$D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ and Weil II

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Outline

1. Weights
2. Weil II
3. Applications: AB$\mathbb{C}$DE
4. Descent

Goals

- Set up weight theory, and $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$.
- Recall Weil II, apply it to show $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ and $Perv_m(X_0, \overline{\mathbb{Q}}_\ell)$ have nice properties.
- Prove the key vanishing of higher Ext^i (yoga of weights).
- State Frobenius "descent": $\mathrm{Perv}(X_0) \hookrightarrow \mathrm{Perv}(X, Fr_q)$.

Notations

- $(-)_0/\mathbb{F}_q \rightsquigarrow (-)/k = \mathbb{F}_q^{alg}$.
- $\Lambda = \overline{\mathbb{Q}}_\ell$ $(\ell \neq p)$.
- a lisse $\overline{\mathbb{Q}}_\ell$-sheaf = a continuous finite dim $\overline{\mathbb{Q}}_\ell$-rep of π_1.
- $p_{1/2}$ = selfdual perversity i.e ${}^pD_c^{\leq 0} = \{K | \dim \mathrm{Supp}\, \mathcal{H}^i(K) \leq -i\}$.

What is the Frobenius Fr_q ?

X_0 scheme of finite type over $\mathbb{F}_q$, $\mathcal{F}_0$ a $\overline{\mathbb{Q}}_\ell$-sheaf on X_0.

$\mathrm{Gal}_{\mathbb{F}_q} \cong \widehat{\mathbb{Z}}$ is generated by **geometric Frobenius** $F = (a \mapsto a^{1/q})$.

Example

On $\operatorname{Spec} \mathbb{F}_q$, a $\overline{\mathbb{Q}}_\ell$-sheaf "=" a vector space V with a linear automorphism F.

Easy way: $x \in X_0(\mathbb{F}_q)$, pullback of $\mathcal{F}_0$ along x gives a vector space $\mathcal{F}_{\bar{x}}$ with $\mathrm{Gal}_{\mathbb{F}_q}$-action.

$Fr_{q,x}^* := F \curvearrowright \mathcal{F}_{\bar{x}}$.

What is the Frobenius Fr_q ?

Another way:

k-linear **relative Frobenius** $Fr_q = Fr_{X/k} : X \to X$, with fixed points $X(\mathbb{F}_q)$. $Fr_X = (Fr_k)_X \circ Fr_q$.

Example

$X_0 = \mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_q[t]$, $(Fr_k)_X : a_i t^i \mapsto a_i^q t^i$, $Fr_q : a_i t^i \mapsto a_i t^{iq}$.

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- Absolute Frob doesn't change etale topology, $Fr_X^* \mathcal{F} = \mathcal{F}$.
- $\mathcal{F}$ comes from $\mathcal{F}_0$, $(Fr_k)_X^* \mathcal{F} \cong \mathcal{F}$.

$\rightsquigarrow$ **a natural isomorphism** $\phi : Fr_q^* \mathcal{F} \cong \mathcal{F}$.

$\rightsquigarrow$ $Fr_q^* = \phi_x \curvearrowright \mathcal{F}_{\bar{x}}$ for any $x \in X_0(\mathbb{F}_q)$. $Fr_q^* = Fr_{q,x}^*$.

Punctual purity

For [BBDG], $w \in \mathbb{Z}$.

- weak q**-Weil number** of weight w: $a \in \overline{\mathbb{Q}}$ s.t $|\iota(a)| = q^{w/2}$, $\forall \iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

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- weak q**-Weil number** of weight w: $a \in \overline{\mathbb{Q}}$ s.t $|\iota(a)| = q^{w/2}$, $\forall \iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.
- A $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_0$ on X_0 is **punctually pure of weight** w, if for any n and $x \in X_0(\mathbb{F}_{q^n})$, the eigenvalues of $Fr^*_{q^n}$ on $\mathcal{F}_{\bar{x}}$ are all weak q^n-Weil numbers of weight w.

$\overline{\mathbb{Q}}_\ell(n)$ is puncturally pure of weight $-2n$.

Mixedness for a sheaf

$\mathcal{F}_0$ is **mixed**, if $\exists$ a finite filtration on $\mathcal{F}_0$ with **punctually pure** successive quotients.

$w(\mathcal{F}_0) :=$ punctural weights of $\mathcal{F}_0$.

By definition, mixedness is stable under extensions, subquotients.

