

cohom smooth

but $Gr_{G, \leq \mu}$ is not in general

6) $\mathcal{E}$ vect bundle on X_S

and all fibers have only positive HN slopes
(>0)

then $BC(\mathcal{E}) \longrightarrow S$

cohomological smooth

not true for slope ≥ 0

$$\mathcal{E} = \mathcal{O}_{X_S}, \quad BC(\mathcal{E}) = \underline{E}$$

not cohom smooth

7) Being cohom smooth can be checked

v-locally on the target

we can define coh smooth for Artin stacks (not on the source)

(+ cohom. smooth localizes) because the source

on source

can always be smooth v-locally
made

Artin stacks

Def'n A small v-stack X is *Artin*

if – $\Delta_X : X \longrightarrow X \times X$

rep in locally spatial diamonds

– $\exists\ f : Y \longrightarrow X$ cohom smooth surj

with Y locally spatial diamonds

schemes $\rightsquigarrow$ alg spaces $\rightsquigarrow$ Artin stacks

perf'd space $\rightsquigarrow$ locally spatial diamonds $\rightsquigarrow$ Artin v-stacks

Thm Bun_G is a cohom. smooth Artin v-stack of $\dim = 0$

Here the degree of the dual complex gives the notion of dimension

(note in practice deg is independent of ι
but fail in general deg is even
$\dim := \frac{\text{even}}{2}$

i.e $\pi : \mathrm{Bun}_G \longrightarrow *$ $R\pi^! \Lambda$ locally isom to $\underbrace{\Lambda[0]}$

Sketch of the proof

Show $\mathrm{Gr}_{G,\mu} / \underline{G(E)} \longrightarrow \mathrm{Bun}_G$

fibers are open in $\mathrm{Gr}_{G,\mu^{-1}}$

is cohom. smooth of dim $\langle 2\rho, \mu \rangle$

and $\mathrm{Gr}_{G,\mu} / \underline{G(E)}$ is also a hom smooth of dim $\langle 2\rho, \mu \rangle$

$\Rightarrow$ open image of this map has desired property (Q: what is the image of it)

(cohom smooth $\Rightarrow$ open) proof ?

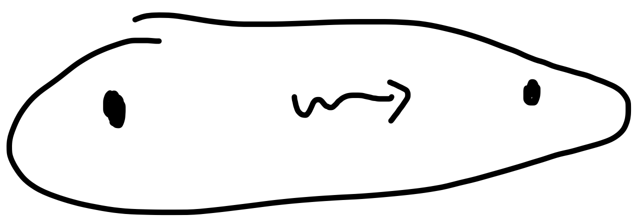
Take $\bigcup_{\mu}$: covers Bun_G $\square$

In particular $[* / \underline{G(E)}] \simeq \mathrm{Bun}_G^1 \subseteq \mathrm{Bun}_G$ is coh smooth $/ *$

To study Bun_G, need better smooth atlas

Example $\quad G = GL_2$

$$\mathcal{O}(\tfrac{1}{2}) \qquad \mathcal{O} \oplus \mathcal{O}(1)$$



$\subseteq \mathrm{Bun}_G$

$\mathcal{U}$ open

want nice atlas for $\mathcal{U}$

$$b \in B(G) \hookrightarrow \mathcal{O} \oplus \mathcal{O}(1)$$

Def'n $\quad M_b$ is the moduli space of extensions

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$$

$\mathcal{L}$: line bundle, $\deg = 0$

$\mathcal{L}'$: line bundle, $\deg = 1$

$\Rightarrow \quad \pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_{GL_2} = \mathrm{Bun}_G$

$$(\mathcal{L} \to \mathcal{E} \to \mathcal{L}') \longmapsto \mathcal{E}$$

Thm π_b cohom smooth, image $= U$

Structure of $\mathcal{M}_b$: $\mathcal{M}_b = \widetilde{\mathcal{M}}_b / \underline{E}^\times \times \underline{E}^\times$