This mixed condition is geometric and motivic.

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We can also require $a \in \overline{\mathbb{Z}}$ i.e a q-Weil number.

[Weil II, Remark 1.2.8] and [KW]: ι-mixedness, uses $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, $w \in \mathbb{R}$.

[Weil II, Conj 1.2.9]: every $\overline{\mathbb{Q}}_\ell$-sheaf on X_0 is ι-mixed $(w \in \mathbb{R})$. This is known by works of L. Lafforgue, V. Drinfeld.

Mixedness for a complex: $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$

$D^b_m(X_0, \overline{\mathbb{Q}}_\ell) \subseteq D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ consists of K_0 s.t the **cohomology sheaves $\mathcal{H}^i K_0$ are mixed**.

By long exact seq for $\mathcal{H}^i$, D^b_m is stable under extensions.

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[BBDG, 5.1.6-5.1.7] (to be proved later)

- $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ is stable under $\mathbb{D}$.
- $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ inherits the perverse t-structure.
- Every sub-quotient of a mixed perverse sheaf is mixed.

Lisse $\mathcal{F}_0$ on smooth X_0: ☺☺

Gluing is non-trivial, we need Weil II (e.g Rj_* for open immersion j).

$D^b_{\leq w}, D^b_{\geq w} \subseteq D^b_m$

K_0 in D^b_m is of weight $\leq w$ if punctural weights of $\mathcal{H}^i K_0$ are $\leq w + i$ for any i.

K_0 in D^b_m is of weight $\geq w$ if its **Verdier's dual** $\mathbb{D}(K_0)$ of weight $\leq -w$.

K_0 is **pure of weight** w if $K_0 \in D^b_{\leq w} \cap D^b_{\geq w}$. By duality,

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K_0 is **pure of weight** w if $K_0 \in D^b_{\le w} \cap D^b_{\ge w}$. By duality,

[BBDG,5.1.9]

$K_0 \in D^b_m$ is in $D^b_{\le w}$ (resp $D^b_{\ge w}$), iff for any closed point $i : x_0 \hookrightarrow X_0$, $i^* K_0$ (resp $i^! K_0$) is of weight $\le w$ (resp $\ge w$).

$D^b_{\le w}[1] = D^b_{\le w+1}$. But $D^b_{\le w} \cap D^b_{\ge w+1} = 0$ (to be proved later) is non-trivial. We need Weil II.

Purity=punctual purity for lisse $\mathcal{F}_0$ on smooth X_0

Assume X_0 is smooth of pure dimension d, so $\omega_{X_0} = \overline{\mathbb{Q}}_\ell[2d](d)$. If $\mathcal{H}^i K_0$ are all lisse, then

Proposition

$\mathcal{H}^i(\mathbb{D}K_0) = (\mathcal{H}^{-2d-i}K_0)^\vee(d)$.

Proof:

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Proof:

$\mathbb{D}(\mathcal{H}^i(K_0)) = R\underline{Hom}(\mathcal{H}^i(K_0), \overline{\mathbb{Q}}_\ell[2d](d)) = R\underline{Hom}(\mathcal{H}^i(K_0), \overline{\mathbb{Q}}_\ell)[2d](d)$

(lisse so higher local Ext sheaves = 0)

$= \underline{Hom}(\mathcal{H}^i(K_0), \overline{\mathbb{Q}}_\ell)[2d](d) = \mathcal{H}^i(K_0)^\vee[2d](d)$.

$\rightsquigarrow E_2^{pq} = \mathcal{H}^p(\mathbb{D}(\mathcal{H}^{-q}K_0)) \Rightarrow \mathcal{H}^{p+q}(\mathbb{D}K_0)$ degenerates,

$\mathcal{H}^i(\mathbb{D}K_0) = \mathcal{H}^{-2d}\mathbb{D}(\mathcal{H}^{-2d-i}(K_0))$.

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Proposition

Assumption as above, K_0 is pure of weight w iff each $\mathcal{H}^i K_0$ is punctually pure of weight $w+i$.

$-(-w-2d-i)-2d = w+i$.

In general, $\mathbb{Q}_l$ is puncturally pure but may not be pure . If X_0 is proper, and $\mathbb{Q}_l$ is pure of weight 0, then Frob eigenvalues on $H^i(X)$ has weights exactly i by Weil II, which is not true in general.

It can be pure in some singular cases.