$\widetilde{\mathcal{M}}_b$ param extensions

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

$\Rightarrow \quad \widetilde{\mathcal{M}}_b = BC(\mathcal{O}(-1)[1])$

negative Banach-Colmez space

$$\begin{tikzcd}
\text{1-dim} \quad \widetilde{\mathcal{M}}_b^{\circ} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{M}}_b \arrow[d, "\substack{\text{1-dim} \\ \text{coh smooth}}"] \arrow[dr, phantom, "\square"] & \supseteq \ast \quad \text{0-dim} \arrow[d] \\
\text{0-dim} \quad \mathrm{Bun}_G^{b'} \arrow[r, phantom, "\overset{j}{\subseteq}"] & \mathcal{U} & \overset{i}{\supseteq} \mathrm{Bun}_G^{b}
\end{tikzcd}$$

$\mathrm{Bun}_G^b \simeq [\ast/G_b]$ $\underline{\underline{-1\text{-dim}}}$

$(\mathrm{Bun}_G^b \simeq [\ast/G_b])$

$i^* R j_*$ can be computed by smooth base change on $\widehat{\mathcal{M}}_b$ which is easier

$$\widetilde{\mathcal{M}}_b^{\circ} = BC(\mathcal{O}(-1)[1]) \setminus \{0\}$$

$$\cong \underline{\mathrm{Spa}\, k((t))^{\diamond}} \,/\, \underline{SL_1(D_{\frac{1}{2}})}$$

trivializing $\mathcal{E} \simeq \mathcal{O}(\tfrac{1}{2})$

$D_{\frac{1}{2}}/E$ quat div alg

$BC(\mathcal{O}(\tfrac{1}{2})) \setminus \{0\}$

$\cong \mathrm{Spa}\, k((t))$

$\mathrm{Aut}(\mathcal{E}_{\frac{1}{2}})$

<u>In general</u> will define cohom smooth

$$\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G \quad \text{for any } b \in B(G)$$

$$\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G$$

" Is the strict henselization of Bun_G

at $[*/\underline{G_b(E)}] \longrightarrow \mathrm{Bun}_G^b \longrightarrow \mathrm{Bun}_G$ "

coh. smooth

$$\begin{tikzcd} \widetilde{\mathcal{M}}_b \arrow[r] \arrow[d] & \mathcal{M}_b \arrow[d] \\ * \arrow[u, dashed, bend left, "\tilde{s}"] \arrow[r] & {[*/\underline{G_b(E)}]} \arrow[u, dashed, bend left, "s"] \end{tikzcd}$$

section = preimage of $\mathrm{Bun}_G^b \subseteq \mathrm{Bun}_G$

$$\widetilde{\mathcal{M}}_b^{\circ} = \widetilde{\mathcal{M}}_b \setminus * \longrightarrow \underline{\mathrm{Bun}_G^{>b}} \subseteq \mathrm{Bun}_G$$

spatial diamond

<u>Note</u> : $\widetilde{\mathcal{M}}_b \longrightarrow *$ is repr. in loc spatial diamonds

but $\widetilde{M}_b$ is not a locally spatial diamond

<u>exa</u> $\mathrm{Spa}\, k[[t]] \longrightarrow *$ is rep in local spatial diamond

functor : $\mathrm{Spa}(R,R^+) \longmapsto R^{\circ\circ}$

(as base change to S get open unit disc$/S$)

but $\mathrm{Spa}\, k[[t]]$ is not a diamond, as it has non-analytic pt $\mathrm{Spa}\, k \subseteq \mathrm{Spa}\, k[[t]]$

$\mathrm{Spa}\, k[[t]] \setminus \mathrm{Spa}\, k = \mathrm{Spa}\, k((t))$ <u>is</u> a spatial diamond

$(X \longrightarrow *$ quasi-compt $\not\Leftarrow$ X quasi-compt$)$

<u>Def'n of M_b</u>

For GL_n $\quad b \in B(GL_n)$

$\rightsquigarrow$ $\mathbb{Q}$-graded vect bundle

$$\mathcal{E} = \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{E}^\lambda$$

$\mathcal{M}_b$: param iterated extensions of the $\mathcal{E}^\lambda$

In general

<u>Def'n</u> Let $\mathcal{M}$ be the moduli space taking $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ to (exact-$\otimes$ functor from $\mathrm{Rep}_E G$ to exact cat of $\mathbb{Q}$-filtered vector bundle $\mathcal{E} \supseteq \mathcal{E}^{\leq \lambda}$ s.t

each $\mathcal{E}^{\lambda} = \mathcal{E}^{\le \lambda} \Big/ \bigcup_{\lambda' < \lambda} \mathcal{E}^{\le \lambda'}$ is semi-stable of slope λ

This is increasing Fil, "opposite" to HN filtration!

$$(\mathcal{E} \supseteq \mathcal{E}^{\lambda}) \longmapsto \mathcal{E}$$

$$\begin{tikzcd}
 & \mathcal{M} \arrow[r] \arrow[d] \arrow[dl, "\text{map}"'] & \mathrm{Bun}_G \\
\bigoplus_{\lambda} \mathcal{E}^{\lambda} & \coprod_{b \in B(G)} [* / \underline{G_b(E)}] &
\end{tikzcd}$$

exact $\otimes$-functor w/ values in isocrystals $\simeq$ $\mathbb{Q}$-graded VB $\mathcal{E}^{\lambda}$ ss of slope λ

$\rightsquigarrow \quad \mathcal{M} = \bigsqcup_{b \in B(G)}$

$\rightsquigarrow \quad \mathcal{M}_b \longrightarrow [* / \underline{G_b(E)}]$

Thm $\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G$

are cohomological smooth

(very hard !!)

Remark For $G = GL_n$: can be proved

by linear alg

other classical gp : not just linear alg

e.g. quadratic hyperset

in BC

Next time: Jacobian Criterion

Q: Why is π_b better?

Quasi-compactness

$$\underline{\mathbb{P}^1 / GL_2(E)} \xrightarrow[\text{smooth}]{\text{coh}} GL_2$$

$$\text{image} = \mathcal{U}$$

$$\mathcal{O}(\tfrac{1}{2}), \ \mathcal{O} \oplus \mathcal{O}(1)$$

$$\text{Strata} = \underline{\mathbb{P}^1(E)} / \underline{GL_2(E)} \overset{i}{\hookrightarrow} \mathbb{P}^1 / \underline{GL_2(E)}$$

$$\Omega^2 / \underline{GL_2(E)} \overset{j}{\hookrightarrow} \mathbb{P}^1 / \underline{GL_2(E)}$$

Hope to compute $i^* Rj_* \mathcal{F}$

$$= \lim_{\substack{\longrightarrow \\ \text{small ball} \\ \mathcal{U} \text{ around} \\ \mathbb{P}^1(E)}} R\Gamma(\underbrace{\mathcal{U} \setminus \mathbb{P}(E)/\cdots, \mathcal{F}}_{\varprojlim_{\substack{V \subseteq \mathcal{U}\setminus\mathbb{P}^1(E) \\ \text{qc open}}} R\Gamma(V/\cdots, \mathcal{F})})$$

(not quasi-compt, out of control)

!!

In $\mathcal{M}_b$-chart

see $(i^* Rj_* \mathcal{F})_{\bar{x}} = R\Gamma(\widehat{\mathcal{M}}_b, \mathcal{F})$

so very explicit!

Spatial so quasi-cpt

finite dim diamond

Q: coh smooth map is open

$\Rightarrow$ $Rf_!$ preserve constructible sheaves

has open supports

So $Rf_! \mathbb{F}_\ell$ support $= \mathrm{Im} f$

is (qc) open

Q: $\mathrm{Im}(Gr_{G, \le \mu} \to Bun_G)$

is $\bigsqcup_{b \in B(G,\mu)} Bun^b$.

Q: Classical analogues of

coh smooth?

for π_b

Thm (Drinfeld – Simpson)

$$\mathrm{Bun}_P^{\text{generic}} \longrightarrow \mathrm{Bun}_G$$

is smooth

$\forall\, P \leq G$ parabolic

$$\mathrm{Spa}\, C' \longrightarrow \mathrm{Spa}\, C$$

not compactifiable

$\underline{G} \longrightarrow *$ is not coh smooth

$\uparrow$ profinite $|\cdot| = \infty$ set

Exa $\mathcal{E}$ vect bundle on X_S

each fiber has only negative slopes

then $BC(\mathcal{E}[1]) \to S$

Q: For LLC

need to understand $D(\mathrm{Bun}_G, \Lambda)$

need to understand how the strata interaces

$\underbrace{i^* R j_* \mathcal{F}}$

compatible with smooth base change

so use $\overset{\text{coh}}{\wedge}$ smooth uniformization
good