Weil II

[Weil II, 3.3.1, 6.2.3]

If $f : X_0 \to Y_0$ is a (separated) morphism between schemes of finite type over $\mathbb{F}_q$, then $Rf_!$ sends $D^b_{\leq w}$ to $D^b_{\leq w} \subseteq D^b_m$.

Corollary [Weil II, 6.1] (induction + proper case+ smooth case)

Rf_* sends D^b_m to D^b_m .

Weil II

Example

$a : X_0 \to \mathbb{F}_q \rightsquigarrow$ Frob eigenvalues on $H^i_c(X, \mathcal{F})$ are q-Weil numbers of weight $\leq w + i$ for any $\mathcal{F}_0 \in D^b_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell)$.

Example

$X \subseteq \mathbb{P}^{d+1}$ is a smooth geometrically irreducible hypersurface over a finite field $\mathbb{F}_q$. Then $\#X(\mathbb{F}_{q^n}) - \#\mathbb{P}^d(\mathbb{F}_{q^n}) = O(q^{nd/2})$.

Now it's time for applications to D^b_m.

Application A: stabilities of D_m^b

By definition, f^* preserves $D^b_{\leq w}$, and $\otimes$ sends $D^b_{\leq w} \times D^b_{\leq w'}$ to $D^b_{\leq w+w'}$.

Proposition

$K_0 \in D_m^b \Leftrightarrow \mathbb{D}(K_0) \in D_m^b$.

Proof: WLOG K_0 is a mixed sheaf. If X_0 is smooth, K_0 is lisse, then $\mathbb{D}K_0 = K_0^\vee[2d](d)$ by previous computation.

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Proof: WLOG K_0 is a mixed sheaf. If X_0 is smooth, K_0 is lisse, then $\mathbb{D}K_0 = K_0^\vee[2d](d)$ by previous computation.

In general, use Noetherian induction. Choose smooth dense open $j : U_0 \hookrightarrow X_0$ s.t j^*K_0 is lisse.

Exact triangle $j_!j^*K_0 \to K_0 \to i_*i^*K_0$.

$\rightsquigarrow i_*\mathbb{D}(i^*K_0) \to \mathbb{D}(K_0) \to j_*\mathbb{D}(j^*K_0)$.

We're done by induction and Weil II for Rj_*.

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By duality and Weil II, we get

[BBDG,5.1.14]

- $\mathbb{D}$ exchanges $D^b_{\leq w}$ and $D^b_{\geq -w}$.
- $f_!, f^*$ sends $D^b_{\leq w}$ to $D^b_{\leq w}$.
- $f^!, f_*$ sends $D^b_{\geq w}$ to $D^b_{\geq w}$.
- $\otimes$ sends $D^b_{\leq w} \times D^b_{\leq w'}$ to $D^b_{\leq w+w'}$.
- $R\underline{\mathrm{Hom}}$ sends $D^b_{\leq w} \times D^b_{\geq w'}$ to $D^b_{\geq -w+w'}$.

$R\underline{\mathrm{Hom}}(A, \mathbb{D}(B)) = R\underline{\mathrm{Hom}}(A, R\underline{\mathrm{Hom}}(B, \omega_{X_0})) = R\underline{\mathrm{Hom}}(A \otimes B, \omega_{X_0}) = \mathbb{D}(A \otimes B)$.

Application A: stabilities of D_m^b

Therefore, the full subcategory $D_m^b(X_0, \overline{\mathbb{Q}}_\ell)$ of $D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$ is stable by all usual operations e.g $Rf_*, Rf_!, f^*, Rf^!, \otimes, R\underline{\mathrm{Hom}}$, $\mathbb{D}$.

Application B: perverse t-structure on D^b_m

[BBDG, 5.1.7. (i)]

D^b_m is stable under ${}^p\tau_{\leq i}$ and ${}^p\tau_{\geq i}$.

D^b_m is stable under $\tau_{\leq i}$ and $\tau_{\geq i}$ by definition.

${}^p\tau_{\leq 0}(-)$ is constructed by j_*, j^*, i_*, i^* plus truncations and taking cones.

More precisely, $F_1 \to F \to Rj_*{}^p\tau_{\geq 1}j^*F$, ${}^p\tau_{\leq 0}(F) \to F_1 \to Ri_*{}^p\tau_{\geq 1}i^*F_1$.

So ${}^p\tau_{\leq i}, {}^p\tau_{\geq i}$ send D^b_m to D^b_m, we're done.

$\rightsquigarrow D^b_m$ is stable under ${}^pH^*$.

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Application C: vanishing of higher Ext^i

$K_0, L_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$.

[BBDG, 5.1.15]

- If $K_0 \in D^b_{\leq w}, L_0 \in D^b_{\geq w}$, then $\mathrm{Hom}^i(K, L)^F = 0$, for $i > 0$.
- If $K_0 \in D^b_{\leq w}, L_0 \in D^b_{> w}$, then $\mathrm{Hom}^i(K_0, L_0) = 0$, for $i > 0$.

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Crucial for semi-simpleness and decomposition theorem in [BBDG].

Lack of weight theory $\rightsquigarrow$ decomposition theorem fails for coefficients like $\mathbb{F}_\ell$.

Proof: $a : X_0 \to \mathrm{Spec}\,\mathbb{F}_q$, apply Weil II to

$M_0 := Ra_* R\underline{\mathrm{Hom}}(K_0, L_0) \in D^b_c(\mathrm{Spec}\,\mathbb{F}_q)$.

Application D: $D^b_{\leq w} \cap D^b_{\geq w+1} = 0$

$$id_K \in \mathrm{Hom}(K, K)^F = \mathrm{Hom}^1(K, K[-1])^F = 0.$$

$M_0 \in D_c^b(\operatorname{Spec}\mathbb{F}_q)$

$M_0 \in D_c^b(\operatorname{Spec}\mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$, $M := i^* M_0 \in D_c^b(\overline{\mathbb{Q}}_\ell)$ with geom Frob action $F : M \to M$.

Global section / fixed points $R\Gamma : D_c^b(\operatorname{Spec}\mathbb{F}_q, \overline{\mathbb{Q}}_\ell) \to D_c^b(\overline{\mathbb{Q}}_\ell)$.

[BBDG, 5.1.2]

Short exact sequence $0 \to (H^{n-1}M)_F \to H^n R\Gamma M_0 \to (H^n M)^F \to 0$.

$E_2^{pq} = H^p(\operatorname{Spec}\mathbb{F}_q, \mathcal{H}^q M_0) = H^p(\mathrm{Gal}_{\mathbb{F}_q}, H^q M) \Rightarrow H^{p+q} R\Gamma M_0$.

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$\mathrm{Gal}_{\mathbb{F}_q} \cong \widehat{\mathbb{Z}}$, generated by the geometric Frobenius F.

$$H^p(\mathrm{Gal}_{\mathbb{F}_q}, -) = \begin{cases} (-)^F, p = 0 \\ (-)_F, p = 1 \\ 0, else \end{cases}.$$

$M_0 := Ra_* R\underline{\mathrm{Hom}}(K_0, L_0) \in D^b_c(\operatorname{Spec} \mathbb{F}_q)$

$a : X_0 \to \operatorname{Spec} \mathbb{F}_q$. $M_0 = Ra_* R\underline{\mathrm{Hom}}(K_0, L_0)$, $M = i^* M_0$.

Proposition

$M = R\mathrm{Hom}(K, L)$ (smooth base change).

$R\Gamma M_0 = R\mathrm{Hom}(K_0, L_0)$ $(R\Gamma Ra_* = R\Gamma_{X_0})$.

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From above,

$$0 \to (\mathrm{Hom}^{i-1}(K, L))_F \to \mathrm{Hom}^i(K_0, L_0) \to \mathrm{Hom}^i(K, L)^F \to 0.$$

- $K_0 \in D^b_{\le w}, L_0 \in D^b_{\ge w} \rightsquigarrow M_0 \in D^b_{\ge 0} \rightsquigarrow w(\mathrm{Hom}^i(K, L)) \ge i$, $\mathrm{Hom}^i(K, L)^F = 0$, for $i > 0$.
- If $K_0 \in D^b_{\le w}, L_0 \in D^b_{> w}$, then $M_0 \in D^b_{> 0}$. $\mathrm{Hom}^i(K_0, L_0) = 0$, for $i > 0$.

Application C is proved.

Application E: $\mathrm{Perv}_m(X_0)$ is stable under subquotient

$\mathrm{Perv}_m(X_0) = \mathrm{Perv}(X_0) \cap D_m^b$.

[BBDG, 5.1.7. (ii)]

$A_0 \subseteq B_0 \in \mathrm{Perv}(X_0),\ B_0 \in \mathrm{Perv}_m(X_0) \Rightarrow A_0, B_0/A_0 \in \mathrm{Perv}_m(X_0)$.

Proof: If A_0, B_0 are concentrated in one degree, this is obvious.

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In general, choose smooth dense open $j : U_0 \hookrightarrow X_0$ s.t $j^*A_0 \hookrightarrow j^*B_0$ are lisse (up to shift). Then $j^*A_0 = {}^p j^* A_0$ is mixed and perverse.

$\rightsquigarrow {}^p j_! j^* A_0, {}^p j_* j^*(B_0/A_0) \in \mathrm{Perv}_m(X_0)$.

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$\rightsquigarrow {}^p j_! j^* A_0, {}^p j_* j^*(B_0/A_0) \in \text{Perv}_m(X_0)$.

$\rightsquigarrow I = Im({}^p j_! j^* A_0 \to B_0), J = Ker(B_0 \to {}^p j_* j^*(B_0/A_0)) \in \text{Perv}_m(X_0)$.

$I \subseteq A_0 \subseteq J \subseteq B_0$. $J/I \in \text{Perv}_m(Z_0)$ gives $A/I \in \text{Perv}_m(Z_0)$ by induction.

A_0 is extension of A_0/I by I, hence also mixed.

Descent

Let $\mathrm{Perv}(X, Fr_q)$ be the category of perverse sheaves $\mathcal{F}$ on X equipped with an isomorphism $\phi : Fr_q^* \mathcal{F} \to \mathcal{F}$. Now Λ is any suitable coefficient.

Descent

Let $\mathrm{Perv}(X, Fr_q)$ be the category of perverse sheaves $\mathcal{F}$ on X equipped with an isomorphism $\phi : Fr_q^* \mathcal{F} \to \mathcal{F}$. Now Λ is any suitable coefficient.

[BBDG, 5.1.2]

1. The functor $\mathrm{Perv}(X_0) \to \mathrm{Perv}(X, Fr_q), \mathcal{F}_0 \mapsto (\mathcal{F}, Fr_q)$ is fully faithful.
2. The category of essential images is stable by extensions and by sub-quotients.

As perversity gives a t-structure, $\mathrm{Hom}^{-1}(K_0, L_0) = 0$. So $\mathrm{Hom}_{D_c^b(X_0)}(K_0, L_0) = \mathrm{Hom}_{D_c^b(X)}(K, L)^F$, part (1) follows.

Thank you!

Stable under extensions

$(*)\, 0 \to (\mathrm{Hom}(K, L))_F \to Ext^1(K_0, L_0) \to Ext^1(K, L)^F \to 0$, is exact,

where $Ext^1(A, B) = \{0 \to B \to C \to A \to 0\}/\simeq$.

Recall $0 \to B \to C_1 \to A \to 0 \simeq 0 \to B \to C_2 \to A \to 0$ iff there is a $f : C_1 \cong C_2$ such that $f|_A = id_A, f|_B = id_B$.

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[BBDG, 5.1.2]

There is another short exact sequence $(**)$

$0 \to (\mathrm{Hom}(K,L))_F \to EXT^1((K,\phi_K),(L,\phi_L)) \stackrel{\text{forget}}{\to} Ext^1(K,L)^F \to 0$,

where EXT^1= extensions in $\mathrm{Perv}(X, Fr_q)$.

Stable under extensions

Proof:

kernel of forget is given by $(L \oplus K, \begin{pmatrix} \phi_L & U\phi \\ 0 & \phi_K \end{pmatrix})$, where the class is determined by U modulo $\phi Fr_q^*(V)\phi^{-1} - V$ for $V : K \to L$. Hence the kernel is the coinvariant.

So $Ext^1(K_0, L_0) \cong EXT^1((K,\phi),(L,\phi))$ by $(*)$ and $(**)$.

Stable under subquotients

Do induction as in application E.

A singular example

References

- Section 5.1 of [BBDG] Beilinson, Bernstein, Deligne, Gabber: Faisceaux pervers (book).
- Notes on a learning Seminar on Deligne's Weil II Theorem, Umich Summer 2016.
- P. Deligne, Weil II.
- V. Drinfeld, On a conjecture of Deligne, Moscow Math. J. 12 (2012), 515-542.